Engineering Functional Quantum Algorithms

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Suppose that a quantum circuit with K elementary gates is known for a unitary matrix U, and assume that U^m is a scalar matrix for some positive integer m. We show that a function of U can be realized on a quantum computer with at most $O(mK + m^2 \log m)$ elementary gates. The functions of U are realized by a generic quantum circuit, which has a particularly simple structure. Among other results, we obtain efficient circuits for the fractional Fourier transform.

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Let U be a unitary matrix, $U \in \mathcal{U}(2^n)$. Suppose that a fast quantum algorithm is known for U, which is given by a factorization of the form

$$U = U_1 U_2 \cdots U_K, \tag{1}$$

where the unitary matrices U_i are realized by controlled-not gates or by single qubit gates [1]. We are interested in the following question:

> *Are there efficient quantum algorithms for unitary matrices, which are functions of U?*

The question is puzzling, because the knowledge of the factorization (1) of U does not seem to be of much help in finding similar factorizations for, say, $V = U^{1/3}$. The purpose of this letter is to give an answer to the above question for a wide range of unitary matrices U.

Our solution to this problem is based on a generic circuit which implements arbitrary functions of U, assuming that U^m is a scalar matrix for some positive integer m. If m is small, then our method provides an efficient quantum circuit for V.

Notations. We denote by $\mathcal{U}(m)$ the group of unitary $m \times m$ matrices, by $\mathbf{1}$ the identity matrix, and by $\mathbf{C}$ the field of complex numbers.

I. PRELIMINARIES

We recall some standard material on matrix functions, see [2, 3, 4] for more details. Let U be a unitary matrix. The spectral theorem states that U is unitarily equivalent to a diagonal matrix D, that is, $U = TDT^\dagger$ for some unitary matrix T. The elements λ_i on the diagonal of $D = \mathrm{diag}(\lambda_1, \dots, \lambda_{2^n})$ are the eigenvalues of U.

Let f be any function of complex scalars such that its domain contains the eigenvalues λ_i, $1 \leq i \leq 2^n$. The matrix function $f(U)$ is then defined by

$$f(U) = T\mathrm{diag}(f(\lambda_1), \dots, f(\lambda_{2^n}))\, T^\dagger,$$

where T denotes the diagonalizing matrix of U, as above.

Notice that any two scalar functions f and g, which take the same values on the spectrum of U, yield the same matrix value $f(U) = g(U)$. In particular, one can find an interpolation polynomial g, which takes the same values as f on the eigenvalues λ_i. It is possible to assume that the degree of g is smaller than the degree of the minimal polynomial of U. In other words, $V = f(U)$ can be expressed by a linear combination of integral powers of the matrix U,

$$V = f(U) = \sum_{i=0}^{m-1} \alpha_i U^i, \tag{2}$$

where m is the degree of the minimal polynomial of the matrix U, and $\alpha_i \in \mathbf{C}$ for $i = 0, \dots, m-1$. In order for V to be unitary, it is necessary and sufficient that the function f maps the eigenvalues λ_i of U to elements on the unit circle.

Remark. There exist several different definitions for matrix functions. The relationshop between these definitions is discussed in detail in [5]. We have chosen the most general definition that allows to express the function values by polynomials.

II. THE GENERIC CIRCUIT

Let U be a unitary $2^n \times 2^n$ matrix with minimal polynomial of degree m. We assume that an efficient quantum circuit is known for U. How can we go about implementing the linear combination (2)? We will use an ancillary system of μ quantum bits, where μ is chosen such that $2^{\mu-1} < m \leq 2^\mu$ holds. This will allow us to create the linear combination by manipulating somewhat larger matrices, which on input $|0\rangle \otimes |\psi\rangle \in \mathbf{C}^{2^\mu} \otimes \mathbf{C}^{2^n}$ produce the state $|0\rangle \otimes V\,|\psi\rangle$.

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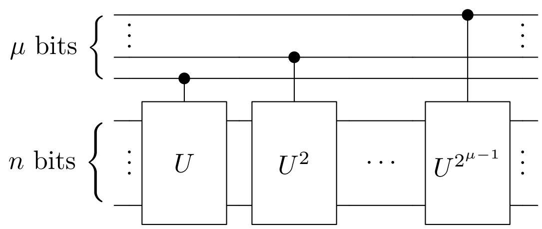

FIG. 1: A quantum circuit realizing the block diagonal matrix $A = \mathrm{diag}(1, U, U^2, \ldots, U^{2^\mu - 1})$.

We first bring the ancillary system into a superposition of the first m computational base states, such that an input state $|0\rangle \otimes |\psi\rangle \in \mathbf{C}^{2^\mu} \otimes \mathbf{C}^{2^n}$ is mapped to the state

$$\frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} |i\rangle \otimes |\psi\rangle \,. \qquad (3)$$

This can be done by acting with a $2^\mu \times 2^\mu$ unitary matrix B on the ancillary system, where the first column of B is of the form $1/\sqrt{m}(1, \ldots, 1, 0, \ldots, 0)^t$. Efficient implementations of B exist.

Notice that there exists an efficient implementation of the block diagonal matrix $A = \mathrm{diag}(1, U, U^2, \ldots, U^{2^\mu - 1})$. Indeed, A can be composed of the matrices U^{2^η}, $0 \leq \eta < \mu$, conditioned on the μ ancillae bits. The resulting implementation is shown in Fig. 1. The state (3) is transformed by this circuit into the state

$$\frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} |i\rangle \otimes U^i \, |\psi\rangle \,. \qquad (4)$$

In the next step, we let a $2^\mu \times 2^\mu$ matrix M act on the ancillae bits. We choose M such that the state (4) is mapped to

$$\frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |k\rangle \otimes U^k V \, |\psi\rangle \qquad (5)$$

It turns out that M can be realized by a unitary matrix, assuming that the minimal polynomial of U is of the form $x^m - \tau$, $\tau \in \mathbf{C}$. This will be explained in some detail in the next section.

We apply the inverse $A^\dagger$ of the block diagonal matrix A. This transforms the state (5) to

$$\frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |k\rangle \otimes V \, |\psi\rangle \,. \qquad (6)$$

We can clean up the ancillae bits by applying the $2^\mu \times 2^\mu$ matrix $B^\dagger$. This yields then the output state

$$|0\rangle \otimes V \, |\psi\rangle = |0\rangle \otimes f(U) \, |\psi\rangle \,. \qquad (7)$$

The steps from the input state $|0\rangle \otimes |\psi\rangle$ to the final output state $|0\rangle \otimes V \, |\psi\rangle$ are illustrated in Fig. 2 for the case $\mu = 2$.

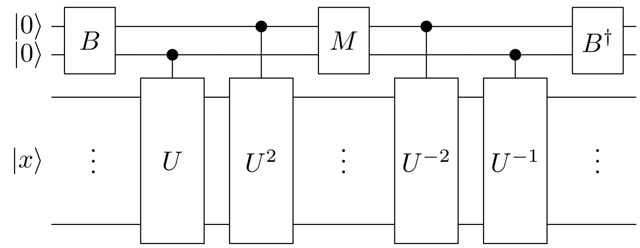

FIG. 2: Generic circuit realizing a linear combination V. The case $\mu = 2$ is shown.

The following theorem gives an upper bound on the complexity of the method. We use the number of elementary gates (that is, the number of single qubit gates and controlled-not gates) as a measure of complexity.

Theorem 1 *Let U be a $2^n \times 2^n$ unitary matrix with minimal polynomial $x^m - \tau$, $\tau \in \mathbf{C}$. Suppose that there exists a quantum algorithm for U using K elementary gates. Then a unitary matrix $V = f(U)$ can be realized with at most $O(mK + m^2 \log m)$ elementary operations.*

Proof. A matrix acting on $\mu \in O(\log m)$ qubits can be realized with at most $O(m^2 \log m)$ elementary operations, cf. [1]. Therefore, the matrices $B, B^\dagger$, and M can be realized with a total of at most $O(3m^2 \log m)$ operations.

If K operations are needed to implement U, then at most $14K$ operations are needed to implement $\Lambda_1(U)$, the operation U controlled by a single qubit. The reason is that a doubly controlled NOT gate can be implemented with 14 elementary gates [6], and a controlled single qubit gate can be implemented with six or fewer elementary gates [1].

We observe that $2^\mu - 1$ copies of $\Lambda_1(U)$ suffice to implement A. Indeed, we certainly can implement $\Lambda_1(U^{2^k})$ by a sequence of 2^k circuits $\Lambda_1(U)$. This bold implementation yields the estimate for A. Typically, we will be able to find much more efficient implementations. Anyway, we can conclude that A and $A^\dagger$ can both be implemented by at most $14(2^\mu - 1)K \in O(14mK)$ operations. Combining our counts yields the result. □

III. UNITARITY OF THE MATRIX M

It remains to show that the state (4) can be transformed into the state (5) by acting with a unitary matrix M on the system of μ ancillae qubits. This is the crucial step in the previously described method.

Let U be a unitary matrix with a minimal polynomial of degree m. A unitary matrix $V = f(U)$ can then be represented by a linear combination

$$V = \sum_{i=0}^{m-1} \alpha_i U^i. \qquad (8)$$

We will motivate the construction of the matrix M by examining in some detail the resulting linear combinations of the matrices U^kV. From (8), we obtain

$$U^kV = \sum_{i=0}^{m-1} \alpha_i U^{i+k}. \qquad (9)$$

Suppose that the minimal polynomial of U is of the form $m(x) = x^m - g(x)$, with $g(x) = \sum_{i=0}^{m-1} g_i x^i$. The right hand side of (9) can be reduced to a polynomial in U of degree less than m using the relation $U^m = g(U)$:

$$U^kV = \sum_{i=0}^{m-1} \beta_{ki} U^i.$$

The coefficients β_{ki} are explicitly given by

$$(\beta_{k0}, \beta_{k1}, \ldots, \beta_{k(m-1)}) = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1})P^k$$

where P denotes the companion matrix of $m(x)$, that is,

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ g_0 & g_1 & g_2 & \cdots & g_{m-1} \end{pmatrix}.$$

The $2^\mu \times 2^\mu$ matrix M is defined by

$$M = \begin{pmatrix} C & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

where $C = (\beta_{ki})_{k,i=0,\ldots,m-1}$, and $\mathbf{1}$ is a $(2^\mu - m) \times (2^\mu - m)$ identity matrix. Under the assumptions of Theorem 1, it turns out that the matrix M is unitary. Before proving this claim, let us formally check that the matrix M transforms the state (4) into the state (5). If we apply the matrix M to the ancillary system, then we obtain from (4) the state

$$\begin{aligned} \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} M\,|i\rangle \otimes U^i\,|\psi\rangle &= \frac{1}{\sqrt{m}} \sum_{k,i=0}^{m-1} \beta_{ki}\,|k\rangle \otimes U^i\,|\psi\rangle \\ &= \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |k\rangle \otimes \sum_{i=0}^{m-1} \beta_{ki} U^i\,|\psi\rangle \\ &= \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |k\rangle \otimes U^kV\,|\psi\rangle \end{aligned}$$

which coincides with (5), as claimed.

Lemma 2 *Let U be a unitary matrix with minimal polynomial $m(x) = x^m - \tau$. Let V be a matrix satisfying (2). If V is unitary, then M is unitary.*

Proof. It suffices to show that the matrix C is unitary. Notice that the assumption on the minimal polynomial $m(x)$ implies that C is of the form

$$C = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{m-2} & \alpha_{m-1} \\ \tau\alpha_{m-1} & \alpha_0 & \cdots & \alpha_{m-3} & \alpha_{m-2} \\ \ddots & \ddots & & \ddots & \ddots \\ \tau\alpha_1 & \tau\alpha_2 & \cdots & \tau\alpha_{m-1} & \alpha_0 \end{pmatrix},$$

that is, C is obtained from a circulant matrix by multiplying every entry below the diagonal by τ. In other words, we have

$$C = \Big([\tau]_{i>j}\, \alpha_{j-i \bmod m} \Big)_{i,j=0,\ldots,m-1}$$

where $[\tau]_{i>j} = \tau$ if $i > j$, and $[\tau]_{i>j} = 1$ otherwise.

Note that the inner product of row a with row b of matrix C is the same as the inner product of row $a+1$ with row $b+1$. Thus, to prove the unitarity of C, it suffices to show that

$$\delta_{a,0} \stackrel{!}{=} \langle \text{row a} | \text{row } 0 \rangle = \sum_{j=0}^{a-1} \overline{\tau}\, \overline{\alpha_{j-a}} \alpha_j + \sum_{j=a}^{m-1} \overline{\alpha_{j-a}} \alpha_j \qquad (10)$$

holds, where $\delta_{a,0}$ denotes the Kronecker delta and the indices of α are understood modulo m.

Consider the equation

$$\mathbf{1} = V^\dagger V = \left(\sum_{i=0}^{m-1} \overline{\alpha_i} U^{-i} \right) \left(\sum_{i=0}^{m-1} \alpha_i U^i \right) \qquad (11)$$

The right hand side can be simplified to a polynomial in U of degree less than m using the identity $\overline{\tau}\, U^m = \mathbf{1}$. The coefficient of U^a in (11) is exactly the right hand side of equation (10). Since the minimal polynomial of U is of degree m, it follows that the matrices $U^0, U^1, \ldots, U^{m-1}$ are linearly independent. Thus, comparing coefficients on both sides of equation (11) shows (10). Hence the rows of C are pairwise orthogonal and of unit norm. □

A Simple Example. Let F_n be the discrete Fourier transform matrix

$$F_n = 2^{-n/2} (\exp(-2\pi i\, k\ell/2^n))_{k,\ell=0,\ldots,2^n-1},$$

with $i^2 = -1$. Recall that the Cooley-Tukey decomposition yields a fast quantum algorithm, which implements F_n with $O(n^2)$ elementary operations. The minimal polynomial of F_n is $x^4 - 1$ if $n \geq 3$. Thus, any unitary matrix V, which is a function of F_n, can be realized with $O(n^2)$ operations.

For instance, if $n \geq 3$, then the fractional power F_n^x, $x \in \mathbf{R}$, can be expressed by

$$F_n^x = \alpha_0(x) I + \alpha_1(x) F_n + \alpha_2(x) F_n^2 + \alpha_3(x) F_n^3,$$

where the coefficients $\alpha_i(x)$ are given by (cf. [7]):

$$\alpha_0(x) = \tfrac{1}{2}(1 + e^{ix}) \cos x, \quad \alpha_1(x) = \tfrac{1}{2}(1 - ie^{ix}) \sin x,$$
$$\alpha_2(x) = \tfrac{1}{2}(-1 + e^{ix}) \cos x, \quad \alpha_3(x) = \tfrac{1}{2}(-1 - ie^{ix}) \sin x.$$

In this case, F_n^x is realized by the circuit in Fig. 2 with $U = F_n$ and $M = (\alpha_{j-i}(x))_{i,j=0,\dots,3}$. The circuit can be implemented with $O(n^2)$ operations.

IV. LIMITATIONS

The previous sections showed that a unitary matrix $f(U)$ can be realized by a linear combination of the powers U^i, $0 \leq i < m$, if the minimal polynomial $m(x)$ of U is of the form $x^m - \tau$, $\tau \in \mathbf{C}$. One might wonder whether the restriction to minimal polynomials of this form is really necessary. The next lemma explains why we had this limitation:

Lemma 3 *Let U be a unitary matrix with minimal polynomial $m(x) = x^m - g(x)$, $\deg g(x) < m$. If $g(x)$ is not a constant, then the matrix M is in general not unitary.*

Proof. Suppose that $g(x) = \sum_{i=0}^{m-1} g_i x^i$. We may choose for instance $V = U^m = g(U)$. Then the norm of first row in M is greater than 1. Indeed, we can calculate this norm to be $|g_0|^2 + |g_1|^2 + \cdots + |g_{m-1}|^2$. However, $|g_0|^2 = 1$, because g_0 is a product of eigenvalues of U. By assumption, there is another nonzero coefficient g_i, which proves the result. □

V. EXTENSIONS

We describe in this section one possibility to extend our approach to a larger class of unitary matrices U. We assumed so far that $f(U)$ is realized by a linear combination (2) of *linearly independent* matrices U^i. The exponents were restricted to the range $0 \leq i < m$, where m is degree of the minimal polynomial of U. We can circumvent the problem indicated in the previous section by allowing m to be larger than the degree of the minimal polynomial.

Theorem 4 *Let $U \in \mathcal{U}(2^n)$ be a unitary matrix such that U^m is a scalar matrix for some positive integer m. Suppose that there exists a quantum circuit which implements U with K elementary gates. Then a unitary matrix $V = f(U)$ can be realized with $O(mK + m^2 \log m)$ elementary operations.*

Proof. By assumption, $U^m = \tau \mathbf{1}$ for some $\tau \in \mathbf{C}$. This means that the minimal polynomial $m(x)$ of U divides the polynomial $x^m - \tau$, that is, $x^m - \tau = m(x) m_2(x)$ for some $m_2(x) \in \mathbf{C}[x]$.

We may assume without loss of generality that the function f is defined at all roots of $x^m - \tau$. Indeed, we can replace f by an interpolation polynomial g satisfying $f(U) = g(U)$ if this is necessary.

Choose any unitary matrix $A \in U(2^n)$ with minimal polynomial $m_2(x)$. The minimal polynomial of the block diagonal matrix $U_A = \operatorname{diag}(U, A)$ is $x^m - \tau$, the least common multiple of the polynomials $m(x)$ and $m_2(x)$. Express $f(U_A)$ by powers of the block diagonal matrix U_A:

$$f(U_A) = \operatorname{diag}(f(U), f(A)) = \sum_{i=0}^{m-1} \alpha_i \operatorname{diag}(U^i, A^i). \quad (12)$$

The approach detailed in Section III yields a unitary matrix M to realize this linear combination. On the other hand, we obtain from (12) the relation

$$f(U) = \sum_{i=0}^{m-1} \alpha_i U^i$$

by ignoring the auxiliary matrices A^i, $0 \leq i < m$. It is clear that a circuit of the type shown in Fig. 2 with μ chosen such that $2^{\mu-1} < m \leq 2^\mu$ implements this linear combination of the matrices U^i, $0 \leq i < m$, provided we use the matrix M constructed above. □

VI. CONCLUSIONS

Few methods are currently known that facilitate the engineering of quantum algorithms. Linear algebra allowed us to derive efficient quantum circuits for $f(U)$, given an efficient quantum circuit for U, as long as U^m is a scalar matrix for some small integer m. This method can be used in conjuction with the Fourier sampling techniques by Shor [8], the eigenvalue estimation technique by Kitaev [9], and the probability amplitude amplification method by Grover [10], to design more elaborate quantum algorithms.

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