# On self-concordant barriers for generalized power cones

Scott Roy[*] Lin Xiao[†]

January 30, 2018

## Abstract

In the study of interior-point methods for nonsymmetric conic optimization and their applications, Nesterov [5] introduced the power cone, together with a 4-self-concordant barrier for it. In his PhD thesis, Chares [2] found an improved 3-self-concordant barrier for the power cone. In addition, he introduced the generalized power cone, and conjectured a "nearly optimal" self-concordant barrier for it. In this short note, we prove Chares' conjecture. As a byproduct of our analysis, we derive a self-concordant barrier for a high-dimensional nonnegative power cone.

## 1 Introduction

Self-concordant barriers play a central role in interior-point methods for convex optimization [6], especially for conic optimization [1]. Let $\mathcal{K}$ be a *normal* convex cone (closed, pointed and with nonempty interior) in $\mathbb{R}^n$. The standard conic optimization problem is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \in \mathcal{K}, \end{array} \tag{1}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A$ is an $m$ by $n$ matrix. For several symmetric cones (including the nonnegative orthant, the second-order cone, and the positive semidefinite cone), their self-concordant barriers are well understood, and efficient interior-point methods have been developed and well tested in practice.

The development of interior-point methods for *nonsymmetric* conic optimization faces several challenges, including the difficulty of computing the conjugate barriers. Nesterov made several progresses for nonsymmetric conic optimization [3, 4, 5], followed by some more recent development by others [2, 7, 8]. As for the symmetric case, the efficiency of interior-point methods for nonsymmetric conic optimization heavily relies on the properties of the self-concordant barriers [4, 8].

---

[*] University of Washington, Department of Mathematics, Seattle, WA 98195. Email: `scottroy@uw.edu`.

[†] Microsoft Research, Redmond, WA 98052. Email: `lin.xiao@microsoft.com`.

Let $X \in \mathbb{R}^n$ be an open convex set. A function $F : X \to \mathbb{R}$ is (standard) self-concordant if it is three-times continuously differentiable and the inequality

$$\left|D^3F(x)[h,h,h]\right| \leq 2D^2F(x)[h,h]^{3/2} \tag{2}$$

holds for any $x \in \operatorname{dom}(F)$ and $h \in \mathbb{R}^n$, where

$$D^kF(x)[h_1,\dots,h_k] = \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1=\cdots=t_k=0} F(x + t_1h_1 + \cdots + t_kh_k)$$

is the $k$th differential of $F$ taken at $x$ along the directions $h_1,\dots,h_k$. In addition, $F$ is a barrier of $X$ if it blows up at the boundary of $X$, i.e., $F(x) \to \infty$ as $x \to \partial X$. For a convex cone $\mathcal{K}$, the natural barriers are *logarithmically homogeneous*:

$$F(\tau x) = F(x) - \nu \log(\tau), \qquad \forall x \in \operatorname{int}\mathcal{K},\ \tau > 0,$$

for some parameter $\nu > 0$. If $F$ is also self-concordant, then we call $F$ a $\nu$-self-concordant barrier of $\mathcal{K}$ [6, Section 2.3.3].

The best known iteration complexity of interior-point methods to generate an $\epsilon$-solution to the conic optimization problem (1) is $O(\sqrt{\nu}\log(1/\epsilon)$, for both symmetric cones [6] and nonsymmetric cones [5, 8]. Therefore, it is desirable to construct self-concordant barriers with a small parameter $\nu$. In this note, we focus on self-concordant barriers of the generalized power cones, a special class of nonsymmetric cones proposed by Chares [2].

## 2 Self-concordant barriers for generalized power cones

Nesterov [5] introduced the three-dimensional power cone

$$\mathcal{K}_{\text{power}} = \left\{(x,y,z) \in \mathbb{R}^2_+ \times \mathbb{R} : x^\theta y^{1-\theta} \geq |z|\right\}$$

where the parameter $\theta \in (0,1)$, to model constraints involving powers . For example, the inequality $|y|^p \leq t$ (with $p > 1$) holds if and only if $(t,1,y)$ lies in the power cone with parameter $\theta = p^{-1}$. Nesterov constructed a 4-self-concordant barrier for the power cone in [5], and Chares found an improved 3-self-concordant barrier for the power cone in [2]. In addition, Chares proposed the $(n,m)$-generalized power cone

$$\mathcal{K}_\alpha^{(n,m)} = \left\{(x,z) \in \mathbb{R}^n_+ \times \mathbb{R}^m : \prod_{i=1}^n x_i^{\alpha_i} \geq \|z\|_2\right\}.$$

where the parameters $\alpha$ belong to the simplex $\Delta_n := \{\alpha \in \mathbb{R}^n : \alpha_i \geq 0,\ \sum_{i=1}^n \alpha_i = 1\}$. When $n = 2$ and $m = 1$, the generalized power cone reduces to the usual power cone.

Chares conjectured that

$$F(x,z) := -\log\left(\prod_{i=1}^n x_i^{2\alpha_i} - \|z\|_2^2\right) - \sum_{i=1}^n (1-\alpha_i)\log(x_i)$$

is an $(n+1)$-self-concordant barrier for $\mathcal{K}_\alpha^{(n,m)}$. Moreover, he proved that any self-concordant barrier for $\mathcal{K}^{(n,m)}$ has parameter at least $n$. Therefore, if his conjecture is true, then this proposed barrier is nearly optimal. In this short note, we prove this conjecture.

One application for the generalized power cone is to model the rotated positive power cone. Let $\alpha \in \Delta_m$ be in the simplex, and let $a_1, \ldots, a_m \in \mathbb{R}^n$ be nonnegative vectors. Nemirovski and Tunçel [9] give a self-concordant barrier for the rotated positive power cone

$$\mathcal{C} = \left\{ (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : \prod_{i=1}^m \langle a_i,\ x\rangle^{\alpha_i} \geq t \right\} \tag{3}$$

with parameter $\nu = 1 + \left(\frac{7}{3}\right)^2 n$. Using Chares' proposed barrier for the generalized power cone, one can construct an $(m+2)$-self-concordant barrier for $\mathcal{C}$ [2, Section 3.1.4]. Indeed, observe the inclusion $(x,t) \in \mathcal{C}$ holds if and only if the inclusions $(Ax,t) \in \mathcal{K}_\alpha^{(m,1)}$ and $t \in \mathbb{R}_+$ hold, where $A$ is a matrix with rows given by the vectors $a_i$. We can therefore construct a self-concordant barrier with parameter $m+1$ for the constraint $(Ax,t) \in \mathcal{K}_\alpha^{(m,1)}$ and another with parameter 1 for the constraint $t \in \mathbb{R}_+$. Their sum is a self-concordant barrier for $\mathcal{C}$ with parameter $m+2$. In conclusion, the approach using Chares' power cone is beneficial compared to Nemirovski's and Tunçcel's barrier when $m \leq \left(\frac{7}{3}\right)^2 n - 1 \approx 5n$.

In fact, we can construct a self-concordant barrier for $\mathcal{C}$ with a slightly better parameter. More specifically, we give an $(n+1)$-self-concordant barrier for the high-dimensional nonnegative power cone

$$\mathcal{K}_\alpha^{(n,+)} = \left\{ (x,z) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : \prod_{i=1}^n x_i^{\alpha_i} \geq z \right\},$$

where $\alpha \in \Delta_n$. This implies an $(m+1)$-self-concordant barrier for $\mathcal{C}$ defined in (3).

Our main results are summarized as follows.

**Theorem 1.** *The function*

$$F(x,z) := -\log\left( \prod_{i=1}^n x_i^{2\alpha_i} - \|z\|_2^2 \right) - \sum_{i=1}^n (1-\alpha_i)\log(x_i)$$

*is an $(n+1)$-self-concordant barrier for the $(n,m)$-generalized power cone*

$$\mathcal{K}_\alpha^{(n,m)} = \left\{ (x,z) \in \mathbb{R}^n_+ \times \mathbb{R}^m : \prod_{i=1}^n x_i^{\alpha_i} \geq \|z\|_2 \right\}.$$

**Theorem 2.** *The function*

$$F(x,z) := -\log\left( \prod_{i=1}^n x_i^{\alpha_i} - z \right) - \sum_{i=1}^n (1-\alpha_i)\log(x_i) - \log(z)$$

*is an $(n+1)$-self-concordant barrier for the high-dimensional nonnegative power cone*

$$\mathcal{K}_\alpha^{+} = \left\{ (x,z) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : \prod_{i=1}^n x_i^{\alpha_i} \geq z \right\}.$$

# 3 Proofs of main results

The rest of this note is devoted to proving Theorem 1 and Theorem 2. In what follows, we use the number of primes to denote the order of differential of a function taken at a point $x$ along the common direction $d$. In other words, we denote $G' = DG(x)[d]$, $G'' = D^2G(x)[d,d]$, and $G''' = D^3G(x)[d,d,d]$. We first prove the following two lemmas.

**Lemma 1** (Composition with logarithm)**.** *Fix a point $x$ and direction $d$. Suppose that $f$ is a positive concave function. Moreover, suppose that $G$ is convex and satisfies $G''' \le 2(G'')^{3/2}$. If $f$ and $G$ satisfy*

$$3(G'')^{1/2} f'' \le f''', \tag{4}$$

*then the function*

$$F := -\log(f) + G$$

*satisfies $F''' \le 2(F'')^{3/2}$.*

*Proof.* Let $\sigma_1 = \left(\frac{f'}{f}\right)^2$, $\sigma_2 = \frac{-f''}{f}$, and $\sigma_3 = G''$. The hypotheses imply each $\sigma_i$ is nonnegative. Now simple calculations yield the following:

$$\begin{aligned}
F'' &= \sigma_1 + \sigma_2 + \sigma_3, \\
F''' &= -2\left(\frac{f'}{f}\right)^3 - 3\sigma_2\left(\frac{f'}{f}\right) - \frac{f'''}{f} + G''' \\
&\le 2\sigma_1^{3/2} + 3\sigma_1^{1/2}\sigma_2 + 3\sigma_3^{1/2}\sigma_2 + 2\sigma_3^{3/2} \\
&= 2(\sigma_1^{1/2} + \sigma_3^{1/2})(\sigma_1 - \sigma_1^{1/2}\sigma_3^{1/2} + \sigma_3) + 3\sigma_2(\sigma_1^{1/2} + \sigma_3^{1/2}) \\
&= (\sigma_1^{1/2} + \sigma_3^{1/2})\left(3F'' - (\sigma_1^{1/2} + \sigma_2^{1/2})^2\right) \\
&\le 2(F'')^{3/2},
\end{aligned}$$

where in the last inequality we used the observation that the positive maximizer of the function $t \mapsto t(3F'' - t^2)$ occurs at $t = (F'')^{1/2}$. $\square$

**Lemma 2.** *Fix a dimension $n \ge 1$ and let $\Delta_n = \{w \in \mathbb{R}^n_+ : \sum_{i=1}^n w_i = 1\}$ be the simplex. Suppose we have $x \in \mathbb{R}^n$ and $w \in \Delta_n$. Define the moments*

$$\begin{aligned}
s_1 &= \sum_{i=1}^n w_i x_i, \\
s_2 &= \sum_{i=1}^n w_i x_i^2, \\
s_3 &= \sum_{i=1}^n w_i x_i^3,
\end{aligned}$$

*and the constants*

$$\begin{aligned}
e_1 &= s_1,\\
e_2 &= s_2 - s_1^2,\\
e_3 &= s_1^3 - 3s_1 s_2 + 2s_3.
\end{aligned}$$

*Then the matrix*

$$M(x,w) = \begin{bmatrix} 6e_1 + 6\|x\|_2 & -3e_2 \\ -3e_2 & e_3 + 3e_2\|x\|_2 \end{bmatrix}$$

*is positive semidefinite.*

*Proof.* We first show that $M$ is positive semidefinite if its determinant $\det(M)$ is nonnegative, and then establish $\det(M)$ is nonnegative by induction on $n$. To this end, suppose that we have $\det(M) \geq 0$. A symmetric matrix is positive semidefinite if all its principal minors are nonnegative, so we need to show the diagonal entries $M_{11}$ and $M_{22}$ are nonnegative. The entry $M_{11}$ is nonnegative because we have

$$\begin{aligned}
|e_1| &= \left|w^T x\right|\\
&\leq \|w\|_2 \|x\|_2\\
&\leq \|w\|_1 \|x\|_2\\
&= \|x\|_2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality, $\|w\|_2 \leq \|w\|_1$, and $w \in \Delta_n$. If $M_{11}$ is strictly positive, then

$$M_{22} = \left(9e_2^2 + \det(M)\right)/M_{11}$$

is also nonnegative. If $M_{11}$ is zero, then we have $e_1 = -\|x\|_2$. This only happens if one $x_i$ is negative, $w_i = 1$, and all other $x_j$ are zero. In this case, $s_1 = x_i$, $s_2 = x_i^2$, and $s_3 = x_i^3$, which imply $e_2 = e_3 = 0$. Therefore $M_{22} = e_3 + 3e_2\|x\|_2$ is also zero.

We now show that $\det(M)$ is nonnegative by induction on $n$. Let $D(x,w)$ denote $\det(M)$, where we emphasize the dependence on $x$ and $w$. The function $D(\cdot, w)$ is positively homogeneous of degree 4; i.e., for $t \geq 0$ we have $D(tx,w) = t^4 D(x,w)$. We therefore assume that $x$ lies on the sphere $S^{n-1}$.

When $n = 2$, a simple yet tedious calculation shows that, in terms of the nonnegative variables $X_i = x_i + 1$ for $i = 1, 2$, the determinant is

$$D(x,w) = 3a(X_1 - X_2)^2(bX_1^2 + cX_1X_2 + dX_2^2),$$

where

$$\begin{aligned}
a &= w_1 - w_1^2\\
b &= w_1 + w_1^2\\
c &= 4 + 2w_1 - 2w_1^2\\
d &= 2 - 3w_1 + w_1^2.
\end{aligned}$$

For $w_1 \in [0,1]$, the coefficients $a$, $b$, $c$, and $d$ are all nonnegative. In addition, since $x$ lies on the sphere $S^{n-1}$, we have $x_i \geq -1$ and $X_i = x_i + 1 \geq 0$, which implies $D(x,w) \geq 0$.

Now suppose we have $n \geq 3$. A simple calculation shows that

$$\begin{aligned} D(x,w) &= -3s_1^4 - 12s_1^3\|x\|_2 - 18s_1^2\|x\|_2^2 + 12s_1s_3 + 18s_2\|x\|_2^2 - 9s_2^2 + 12s_3\|x\|_2 \\ &= -3s_1^4 - 12s_1^3 - 18s_1^2 + 12s_1s_3 + 18s_2 - 9s_2^2 + 12s_3. \end{aligned}$$

The function $D(\cdot, w)$ is continuous so it suffices to establish $D(\cdot, w)$ is nonnegative on the intersection of the sphere $S^{n-1}$ and the set $\{x \in \mathbb{R}^n : \text{all } x_i \text{ are distinct}\}$. Fix a vector $x \in \mathbb{R}^n$ with distinct components and unit norm, and let $w$ be the minimizer of $D(x, \cdot)$ over $\Delta_n$. We show that $D(x,w)$ is nonnegative.

We claim that $w$ does not have full support. Before we show this, let's first see how this completes the argument. Let $J$ be the support of $w$ so that $w_J \in \Delta_{n-1}$. If we have $|J| < n$, then by induction we know that $M(x_J, w_J)$ is positive semidefinite, and therefore

$$M(x,w) = M(x_J, w_J) + \begin{bmatrix} 6\,(1 - \|x_J\|_2) & 0 \\ 0 & 3e_2\,(1 - \|x_J\|_2) \end{bmatrix} \tag{5}$$

(noticing that $\|x\|_2 = 1$ since $x \in S^{n-1}$) is also positive semidefinite.

Now we show that $w$ does not have full support. To the contrary, suppose that $w$ does have full support, i.e., $w$ belongs to the relative interior of $\Delta_n$. By the optimality condition that $w$ minimizes $D(x, \cdot)$ over $\Delta_n$, the gradient $\nabla_w D(x,w)$ is a normal vector to $\Delta_n$. Since any normal vector of $\Delta_n$ at a non-boundary point is proportional to the all-one vector $[1, \ldots, 1] \in \mathbb{R}^n$, the partial derivatives of $D(x, \cdot)$ at $w$ are all equal. Thus there exists a scalar $v \in \mathbb{R}$ such that

$$v = q_i := \frac{1}{6}\,\frac{\partial}{\partial w_i}\,D = x_i(ax_i^2 + bx_i + c), \qquad i = 1, \ldots, n, \tag{6}$$

where $a = 2(s_1 + 1)$, $b = 3(1 - s_2)$, and $c = 2(s_3 - s_1^3 - 3s_1^2 - 3s_1)$. We derive contradictions in the following two cases.

- *Case 1: $n \geq 4$.* The numbers $x_1$, $x_2$, $x_3$, and $x_4$ are distinct roots of the cubic $t \mapsto at^3 + bt^2 + ct - v$, and therefore we have $a = b = c = v = 0$. Since $b = 0$, we have $s_2 = 1$; on the other hand, the assumption that $x$ has distinct components and unit norm implies that $s_2$ is strictly less than 1.

- *Case 2: $n = 3$.* Since the $q_i$ are equal and the $x_i$ are distinct, we have

  $$\frac{(x_2 - x_3)(q_1 - q_3) - (x_1 - x_3)(q_2 - q_3)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} = 0.$$

  Substituting $q_1$, $q_2$ and $q_3$ in the above equation by their definitions in (6) and simplifying the resulting expression, we obtain

  $$a\Sigma + b = 0,$$

where $\Sigma = x_1 + x_2 + x_3$. Next, we get a contradiction by showing that $a\Sigma + b$ is strictly positive. First observe the bound

$$\begin{aligned} a\Sigma + b &= 2\Sigma + 3 + \sum_{1=1}^{3} (2\Sigma x_i - 3x_i^2) w_i \\ &\geq \min_{i=1}^{3} \; 2\Sigma + 3 + 2\Sigma x_i - 3x_i^2 \\ &= \min_{i=1}^{3} \; (1 + x_i)(3 + 2x_1 + 2x_2 + 2x_3 - 3x_i). \end{aligned}$$

For any $i$, we claim both $1 + x_i$ and $3 + 2x_1 + 2x_2 + 2x_3 - 3x_i$ are strictly positive. For concreteness, we focus on the case where $i = 1$. For $z \in S^2$, we have $z_1 \geq -1$ with $z_1 = -1$ if and only if $z = (-1, 0, 0)$. Thus we have $1 + x_1 > 0$ since we assumed $x_2 \neq x_3$. Similarly, the affine function $z \mapsto 3 + \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^T z$ has unique minimizer over $S^2$ at $z = \frac{-1}{3}(-1, 2, 2)$ with minimum value 0, and so we have $3 - x_1 + 2x_2 + 2x_3 > 0$ since we assumed $x_2 \neq x_3$.

Based on the above two contradictions, we conclude that for $n \geq 3$ and any $x \in S^{n-1}$ that has distinct components, the minimizer of $D(x, \cdot)$ over $\Delta_n$ does not have full support. This implies that $M(x, w)$ is positive semidefinite for all $n \geq 2$ by the induction through (5). $\square$

*Proof of Theorem 1.* The function $F$ is $(n+1)$-logarithmically homogeneous, so the only difficulty is showing self-concordance. Define $\xi = \prod_{i=1}^n x_i^{\alpha_i}$, $f = \xi - \frac{\|z\|_2^2}{\xi}$, and $G = -\sum_{i=1}^n \log(x_i)$. The proposed barrier is then

$$F = -\log(f) + G$$

and we can show self-concordance by establishing Inequality (4) and appealing to Lemma 1.

Let $\Delta x \in \mathbb{R}^n$ be a direction starting at $x$. Denote $\delta_i = \frac{\Delta x_i}{x_i}$ and $s_j = \sum_{i=1}^n \alpha_i \delta_i^j$. The derivatives of $\xi$ at $x$ in direction $\Delta x$ are

$$\begin{aligned} \xi' &= e_1 \xi = s_1 \xi \\ \xi'' &= -e_2 \xi = -(s_2 - s_1^2)\xi \\ \xi''' &= e_3 \xi = (s_1^3 - 3s_1 s_2 + 2s_3)\xi, \end{aligned}$$

where we adopted the definitions of $e_1$, $e_2$ and $e_3$ in Lemma 2. The derivatives of $f$ at $(x, z)$ in direction $(\Delta x, \Delta z)$ are

$$\begin{aligned} f' &= \xi' + \frac{1}{\xi}\left(e_1 \|z\|_2^2 - 2z \cdot \Delta z\right) \\ f'' &= \xi'' - \frac{e_2}{\xi}\|z\|_2^2 - \frac{2}{\xi}\|e_1 z - \Delta z\|_2^2 \\ f''' &= \xi''' + \frac{e_3}{\xi}\|z\|_2^2 + \frac{6}{\xi}\left[e_1 \|e_1 z - \Delta z\|_2^2 + e_2 z \cdot (e_1 z - \Delta z)\right]. \end{aligned}$$

Let

$$g := (G'')^{1/2} = \sqrt{\sum_{i=1}^{n} \delta_i^2}.$$

Inequality (4) is equivalent to nonnegativity of

$$f''' - 3gf'' = \underbrace{\xi''' - 3g\xi''}_{A} + \frac{1}{\xi} \underbrace{\left[(6e_1 + 6g)\|e_1 z - \Delta z\|_2^2 + 6e_2 z \cdot (e_1 z - \Delta z) + (e_3 + 3ge_2)\|z\|_2^2\right]}_{B}.$$

We show that both $A$ and $B$ are nonnegative. We can write $A = \xi(e_3 + 3ge_2)$, and because $e_2$ is nonnegative, Cauchy-Schwarz yields a lower bound on $B$:

$$B \geq \begin{bmatrix}\|e_1 z - \Delta z\|_2 & \|z\|_2\end{bmatrix} \begin{bmatrix} 6e_1 + 6g & -3e_2 \\ -3e_2 & e_3 + 3ge_2 \end{bmatrix} \begin{bmatrix} \|e_1 z - \Delta z\|_2 \\ \|z\|_2 \end{bmatrix}.$$

As $\xi$ is positive, nonnegativity of $A$ and $B$ follow from Lemma 2. □

*Proof of Theorem 2.* The proposed barrier is

$$\begin{aligned} F(x, w) &= -\log(\xi - z) + \log(\xi) - \sum_{i=1}^{n} \log(x_i) - \log(z) \\ &= -\log(f) + G, \end{aligned}$$

where $f = z - \frac{z^2}{\xi}$ and $G = -\sum_{i=1}^{m} \log(x_i)$. By Lemma 1, it suffices to show

$$f''' - 3gf''$$

is nonnegative. The derivatives of $\xi$ at $x$ in direction $\Delta x$ are

$$\begin{aligned} \xi' &= e_1\xi = s_1\xi \\ \xi'' &= -e_2\xi = -(s_2 - s_1^2)\xi \\ \xi''' &= e_3\xi = (s_1^3 - 3s_1 s_2 + 2s_3)\xi, \end{aligned}$$

where $s_j = \sum_{i=1}^{n} \alpha_i \delta_i^j$ and $\delta_i = \frac{\Delta x_i}{x_i}$. The derivatives of $f$ at $(x, z)$ in direction $(\Delta x, \Delta z)$ are

$$\begin{aligned} f' &= \frac{\xi' z^2}{\xi^2} - \frac{2z\Delta z}{\xi} \\ f'' &= \frac{-1}{\xi}\left(2(e_1 z - \Delta z)^2 + e_2 z^2\right) \\ f''' &= \frac{1}{\xi}\left(e_3 z^2 + 6e_1(e_1 z - \Delta z)^2 + 6e_2 z(e_1 z - \Delta z)\right). \end{aligned}$$

Let

$$g := (G'')^{1/2} = \sqrt{\sum_{i=1}^{n} \delta_i^2}.$$

We must show

$$f''' - 3(G'')^{1/2} f'' = \frac{1}{\xi} \left( 6\left(e_1 + g\right)\left(e_1 z - \Delta z\right)^2 + 6 e_2 z (e_1 z - \Delta z) + \left(e_3 + 3 g e_2\right) z^2 \right),$$

a quadratic form in $z$ and $e_1 z - \Delta z$, is nonnegative. It suffices to note the matrix

$$M := \begin{bmatrix} 6\left(e_1 + g\right) & 3e_2 \\ 3e_2 & e_3 + 3ge_2 \end{bmatrix}$$

is positive semidefinite by Lemma 2. (The off-diagonal entries of $M$ are negated in Lemma 2, but this does not affect it being positive-semidefinite.) ☐

# References

[1] Aharon Ben-Tal and Arkadi Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* MPS-SIAM Series on Optimization. SIAM, 2001.

[2] Peter Robert Chares. *Cones and Interior-Point Algorithms for Structured Convex Optimization involving Powers and Exponentials.* PhD thesis, Universite catholique de Louvain, 2009.

[3] Yurii Nesterov. Constructing self-concordant barriers for convex cones. CORE discussion paper 2006/30, Center for Operations Research and Econometrics, Catholic University of Louvain, 2006.

[4] Yurii Nesterov. Primal-dual interior-point methods with asymmetric barriers. CORE discussion paper 2008/57, Center for Operations Research and Econometrics, Catholic University of Louvain, 2008.

[5] Yurii Nesterov. Towards non-symmetric conic optimization. *Optim. Methods Softw.*, 27(4-5):893–917, 2012.

[6] Yurii Nesterov and Arkadii Nemirovskii. *Interior-point polynomial algorithms in convex programming*, volume 13 of *SIAM Studies in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[7] Santiago Akle Serrano. *Algorithms for unsymmetric cone optimization and implementation for problems with the exponential cone.* PhD thesis, Standford University, 2015.

[8] Anders Skajaa and Yinyu Ye. A homogeneous interior-point algorithm for nonsymmetric convex conic optimization. *Mathematical Programming*, 150(2):391–422, 2015.

[9] Levent Tunçel and Arkadi Nemirovski. Self-concordant barriers for convex approximations of structured convex sets. *Foundations of Computational Mathematics*, 10(5):485–525, 2010.