Graphs, Quadratic Forms, and Quantum Codes

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***Abstract*— We show that any stabilizer code over a finite field is equivalent to a graphical quantum code. Furthermore we prove that a graphical quantum code over a finite field is a stabilizer code. The technique used in the proof establishes a new connection between quantum codes and quadratic forms. We provide some simple examples to illustrate our results.**

***Index Terms*— Graphs, quadratic forms, quantum error-correcting codes.**

I. Graphical Quantum Codes

Let A be the additive group of a finite field $\mathbb{F}_{p^m}$. Denote by $\mathcal{H}$ the complex vector space $\mathbb{C}^\alpha$ of dimension $\alpha = |A|$. Let B be an orthonormal basis of $\mathcal{H}^{\otimes n}$ consisting of basis vectors $|y\rangle$ labeled by elements of the group A^n. Let $K \cong A^k$ and $N \cong A^n$ be subgroups of A^{k+n} such that $A^{k+n} = K \times N$.

Following the definition of Schlingemann and Werner in [9], a *graphical quantum code* is an α^k-dimensional subspace Q of $\mathcal{H}^{\otimes n}$, which is spanned by the vectors

$$|x\rangle = \frac{1}{\sqrt{\alpha^n}} \sum_{y\in N} \Big(\prod_{\substack{i,j=1\\ i<j}}^{k+n} \chi(z_i, z_j)^{\Gamma_{ij}} \Big) |y\rangle, \tag{1}$$

where $x \in K$ and $z = x + y \in K \times N \cong A^{k+n}$. The coefficients on the right hand side are given by the values of a non-degenerate symmetric bicharacter χ on $A \times A$. The exponents Γ_{ij} are given by the adjacency matrix Γ of a weighted undirected graph with integral weights, $\Gamma_{ij} \in \mathbb{Z}$. As (1) is independent of the diagonal elements Γ_{ii}, we can assume without loss of generality that the graph has no loops.

In [9] the authors raised the question whether or not every stabilizer code is equivalent to a graphical quantum code. Our main result gives an affirmative answer to this question:

Theorem 1: Any stabilizer code over the alphabet $A = \mathbb{F}_{p^m}$ is equivalent to a graphical quantum code. Conversely, any graphical quantum code over A is a stabilizer code.

In the sequel, we will prove this theorem. First, we show that any graphical code over an extension field $\mathbb{F}_{p^m}$ can be reformulated as a graphical code over the prime field $\mathbb{F}_p$. Then we compute the stabilizer associated with a graphical code, followed by the construction of a graphical representation of a stabilizer code. We conclude by giving examples which illustrate both directions of our main theorem.

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Lemma 1: Any symmetric bicharacter χ over the abelian group $A \cong \mathbb{F}_p^m$ can be written as

$$\chi(h, g) = \exp\left(\frac{2\pi i}{p} b(h, g)\right), \tag{2}$$

where b is a symmetric bilinear form over $\mathbb{F}_p$, i.e.,

$$b(h, g) = h^{\mathsf{t}} M g$$

where M is a symmetric matrix over $\mathbb{F}_p$.

Proof: For fixed $h \in A$, the mapping $g \mapsto \chi(h, g)$ is a character of A. Any character ζ of A can be written as $\zeta(g) = \exp(2\pi i/p \cdot h^{\mathsf{t}} g)$ where $h^{\mathsf{t}} g$ denotes the inner product of the group element g identified with a vector in $\mathbb{F}_p^m$ and the vector $h \in \mathbb{F}_p^m$. As $\chi(h_1 + h_2, g) = \chi(h_1, g)\chi(h_2, g)$ and the group A is (non-canonically) isomorphic to its character group A^*, the bicharacter χ can be written as

$$\chi(h, g) = \exp\left(\frac{2\pi i}{p} (Mh)^{\mathsf{t}} g\right),$$

where M is an $m \times m$ matrix over $\mathbb{F}_p$. Symmetry of the bicharacter implies symmetry of M. ∎

Using this lemma, eq. (1) can be rewritten as

$$\begin{aligned} |x\rangle &= \frac{1}{\sqrt{\alpha^n}} \sum_{y\in N} \Big(\prod_{\substack{i,j=1\\ i<j}}^{k+n} \exp(2\pi i/p \cdot (z_i^{\mathsf{t}} M z_j))^{\Gamma_{ij}} \Big) |y\rangle \\ &= \frac{1}{\sqrt{\alpha^n}} \sum_{y\in N} \exp\left(\frac{2\pi i}{p} q(v)\right) |y\rangle. \end{aligned} \tag{3}$$

Here we identify $x + y \in \mathbb{F}_{p^m}^{k+n}$ with $v = (v_i) \in \mathbb{F}_p^{m(k+n)}$. Furthermore, q is the quadratic form

$$q(v) := \sum_{\substack{i,j=1\\ i<j}}^{m(k+n)} \Gamma'_{ij} v_i v_j \tag{4}$$

on $\mathbb{F}_p^{m(k+n)}$ defined by the symmetric matrix $\Gamma' := \Gamma \otimes M$. Hence the states (1) of the graphical quantum code Q can be expressed in the form

$$|x\rangle = \frac{1}{\sqrt{\alpha^n}} \sum_{y\in N} \zeta(q(x + y))|y\rangle, \tag{5}$$

where ζ is a non-trivial additive character of $\mathbb{F}_p$ and q is the quadratic form (4) on $\mathbb{F}_p^{m(k+n)}$. We will take advantage of this presentation in the following sections.

Finally, identifying the vector space $\mathbb{C}^{p^m}$ with $(\mathbb{C}^p)^{\otimes m}$ allows us to reformulate (1) in the form

$$|x\rangle = \frac{1}{\sqrt{\alpha^n}} \sum_{y\in \mathbb{F}_p^{mn}} \Big(\prod_{\substack{i,j=1\\ i<j}}^{m(k+n)} \tilde\chi(v_i, v_j)^{\Gamma'_{ij}} \Big) |y\rangle, \tag{6}$$

where $\tilde{\chi}$ is a (non-trivial) bicharacter on $\mathbb{F}_p$ and $x \in \mathbb{F}_p^{mk}$, $v = (x, y) \in \mathbb{F}_p^{m(k+n)}$. Therefore, it is sufficient to study graphical quantum codes over the prime field $\mathbb{F}_p$.

II. Orthonormal Basis of a Graphical Quantum Code

In general, the vectors defined by (1) need not form a basis of the graphical quantum code. In this section, we derive conditions for the bicharacter χ and the graph Γ under which the vectors form an orthonormal basis of the code.

From the preceding it is sufficient to consider a graphical quantum code Q defined over the additive group A of the prime field $\mathbb{F}_p$. The code is spanned by the vectors $|x\rangle$ according to (6). We can associate with the quadratic form q of (4) a symmetric bilinear form b on A^{n+k} given by

$$b(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2).$$

For $v_1 = x \in K$ and $v_2 = y \in N$, this implies a block structure of the adjacency matrix Γ, namely

$$\Gamma = \left(\begin{array}{c|c} M_x & B \\ \hline B^{\mathsf{t}} & M_y \end{array}\right), \tag{7}$$

where the symmetric matrices M_x and M_y correspond to the restriction of the quadratic form q to K and N, resp., and the $k \times n$ matrix B corresponds to the bilinear form $b(x, y) = x^{\mathsf{t}} B y$. From (5) we get

$$\begin{aligned} |x\rangle &= \frac{1}{\sqrt{\alpha^n}} \sum_{y \in N} \zeta(q(x+y))|y\rangle \\ &= \frac{1}{\sqrt{\alpha^n}} \sum_{y \in N} \zeta(b(x,y) + q(x) + q(y))|y\rangle \\ &= \frac{\zeta(q(x))}{\sqrt{\alpha^n}} \sum_{y \in N} \zeta(b(x,y) + q(y))|y\rangle. \end{aligned} \tag{8}$$

Notice that $\zeta(q(x))$ yields an insignificant phase factor and thus we can assume without loss of generality that K is totally isotropic, i.e., $q(x) = 0$ for all $x \in K$. For the adjacency matrix Γ, this implies $M_x = 0$. If ζ is the trivial character, the coefficients of the right hand side are independent of x. In this case the code is one-dimensional. Hence we require ζ to be non-trivial, say $\zeta(g) = \exp(2\pi i/p \cdot g)$.

The inner product of two base states $|x\rangle$ and $|x'\rangle$ of the code is given by

$$\begin{aligned} \langle x'|x\rangle &= \frac{1}{|N|} \sum_{y \in N} \overline{\zeta(b(x', y) + q(y))} \zeta(b(x, y) + q(y)) \\ &= \frac{1}{|N|} \sum_{y \in N} \zeta(b(x, y) - b(x', y)) \\ &= \frac{1}{|N|} \sum_{y \in N} \zeta(b(x - x', y)). \end{aligned}$$

This sum is either zero or one, that is, the vectors are either orthogonal or identical. The sum vanishes unless $b(x - x', y) = (x - x')^{\mathsf{t}} B y = 0$ for all $y \in N$, that is, unless $x - x'$ lies in the kernel of B. Imposing orthogonality of different states, i.e., $\langle x'|x\rangle = 0$ unless $x' = x$, implies that this kernel needs to be trivial. In other words, if we view B as a matrix over $\mathbb{F}_p$ then the rank of this matrix is k.

III. The Stabilizer of a Graphical Quantum Code

For $a, d \in \mathbb{F}_p^n$, we define the following operators:

$$\begin{aligned} X^a &:= \sum_{y \in \mathbb{F}_p^n} |y + a\rangle\langle y| \\ \text{and} \quad Z^d &:= \sum_{y \in \mathbb{F}_p^n} \omega^{d^{\mathsf{t}} y} |y\rangle\langle y|, \end{aligned}$$

where $\omega \in C$ is a primitive p^{th} root of unity. The set of unitary operators $\mathcal{E} := \{X^a Z^d \colon a, d \in \mathbb{F}_p^n\}$ is an orthonormal basis for the vector space of $p^n \times p^n$ matrices with respect to the trace inner product $\langle A|B\rangle = \operatorname{tr}(A^\dagger B)/p^n$. What is more, $\mathcal{E}$ defines a *nice error basis* [6]. The group G generated by $\mathcal{E}$ is an extra-special p-group. The elements $X^a Z^d$ and $X^{a'} Z^{d'}$ commute up to scalars as follows [1]

$$(X^a Z^d)(X^{a'} Z^{d'}) = \omega^{\langle a, d' \rangle - \langle a', d \rangle} (X^{a'} Z^{d'})(X^a Z^d), \tag{9}$$

where we have used the standard inner product $\langle a, d \rangle := \sum_{i=1}^n a_i d_i \in \mathbb{F}_p$. Equivalently, two elements of the group G commute if the vectors $(a, d), (a', d') \in \mathbb{F}_p^{2n}$ are orthogonal with respect to the symplectic inner product [7]

$$\langle a, d' \rangle - \langle a', d \rangle. \tag{10}$$

Hence, the elements of G are given by $\{\omega^\gamma X^a Z^d \colon \gamma \in \mathbb{F}_p, a, d \in \mathbb{F}_p^n\}$.

A *stabilizer code* $Q \subseteq \mathcal{H}^{\otimes n}$ with respect to the nice error basis $\mathcal{E}$ corresponds to a joint eigenspace of an abelian normal subgroup of G. In order to compute the stabilizer group of the graphical code, we consider the action of an operator $\omega^\gamma X^a Z^d$ on the states (5). Recall that $\omega^\gamma X^a Z^d$ belongs to the stabilizer of the graphical quantum code Q if and only if $\omega^\gamma X^a Z^d |x\rangle = |x\rangle$ for all $x \in K$. We claim that the stabilizer of the graphical quantum code Q is given by the following set of operators

$$S_Q = \{\omega^{q(a)} X^a Z^{a M_y} \mid a \in N \text{ such that } Ba = 0\}, \tag{11}$$

where M_y denotes the $n \times n$ submatrix of the adjacency matrix Γ defined in (7). Indeed, using the character $\zeta(\gamma) := \omega^\gamma$ in (5), straightforward calculation shows that

$$\begin{aligned} &\omega^\gamma X^a Z^d |x\rangle \\ &= \frac{1}{\sqrt{\alpha^n}} \sum_{y \in N} \zeta(\gamma + b(x, y - a) + q(y - a) + d^{\mathsf{t}}(y - a))|y\rangle. \end{aligned} \tag{12}$$

Comparing equations (12) and (8) yields

$$\zeta(\gamma + b(x, y-a) + q(y-a) + d^{\mathsf{t}}(y-a)) = \zeta(b(x,y) + q(y))$$

for all $x \in K$ and $y \in N$. As the character χ is faithful, we can simplify this to

$$\gamma - b(x, a) - b(a, y) + q(a) + d^{\mathsf{t}}(y - a) = 0. \tag{13}$$

This formula holds for any choice of $x \in K$, thus $b(x, a) = 0$ for all $x \in K$. Whence the argument $a \in N$ satisfies the constraint $b(x, a) = x^{\mathsf{t}} B a = 0$ for all $x \in K$, implying $Ba = 0$ as claimed. Moreover, equation (13) can be simplified to

$$\gamma - b(a, y) + q(a) + d^{\mathsf{t}}(y - a) = 0. \tag{14}$$

Since this holds for all $y \in N$, we get $-b(a, y) + d^{\mathsf{t}} y = 0$ for all $y \in N$; hence $-b(a, y) + d^{\mathsf{t}} y = (d - a M_y)^{\mathsf{t}} y = 0$ for all $y \in N$. This shows that $Z^d = Z^{aM_y}$. Finally, substituting $y = a$ in (14) yields $\gamma = q(a)$, which proves the claim.

Two different elements $\omega^{q(a)} X^a Z^{aM_y}, \omega^{q(a')} X^{a'} Z^{a' M_y} \in S_Q$ commute, since the symplectic inner product (10) of (a, aM_y) and $(a', a'M_y)$ vanishes as the matrix M_y is symmetric. As the rank of the matrix B is assumed to be k, there are $n-k$ different vectors a that satisfy the constraint. Hence the group generated by S_Q has at least p^{n-k} elements.

A projection onto the joint eigenspace $E_1 \subseteq \mathcal{H}^{\otimes n}$ with eigenvalue 1 of the operators in S_Q generating the group S is given by

$$P = \frac{1}{|S|} \sum_{M \in S} M.$$

The dimension of E_1 is bounded from above by $\dim E_1 = \operatorname{tr} P = p^n/|S| \leq p^n/|S_Q| = p^k$. Since E_1 contains Q and $\dim Q = p^k$, this shows that $E_1 = Q$. We can conclude that Q is a stabilizer code.

Omitting the phase factors $\omega^{q(a)}$, we obtain an equivalent code Q' whose stabilizer group is

$$S_{Q'} := \{X^a Z^{aM_y} \colon a \in \langle B \rangle^\perp = \langle D \rangle\}. \tag{15}$$

Here $\langle B \rangle^\perp = \langle D \rangle$ denotes the linear space of elements $a \in N$ such that $Ba = 0$, and the $(n-k) \times n$ matrix D is formed by a basis of that space. The stabilizer group $S_{Q'}$ gives rise to a symplectic code over $\mathbb{F}_p \times \mathbb{F}_p$ [7]. This code is generated by the matrix $D(I + \alpha \cdot M_y)$ where $\{1, \alpha\}$ is a basis of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$. Additionally, we have used the isomorphism of $\mathbb{F}_p \times \mathbb{F}_p$ and $\mathbb{F}_{p^2}$ as vector spaces.

IV. The Graph of a Stabilizer Code

In [1] it is shown that any stabilizer code that is defined via a code over $\mathbb{F}_{p^m}$ can be regarded as a stabilizer code over $\mathbb{F}_p$. Hence it is sufficient to consider stabilizer codes over a space $\mathcal{H}$ of prime dimension. Those codes correspond to symplectic codes over $\mathbb{F}_{p^2}$ [7].

Let $\mathcal{C}$ be a symplectic code over $\mathbb{F}_{p^2}$ which is generated by the matrix $X + \alpha Z =: (X|Z)$ where $\{1, \alpha\}$ is a basis of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$ and $X, Z \in \mathbb{F}_p^{(n-k)\times n}$. Furthermore, let $\mathcal{C}^\perp$ denote the orthogonal code of $\mathcal{C}$ with respect to the symplectic inner product on $\mathbb{F}_{p^2}^n$. As $\mathcal{C} \leq \mathcal{C}^\perp$, there exists a self-dual code $\mathcal{D}$ with $\mathcal{C} \leq \mathcal{D} = \mathcal{D}^\perp \leq \mathcal{C}^\perp$. Identifying $\mathbb{F}_{p^2}^n$ and $\mathbb{F}_p^{2n}$ and rearranging the coordinates, we can choose a generator matrix for $\mathcal{D}$ of the form

$$G' := (X'|Z') = \left(\begin{array}{c|c} X & Z \\ \hline \widetilde{X} & \widetilde{Z} \end{array} \right). \tag{16}$$

The group of isometries of the symplectic space $\mathbb{F}_{p^2}^n$ that additionally preserve the Hamming weight is given by the wreath product of the symplectic group $\mathrm{Sp}_2(p)$ and the symmetric group S_n [7]. The symmetric group acts on the generator matrix (16) by simultaneously permuting columns of X' and Z'. The elements of $\mathrm{Sp}_2(p)$ operate from the right on the i^{th} column of the submatrix X and the i^{th} column of the submatrix Z. On the rows of G' operates the full linear group. Similar to Gauß' algorithm, G' can assumed to be of the form $G' = (I|P)$ [4]. As $\mathcal{D}$ is self-dual with respect to the symplectic inner product (10), we have $I \cdot P^{\mathrm{t}} - P \cdot I = 0$, i. e., P is symmetric.

In the next step we perform column operations in order to obtain a matrix of the form $(I|C)$ where C is symmetric and all entries on the diagonal of C are zero. The code generated by $(I|C)$ is equivalent to $\mathcal{D}$, and in particular, it contains a subcode that is equivalent to $\mathcal{C}$. Let this subcode be generated by $D \cdot (I|C)$. Furthermore, let B be a $k \times n$ parity check matrix for the linear code $[n, n-k]$ over $\mathbb{F}_p$ generated by D. Then the matrix

$$\Gamma = \left(\begin{array}{c|c} 0 & B \\ \hline B^{\mathrm{t}} & C \end{array} \right)$$

is of the form (7). The entries of Γ are elements of $\mathbb{F}_p$ which can be interpreted as integers modulo p. As Γ is symmetric and the diagonal entries are zero, Γ is the adjacency matrix of a weighted undirected graph. Repeating the arguments of Section I and Section III, it can be shown that the graphical quantum code defined by Γ is equivalent to $\mathcal{C}$. Hence, for any stabilizer code over $\mathbb{F}_{p^m}$ there exists an equivalent graphical quantum code.

V. Examples

A. *The stabilizer of a graphical quantum code*

Consider the highly symmetric graph depicted in Fig. 1. This graph is called the wheel W_7. All edges in this graph have the same weight, hence its adjacency matrix is given by

$$\Gamma_{W_7} = \left(\begin{array}{c|ccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right),$$

where we have indicated the block structure as in eq. (7). The 7×1 submatrix B corresponds to the repetition code, so the matrix D (cf. eq. (15)) generates the even weight code of length 7. Using the notation of (16), we obtain

$$\begin{aligned} G &= D \cdot (I|M_y) \\ &= \left(\begin{array}{ccccccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right), \end{aligned}$$

where M_y is the lower right 7×7 submatrix of Γ_{W_7}. The corresponding additive code over $\mathrm{GF}(4)$ [2] is generated by

$$G_4 := \begin{pmatrix} \alpha^2 & \alpha & 0 & 0 & 0 & \alpha & \alpha^2 \\ 0 & 1 & \alpha & 0 & 0 & \alpha & 1 \\ \alpha & \alpha & 1 & \alpha & 0 & \alpha & 1 \\ \alpha & 0 & \alpha & 1 & \alpha & \alpha & 1 \\ \alpha & 0 & 0 & \alpha & 1 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \alpha & \alpha^2 & \alpha^2 \end{pmatrix},$$

where α denotes a primitive element of $\mathrm{GF}(4)$. The additive code $C_4 = (7, 2^6)$ generated by G_4 is not $\mathrm{GF}(4)$-linear as G_4

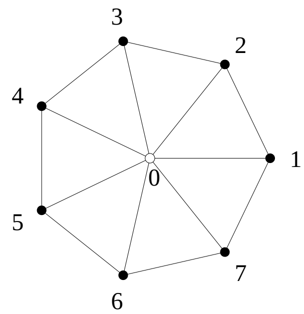

Fig. 1. The wheel W_7 with 7 vertices of degree 3 yields a $[\![7,1,3]\!]$ QECC.

has rank 6 over $\mathrm{GF}(4)$. The weight distribution of C_4 and its dual $C_4^\perp$ are

$$W_{C_4}(x,y) = x^7 + 21x^3y^4 + 42xy^6$$
$$W_{C_4^\perp}(x,y) = x^7 + 21x^4y^3 + 21x^3y^4 + 126x^2y^5 + 42xy^6 + 45y^7.$$

Thus the corresponding stabilizer code has minimum distance 3.

B. The graph of a stabilizer code

Consider the CSS code (cf. [3], [10]) $[\![7,1,3]\!]$ derived from the $[7,4,3]$ Hamming code over $\mathbb{F}_2$. The corresponding additive code $\mathcal{C}_7$ is generated by

$$G = \left(\begin{array}{ccccccc|ccccccc}
1&0&0&1&0&1&1&&&&&&&\\
0&1&0&1&1&1&0&&&&&&&\\
0&0&1&0&1&1&1&&&&&&&\\
&&&&&&&1&0&0&1&0&1&1\\
&&&&&&&0&1&0&1&1&1&0\\
&&&&&&&0&0&1&0&1&1&1
\end{array}\right),$$

which has block diagonal form. In order to obtain a generator matrix G' for the self-orthogonal code $\mathcal{D}$ with $\mathcal{C} \le \mathcal{D} \le \mathcal{C}^\perp$, we have to add a non-zero vector of the complement of $\mathcal{C}$ in $\mathcal{C}^\perp$ to G. In our example, we choose the all-ones vector. Next we transform G' into the form $(I|C)$ where C is symmetric. We obtain

$$\begin{aligned}(I|C) &= T \cdot G' \cdot S\\
&= \left(\begin{array}{ccccccc|ccccccc}
1&0&0&0&0&0&0&0&0&1&1&1&0&0\\
0&1&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&1&0&0&0&0&1&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&1&1\\
0&0&0&0&1&0&0&1&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&1&0&0
\end{array}\right),\end{aligned} \tag{17}$$

where $S \in \mathrm{Sp}_2(2) \wr \mathrm{S}_7$, and $T \in \mathrm{GL}_7(2)$ is given by

$$T = \begin{pmatrix}
1&0&1&1&0&1&0\\
1&0&0&1&0&0&1\\
0&0&0&1&1&1&0\\
0&0&0&1&0&0&0\\
0&0&0&1&1&0&0\\
1&1&0&1&1&0&1\\
1&1&1&1&1&1&1
\end{pmatrix}.$$

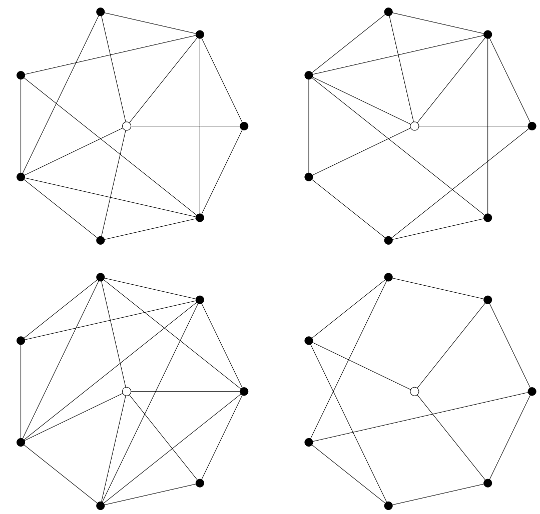

Fig. 2. Four non-isomorphic graphs which yield graphical quantum codes that are equivalent to the CSS code $[\![7,1,3]\!]$.

In our example, the matrix B^{t} is given by the last column of T. This leads to the adjacency matrix

$$\Gamma_{\text{Hamming}} = \left(\begin{array}{c|ccccccc}
0&0&1&0&0&0&1&1\\ \hline
0&0&0&1&1&1&0&0\\
1&0&0&1&0&1&0&0\\
0&1&1&0&0&0&1&0\\
0&1&0&0&0&0&1&1\\
0&1&1&0&0&0&0&1\\
1&0&0&1&1&0&0&0\\
1&0&0&0&1&1&0&0
\end{array}\right).$$

Note that the “normal” form (17) is not unique, wherefore the corresponding graph is not unique either. In Fig. 2 we have depicted four obviously non-isomorphic graphs which all lead to graphical quantum codes that are equivalent to the CSS code $[\![7,1,3]\!]$. The lower right graph is a permuted version of Γ_{Hamming}. As in Fig. 1, the first node is drawn as an open circle. Furthermore, none of the graphs reflects the cyclic symmetry of the quantum code.

VI. Conclusions

We have shown that any stabilizer code over a finite field has an equivalent representation as a graphical quantum code. Unfortunately, this representation is not unique, neither does it reflect all the properties of the quantum code. However, the construction of good quantum codes with the help of graphs is a promising avenue for further research. It should be noted that independent of this work, Dirk Schlingemann has also established the equivalence of graphical quantum codes and stabilizer codes [8].

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