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# Global Non-convex Optimization with Discretized Diffusions

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## Abstract

An Euler discretization of the Langevin diffusion is known to converge to the global minimizers of certain convex and non-convex optimization problems. We show that this property holds for any suitably smooth diffusion and that different diffusions are suitable for optimizing different classes of convex and non-convex functions. This allows us to design diffusions suitable for globally optimizing convex and non-convex functions not covered by the existing Langevin theory. Our non-asymptotic analysis delivers computable optimization and integration error bounds based on easily accessed properties of the objective and chosen diffusion. Central to our approach are new explicit Stein factor bounds on the solutions of Poisson equations. We complement these results with improved optimization guarantees for targets other than the standard Gibbs measure.

## 1 Introduction

Consider the unconstrained and possibly non-convex optimization problem

$$\underset{x \in \mathbb{R}^d}{\text{minimize}} f(x).$$

Recent studies have shown that the *Langevin algorithm* – in which an appropriately scaled isotropic Gaussian vector is added to a gradient descent update – globally optimizes  $f$  whenever the objective is *dissipative* ( $\langle \nabla f(x), x \rangle \geq \alpha \|x\|_2^2 - \beta$  for  $\alpha > 0$ ) with a Lipschitz gradient [14, 26, 30]. Remarkably, these globally optimized objectives need not be convex and can even be multimodal. The intuition behind the success of the Langevin algorithm is that the stochastic optimization method approximately tracks the continuous-time Langevin diffusion which admits the *Gibbs measure* – a distribution defined by  $p_\gamma(x) \propto \exp(-\gamma f(x))$  – as its invariant distribution. Here,  $\gamma > 0$  is an inverse temperature parameter, and when  $\gamma$  is large, the Gibbs measure concentrates around its modes. As a result, for large values of  $\gamma$ , a rapidly mixing Langevin algorithm will be close to a global minimum of  $f$ . In this case, rapid mixing is ensured by the Lipschitz gradient and dissipativity. Due to its simplicity, efficiency, and well-understood theoretical properties, the Langevin algorithm and its derivatives have found numerous applications in machine learning [see, e.g., 7, 29].

In this paper, we prove an analogous global optimization property for the Euler discretization of any smooth and dissipative diffusion and show that different diffusions are suitable for solving different classes of convex and non-convex problems. Our non-asymptotic analysis, based on a multidimensional version of Stein’s method, establishes explicit bounds on both integration and optimization error. Our contributions can be summarized as follows:

- For any function  $f$ , we provide explicit  $\mathcal{O}(\frac{1}{\epsilon^2})$  bounds on the numerical integration error of discretized dissipative diffusions. Our bounds depend only on simple properties of the

diffusion’s coefficients and Stein factors, i.e., bounds on the derivatives of the associated Poisson equation solution.

- For pseudo-Lipschitz  $f$ , we derive explicit first through fourth-order Stein factor bounds for every fast-coupling diffusion with smooth coefficients. Since our bounds depend on Wasserstein coupling rates, we provide user-friendly, broadly applicable tools for computing these rates. The resulting computable integration error bounds recover the known Markov chain Monte Carlo convergence rates of the Langevin algorithm in both convex and non-convex settings but apply more broadly.
- We introduce new explicit bounds on the expected suboptimality of sampling from a diffusion. Together with our integration error bounds, these yield computable and convergent bounds on global optimization error. We demonstrate that improved optimization guarantees can be obtained by targeting distributions other than the standard Gibbs measure.
- We show that different diffusions are appropriate for different objectives  $f$  and detail concrete examples of global non-convex optimization enabled by our framework but not covered by the existing Langevin theory. For example, while the Langevin diffusion is particularly appropriate for dissipative and hence quadratic growth  $f$  [26, 30], we show alternative diffusions are appropriate for “heavy-tailed”  $f$  with subquadratic or sublinear growth.

We emphasize that, while past work has assumed the existence of finite Stein factors [5, 30], focused on deriving convergence rates with inexplicit constants [23, 27, 30], or concentrated singularly on the Langevin diffusion [7, 10, 26, 30], the goals of this work are to provide the reader with tools to (a) check the appropriateness of a given diffusion for optimizing a given objective and (b) compute explicit optimization and integration error bounds based on easily accessed properties of the objective and chosen diffusion. The rest of the paper is organized as follows. Section 1.1 surveys related work. Section 2 provides an introduction to diffusions and their use in optimization and reviews our notation. Section 3 provides explicit bounds on integration error in terms of Stein factors and on Stein factors in terms of simple properties of  $f$  and the diffusion. In Section 4, we provide explicit bounds on optimization error by targeting Gibbs and non-Gibbs invariant measures and discuss how to obtain better optimization error using non-Gibbs invariant measures. We give concrete examples of applying these tools to non-convex optimization problems in Section 5 and conclude in Section 6.

## 1.1 Related work

The Euler discretization of the Langevin diffusion is commonly termed the Langevin algorithm and has been studied extensively in the context of sampling from a log concave distribution. Non-asymptotic integration error bounds for the Langevin algorithm are studied in [8–11]. A representative bound follows from combining the ergodicity of the diffusion with a discretization error analysis and yields  $\epsilon$  error in  $\mathcal{O}(\frac{1}{\epsilon^2} \text{poly}(\log(\frac{1}{\epsilon})))$  steps for the strongly log concave case and  $\mathcal{O}(\frac{1}{\epsilon^2} \text{poly}(\log(\frac{1}{\epsilon})))$  steps for the general log concave case [8, 10].

Our work is motivated by a line of research that uses the Langevin algorithm to globally optimize non-convex functions. Gelfand and Mitter [14] established the global convergence of an appropriate variant of the algorithm, and Raginsky et al. [26] subsequently used optimal transport theory to prove optimization and integration error bounds. For example, [26] provides an integration error bound of  $\epsilon$  after  $\mathcal{O}(\frac{1}{\epsilon^2} \text{poly}(\log(\frac{1}{\epsilon}))) \frac{1}{\lambda_*}$  steps under the quadratic-growth assumptions of dissipativity and a Lipschitz gradient; the estimate involves the inverse spectral gap parameter  $\lambda_*^{-1}$ , a quantity that is often unknown and sometimes exponential in both inverse temperature and dimension. In this work, we accommodate “heavy-tailed” objectives that grow subquadratically and trade the often unknown and hence inexplicit spectral gap parameter of [26] for the more user-friendly distant dissipativity condition (Prop. 3.4) which provides a straightforward and explicit certification of fast coupling and hence the fast mixing of a diffusion. For distantly dissipative diffusions, the size of our error bounds is driven primarily by a computable distance parameter; in the Langevin setting, an analogous quantity is studied in place of the spectral gap in the contemporaneous work of [6].

Cheng et al. [6] provide integration error bounds for sampling with the overdamped Langevin algorithm under a distant strong convexity assumption (a special case of distant dissipativity). The authors build on the results of [10, 12] and establish  $\epsilon$  error in  $\mathcal{O}(\frac{1}{\epsilon^2} \log(\frac{1}{\epsilon}))$  steps. We consider general distantly dissipative diffusions and establish an integration error of  $\epsilon$  in  $\mathcal{O}(\frac{1}{\epsilon^2})$  steps under mild assumptions on the objective function  $f$  and smoothness of the diffusion.

Vollmer et al. [27] used the solution of the Poisson equation in their analysis of stochastic Langevin gradient descent, invoking the bounds of Pardoux and Veretennikov [25, Thms. 1 and 2] to obtain Stein factors. However, Thms. 1 and 2 of [25] yield only inexplicit constants and require bounded diffusion coefficients, a strong assumption violated by the examples treated in Section 5. Chen et al. [5] considered a broader range of diffusions but assumed, without verification, that Stein factor and Markov chain moment were universally bounded by constants independent of all problem parameters. One of our principal contributions is a careful enumeration of the dependencies of these Stein factors and Markov chain moments on the objective  $f$  and the candidate diffusion. Our convergence analysis builds on the arguments of [15, 23], and our Stein factor bounds rely on distant and uniform dissipativity conditions for  $L_1$ -Wasserstein rate decay [12, 15] and the smoothing effect of the Markov semigroup [4, 15]. Our Stein factor results significantly generalize the existing bounds of [15] by accommodating pseudo-Lipschitz objectives  $f$  and quadratic growth in the covariance coefficient and deriving the first four Stein factors explicitly.

## 2 Optimization with Discretized Diffusions: Preliminaries

Consider a target objective function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . Our goal is to carry out unconstrained minimization of  $f$  with the aid of a candidate diffusion defined by the stochastic differential equation (SDE)

$$dZ_t^z = b(Z_t^z)dt + \sigma(Z_t^z)dB_t \quad \text{with } Z_0^z = z. \quad (2.1)$$

Here,  $(B_t)_{t \geq 0}$  is an  $l$ -dimensional Wiener process, and  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times l}$  represent the drift and the diffusion coefficients, respectively. The diffusion  $Z_t^z$  starts at a point  $z \in \mathbb{R}^d$  and, under the conditions of Section 3, admits a limiting invariant distribution  $P$  with (Lebesgue) density  $p$ . To encourage sampling near the minima of  $f$ , we would like to choose  $p$  so that the maximizers of  $p$  correspond to minimizers of  $f$ . Fortunately, under mild conditions, one can construct a diffusion with target invariant distribution  $P$  (see, e.g., [15, 20, Thm. 2]), by selecting the drift coefficient

$$b(x) = \frac{1}{2p(x)} \langle \nabla, p(x)(a(x) + c(x)) \rangle, \quad (2.2)$$

where  $a(x) \triangleq \sigma(x)\sigma(x)^\top$  is the covariance coefficient,  $c(x) = -c(x)^\top \in \mathbb{R}^{d \times d}$  is the skew-symmetric stream coefficient, and  $\langle \nabla, m(x) \rangle = \sum_j e_j \sum_k \frac{\partial m_{jk}(x)}{\partial x_k}$  denotes the divergence operator with  $\{e_j\}_j$  as the standard basis of  $\mathbb{R}^d$ . As an illustration, consider the (overdamped) Langevin diffusion for the Gibbs measure with inverse temperature  $\gamma > 0$  and density

$$p_\gamma(x) \propto \exp(-\gamma f(x)) \quad (2.3)$$

associated with our objective  $f$ . Inserting  $\sigma(x) = \sqrt{2/\gamma} I$  and  $c(x) = 0$  into the formula (2.2) we obtain

$$b_j(x) = \frac{1}{2p_\gamma(x)} \langle \nabla, p_\gamma(x)(a(x) + c(x)) \rangle_j = \frac{1}{\gamma p_\gamma(x)} \sum_k \frac{\partial p_\gamma(x) I_{jk}}{\partial x_k} = \frac{1}{\gamma p_\gamma(x)} \frac{\partial p_\gamma(x)}{\partial x_j} = -\frac{\partial_j f(x)}{\partial x_j},$$

which reduces to  $b = -\nabla f$ . We emphasize that the choice of the Gibbs measure is arbitrary, and we will consider other measures that yield superior guarantees for certain minimization problems.

In practice, the diffusion (2.1) cannot be simulated in continuous time and is instead approximated by a discrete-time numerical integrator. We will show that a particular discretization, the Euler method, can be used as a global optimization algorithm for various families of convex and non-convex  $f$ . The Euler method is the most commonly used discretization technique due to its explicit form and simplicity; however, our analysis can be generalized to other numerical integrators as well. For  $m = 0, 1, \dots$ , the Euler discretization of the SDE (2.1) corresponds to the Markov chain updates

$$X_{m+1} = X_m + \eta b(X_m) + \sqrt{\eta} \sigma(X_m) W_m,$$

where  $\eta$  is the step size, and  $W_m \sim \mathcal{N}_d(0, I)$  is an isotropic Gaussian vector that is independent from  $X_m$ . This update rule defines a Markov chain which typically has an invariant measure that is different from the invariant measure of the continuous time diffusion. However, when the step size  $\eta$  is sufficiently small, the difference between two invariant measures becomes small and can be quantitatively characterized [see, e.g., 22]. Our optimization algorithm is simply to evaluate the function  $f$  at each the Markov chain iterate  $X_m$  and report the point with the smallest function value.

Denoting by  $p(f)$  the expectation of  $f$  under the density  $p$  – i.e.,  $p(f) = \mathbb{E}_{Z \sim p}[f(Z)]$  – we decompose the optimization error after  $M$  steps of our Markov chain into two components,

$$\min_{m=1, \dots, M} \mathbb{E}[f(X_m)] - \min_x f(x) \leq \underbrace{\frac{1}{M} \sum_{m=1}^M \mathbb{E}[f(X_m) - p(f)]}_{\text{integration error}} + \underbrace{p(f) - \min_x f(x)}_{\text{expected suboptimality}}, \quad (2.4)$$

and bound each term on the right-hand side separately. The integration error—which captures both the short-term non-stationarity of the chain and the long-term bias due to discretization—is the subject of Section 3; we develop explicit bounds using techniques that build upon [15, 23]. The expected suboptimality quantifies how well exact samples from  $p$  minimize  $f$  on average. In Section 4, we extend the Gibbs measure Langevin diffusion bound of Raginsky et al. [26], to more general invariant measures and associated diffusions and demonstrate the benefits of targeting non-Gibbs measures.

**Notation** We say a function  $g$  is pseudo-Lipschitz continuous of order  $n$  if it satisfies

$$|g(x) - g(y)| \leq \tilde{\mu}_{1,n}(g)(1 + \|x\|_2^n + \|y\|_2^n)\|x - y\|_2, \quad \text{for all } x, y \in \mathbb{R}^d, \quad (2.5)$$

where  $\|\cdot\|_2$  denotes the Euclidean norm, and  $\tilde{\mu}_{1,n}(g)$  is the smallest constant satisfying (2.5). This assumption, which relaxes the more stringent Lipschitz assumption, allows  $g$  to exhibit polynomial growth of order  $n$ . For example,  $g(x) = x^2$  is not Lipschitz but satisfies (2.5) with  $\tilde{\mu}_{1,1}(g) \leq 1$ . In all of our examples of interest,  $n \leq 1$ . For operator and Frobenius norms  $\|\cdot\|_{\text{op}}$  and  $\|\cdot\|_{\text{F}}$ , we use

$$\phi_1(g) = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{\|g(x) - g(y)\|_{\text{F}}}{\|x - y\|_2}, \quad \mu_0(g) = \sup_{x \in \mathbb{R}^d} \|g(x)\|_{\text{op}},$$

and  $\mu_i(g) = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{\|\nabla^{i-1} g(x) - \nabla^{i-1} g(y)\|_{\text{op}}}{\|x - y\|_2}$

for the  $i$ -th order Lipschitz coefficients of a sufficiently differentiable function  $g$ . We denote the degree  $n$  polynomial coefficient of the  $i$ -th derivative of  $g$  by  $\tilde{\pi}_{i,n}(g)$ , i.e.,

$$\|\nabla^i g(x)\|_{\text{op}} \leq \tilde{\pi}_{i,n}(g)(1 + \|x\|_2^n) \quad \text{where} \quad \tilde{\pi}_{i,n}(g) = \sup_{x \in \mathbb{R}^d} \frac{\|\nabla^i g(x)\|_{\text{op}}}{1 + \|x\|_2^n}. \quad (2.6)$$

### 3 Explicit Bounds on Integration Error

We develop our explicit bounds on integration error in three steps. In Theorem 3.1, we bound integration error in terms of the polynomial growth and dissipativity of diffusion coefficients (Conditions 1 and 2) and Stein factors bounds on the derivatives of solutions to the diffusion’s Poisson equation (Condition 3). Condition 3 is a common assumption in the literature but is typically not verified. To address this shortcoming, Theorem 3.2 shows that any smooth, fast-coupling diffusion admits finite Stein factors expressed in terms of diffusion coupling rates (Condition 4). Finally, in Section 3.1, we provide user-friendly tools for explicitly bounding those diffusion coupling rates. We begin with our conditions.

**Condition 1** (Polynomial growth of coefficients). *For some  $r \in \{1, 2\}$  and  $\forall x \in \mathbb{R}^d$ , the drift and the diffusion coefficients of the diffusion (2.1) satisfy the following growth condition*

$$\|b(x)\|_2 \leq \frac{\lambda_b}{4}(1 + \|x\|_2), \quad \|\sigma(x)\|_{\text{F}} \leq \frac{\lambda_\sigma}{4}(1 + \|x\|_2), \quad \text{and} \quad \|\sigma\sigma^\top(x)\|_{\text{op}} \leq \frac{\lambda_\sigma}{4}(1 + \|x\|_2^r).$$

The existence and uniqueness of the solution to the diffusion SDE (2.1) is guaranteed under Condition 1 [19, Thm 3.5]. The cases  $r = 1$  and  $r = 2$  correspond to linear and quadratic growth of  $\|\sigma\sigma^\top(x)\|_{\text{op}}$ , and we will explore examples of both  $r$  settings in Section 5. As we will see in each result to follow, the quadratic growth case is far more delicate.

**Condition 2** (Dissipativity). *For  $\alpha, \beta > 0$ , the diffusion (2.1) satisfies the dissipativity condition*

$$\mathcal{A}\|x\|_2^2 \leq -\alpha\|x\|_2^2 + \beta \quad \text{for} \quad \mathcal{A}g(x) \triangleq \langle b(x), \nabla g(x) \rangle + \frac{1}{2} \langle \sigma(x)\sigma(x)^\top, \nabla^2 g(x) \rangle. \quad (3.1)$$

*$\mathcal{A}$  is the generator of the diffusion with coefficients  $b$  and  $\sigma$ , and  $\mathcal{A}\|x\|_2^2 = 2\langle b(x), x \rangle + \|\sigma(x)\|_{\text{F}}^2$ .*

Dissipativity is a standard assumption that ensures that the diffusion does not diverge but rather travels inward when far from the origin [22]. Notably, a linear growth bound on  $\|\sigma(x)\|_{\text{F}}$ , and a quadratic growth bound on  $\|\sigma\sigma^\top(x)\|_{\text{op}}$  follow directly from the linear growth of  $\|b(x)\|$  and Condition 2. However, in many examples, tighter growth constants can be obtained by inspection.

Our final condition concerns the solution of the Poisson equation (also known as the Stein equation in the Stein’s method literature) associated with our candidate diffusion.

**Condition 3** (Finite Stein factors). *The function  $u_f$  solves the Poisson equation with generator (3.1)*

$$f - p(f) = Au_f, \quad (3.2)$$

*is pseudo-Lipschitz of order  $n$  with constant  $\zeta_1$ , and has  $i$ -th order derivative with degree- $n$  polynomial growth for  $i = 2, 3, 4$ , i.e.,*

$$\|\nabla^i u_f(x)\|_{\text{op}} \leq \zeta_i(1 + \|x\|_2^n) \text{ for } i \in \{2, 3, 4\} \text{ and all } x \in \mathbb{R}^d.$$

*In other words,  $\tilde{\mu}_{1,n}(u_f) = \zeta_1$ , and  $\tilde{\pi}_{i,n}(u_f) = \zeta_i$  for  $i = 2, 3, 4$  with  $\max_i \zeta_i < \infty$ .*

The coefficients  $\zeta_i$  govern the regularity of the Poisson equation solution  $u_f$  and are termed Stein factors in the Stein's method literature. Although variants of Condition 3 have been assumed in previous work [5, 27], we emphasize that this assumption is not easily verified, and frequently only empirical evidence is provided as justification for the assumption [5]. We will ultimately derive explicit expressions for the Stein factors  $\zeta_i$  for a wide variety of diffusions and functions  $f$ , but first we will use the factors bound integration error of our discretized diffusion.

**Theorem 3.1** (Integration error of discretized diffusions). *Let Conditions 1 to 3 hold for some  $r \in \{1, 2\}$ . For any even integer<sup>1</sup>  $n_e \geq n+4$  and a step size satisfying  $\eta < 1 \wedge \frac{\alpha}{2(n_e-1)!!(1+\lambda_b/2+\lambda_\sigma/2)^{n_e}}$ ,*

$$\left| \frac{1}{M} \sum_{m=1}^M \mathbb{E}[f(X_m)] - p(f) \right| \leq \left( c_1 \frac{1}{\eta^M} + c_2 \eta + c_3 \eta^{1+|1 \wedge n/2|} \right) (\kappa_r(n_e) + \mathbb{E}[\|X_0\|_2^{n_e}]),$$

where

$$\begin{aligned} c_1 &= 6\zeta_1, & c_2 &= \frac{1}{16} \left[ 2\zeta_2 \lambda_b^2 + \zeta_3 \lambda_b \lambda_\sigma^2 + \zeta_4 (1 + 3^{n-1}) \lambda_\sigma^4 \right], \\ c_3 &= \frac{1}{48} \left[ \zeta_3 \lambda_b^3 + \zeta_4 \lambda_b^4 (1 + 3^{n-1}) + 4\zeta_4 \frac{1.5^n}{n^4} (\lambda_b^4 + n_e^2 \lambda_\sigma^4) (\lambda_b^n + n!! \lambda_\sigma^n) \right], \\ \kappa_r(n) &= 2 + \frac{2\beta}{\alpha} + \frac{n\lambda_a}{4\alpha} + \frac{\tilde{\alpha}_r}{\alpha} \left( \frac{n\lambda_a + 6r\beta}{2r\tilde{\alpha}_r} \right)^n, \quad \text{with } \tilde{\alpha}_1 = \alpha, \quad \tilde{\alpha}_2 = [\alpha - n_e \lambda_a / 4]_+. \end{aligned}$$

This integration error bound, proved in Appendix A, is  $\mathcal{O}(\frac{1}{\eta^M} + \eta)$  since the higher order term  $c_3 \eta^{1+|1 \wedge n/2|}$  can be combined with the dominant term  $c_2 \eta$  yielding  $(c_2 + c_3)\eta$  as  $\eta < 1$ . We observe that one needs  $\mathcal{O}(\epsilon^{-2})$  steps to reach a tolerance of  $\epsilon$ . Theorem 3.1 seemingly makes no assumptions on the objective function  $f$ , but in fact the dependence on  $f$  is present in the growth parameters, the Stein factors, and the polynomial degree of the Poisson equation solution. For example, we will show in Theorem 3.2 that this polynomial degree is upper bounded by that of the objective function  $f$ . To characterize the function classes covered by Theorem 3.1, we next turn to dissecting the Stein factors.

While verifying Conditions 1 and 2 for a given diffusion is often straightforward, it is not immediately clear how one might verify Condition 3. As our second principal contribution, we derive explicit values for the Stein factors  $\zeta_i$  for any smooth and dissipative diffusion exhibiting fast  $L_1$ -Wasserstein decay:

**Condition 4** (Wasserstein rate). *The diffusion  $Z_t^x$  has  $L_p$ -Wasserstein rate  $\varrho_p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  if*

$$\inf_{\text{couplings } (Z_t^x, Z_t^y)} \mathbb{E}[\|Z_t^x - Z_t^y\|_2^{p/2}]^{1/p} \leq \varrho_p(t) \|x - y\|_2 \quad \text{for all } x, y \in \mathbb{R}^d \text{ and } t \geq 0,$$

where infimum is taken over all couplings between  $Z_t^x$  and  $Z_t^y$ . We further define the relative rates

$$\tilde{\varrho}_1(t) = \log(\varrho_2(t)/\varrho_1(t)) \quad \text{and} \quad \tilde{\varrho}_2(t) = \log(\varrho_1(t)/[\varrho_1(0)\varrho_2(t)]) / \log(\varrho_1(t)/\varrho_1(0)).$$

**Theorem 3.2** (Finite Stein factors from Wasserstein decay). *Assume that Conditions 1, 2 and 4 hold and that  $f$  is pseudo-Lipschitz continuous of order  $n$  with, for  $i = 2, 3, 4$ , at most degree- $n$  polynomial growth of its  $i$ -th order derivatives. Then, the Stein factors in Condition 3 are given as*

$$\begin{aligned} \zeta_i &= \tau_i + \xi_i \int_0^\infty \varrho_1(t) \omega_r(t+i-2) dt \quad \text{for } i = 1, 2, 3, 4, \quad \text{where,} \\ \omega_r(t) &= 1 + 4\varrho_1(t)^{1-1/r} \varrho_1(0)^{1/2} \left( 1 + \frac{2}{\tilde{\alpha}_r} \{ [1 \vee \tilde{\varrho}_r(t)] 2\lambda_a n + 3r\beta \}^n \right), \end{aligned}$$

<sup>1</sup>In a typical example where  $f$  is bounded by a quadratic polynomial, we have  $n = 1$  and  $n_e = 6$ . We also remind the reader that the double factorial  $(n_e - 1)!! = 1 \cdot 3 \cdot 5 \cdots (n_e - 1)$  is of order  $\sqrt{n_e!}$ .

with  $\tilde{\alpha}_1 = \alpha$ ,  $\tilde{\alpha}_2 = \inf_t [\alpha - n\lambda_a(1 \vee \tilde{\varrho}_2(t))]_+$ , and

$$\begin{aligned} \tau_1 &= 0 & \text{and } \tau_i &= \tilde{\mu}_{1,n}(f)\tilde{\pi}_{2:i,n}(f)\tilde{\nu}_{1:i}(b)\tilde{\nu}_{1:i}(\sigma)\kappa_r(6n) & \text{for } i = 2, 3, 4, \\ \xi_1 &= \tilde{\mu}_{1,n}(f) & \text{and } \xi_i &= \tilde{\mu}_{1,n}(f)\tilde{\nu}_{1:i}(b)\tilde{\nu}_{1:i}(\sigma)\tilde{\nu}_{0:i-2}(\sigma^{-1})\varrho_1(0)\omega_r(1)\kappa_r(6n)^{i-1} & \text{for } i = 2, 3, 4, \end{aligned}$$

where  $\kappa_r(n)$  is as in Theorem 3.1,  $\tilde{\pi}_{2:i,n}(f) = \max_{j=2,\dots,i} \tilde{\pi}_{j,n}(f)$ , and for a function  $g$ ,  $\tilde{\nu}_{k:l}(g)$  denotes an upper bound on its derivatives of order  $k$  through  $l$ .

A more detailed version of the above theorem is given as Theorem C.6 in Section C along with its proof. We emphasize that, to provide finite Stein factors, Theorem 3.2 only requires  $L_1$ -Wasserstein decay and allows the  $L_2$ -Wasserstein rate to grow. An integrable Wasserstein rate is an indication that a diffusion mixes quickly to its stationary distribution. Hence, Theorem 3.2 suggests that, for a given  $f$ , one should select a diffusion that mixes quickly to a stationary measure, like the Gibbs measure (2.3), with modes at the minimizers of  $f$ . We explore user-friendly conditions implying fast Wasserstein decay in Section 3.1 and detailed examples deploying these tools in Section 5. Crucially for the ‘‘heavy-tailed’’ examples given in Section 5, Theorem 3.2 allows for an unbounded diffusion coefficient  $\sigma$ , unlike the classic results of [25].

### 3.1 Sufficient conditions for Wasserstein decay

A simple condition that leads to exponential  $L^1$  and  $L^2$ -Wasserstein decay is *uniform dissipativity* (3.3). The next result from [28] (see also [3, Sec. 1], [15, Thm. 10]) makes the relationship precise.

**Proposition 3.3** (Wasserstein decay from uniform dissipativity [28, Thm. 2.5]). *A diffusion with drift and diffusion coefficients  $b$  and  $\sigma$  has Wasserstein rate  $\varrho_p(t) = e^{-kt/2}$  if, for all  $x, y \in \mathbb{R}^d$ ,*

$$2\langle b(x) - b(y), x - y \rangle + \|\sigma(x) - \sigma(y)\|_F^2 + (p - 2)\|\sigma(x) - \sigma(y)\|_{op}^2 \leq -k\|x - y\|_2^2. \quad (3.3)$$

In the Gibbs measure Langevin case, where  $b = -\nabla f$  and  $\sigma \equiv \sqrt{2/\gamma}I$ , uniform dissipativity is equivalent to the strong convexity of  $f$ . As we will see in Section 5, the extra degree of freedom in the diffusion coefficient  $\sigma$  will allow us to treat non-convex and non-strongly convex functions  $f$ .

A more general condition leading to exponential  $L_1$ -Wasserstein decay is the *distant dissipativity* condition (3.4). The following result of [15] builds upon the pioneering analyses of Eberle [12, Cor. 2] and Wang [28, Thm. 2.6] to provide explicit Wasserstein decay.

**Proposition 3.4** (Wasserstein decay from distant dissipativity [15, Cor. 4.2]). *A diffusion with drift and diffusion coefficients  $b$  and  $\sigma$  satisfying  $\tilde{\sigma}(x) \triangleq (\sigma(x)\sigma(x)^\top - s^2I)^{1/2}$  and*

$$\frac{\langle b(x) - b(y), x - y \rangle}{s^2\|x - y\|_2^2/2} + \frac{\|\tilde{\sigma}(x) - \tilde{\sigma}(y)\|_F^2}{s^2\|x - y\|_2^2} - \frac{\|(\tilde{\sigma}(x) - \tilde{\sigma}(y))^\top(x - y)\|_2^2}{s^2\|x - y\|_2^4} \leq \begin{cases} -K & \text{if } \|x - y\|_2 > R \\ L & \text{if } \|x - y\|_2 \leq R \end{cases} \quad (3.4)$$

for  $R, L \geq 0$ ,  $K > 0$ , and  $s \in (0, 1/\mu_0(\sigma^{-1}))$  has Wasserstein rate  $\varrho_1(t) = 2e^{LR^2/8}e^{-kt/2}$  for

$$s^2k^{-1} \leq \begin{cases} \frac{e-1}{2}R^2 + e\sqrt{8K^{-1}}R + 4K^{-1} & \text{if } LR^2 \leq 8 \\ 8\sqrt{2\pi}R^{-1}L^{-1/2}(L^{-1} + K^{-1})\exp(\frac{LR^2}{8}) + 32R^{-2}K^{-2} & \text{if } LR^2 > 8. \end{cases}$$

Conveniently, both uniform and distant dissipativity imply our dissipativity condition, Condition 2. The Prop. 3.4 rates feature the distance-dependent parameter  $e^{LR^2/8}$ . In the pre-conditioned Langevin Gibbs setting ( $b = -\frac{1}{2}a\nabla f$  and  $\sigma$  constant) when  $f$  is the negative log likelihood of a multimodal Gaussian mixture,  $R$  in (3.4) represents the maximum distance between modes [15]. When  $R$  is relatively small, the convergence of the diffusion towards its stationary distribution is rapid, and the non-uniformity parameter is small; when  $R$  is relatively large, the parameter grows exponentially in  $R^2$ , as would be expected due to infrequent diffusion transitions between modes.

Our next result, proved in Appendix E, provides a user-friendly set of sufficient conditions for verifying distant dissipativity and hence exponential Wasserstein decay in practice.

**Proposition 3.5** (User-friendly Wasserstein decay). *Fix any diffusion and skew-symmetric stream coefficients  $\sigma$  and  $c$  satisfying  $L^* \triangleq F_1(\tilde{\sigma})^2 + \sup_x \lambda_{\max}(\nabla\langle\nabla, m(x)\rangle) < \infty$  for  $m(x) \triangleq \sigma(x)\sigma(x)^\top + c(x)$ ,  $\tilde{\sigma}(x) \triangleq (\sigma(x)\sigma(x)^\top - s_0^2I)^{1/2}$ , and  $s_0 \in (0, 1/\mu_0(\sigma^{-1}))$ . If*

$$\frac{-\langle m(x)\nabla f(x) - m(y)\nabla f(y), x - y \rangle}{\|x - y\|_2^2} \leq \begin{cases} -K_m & \text{if } \|x - y\|_2 > R_m \\ L_m & \text{if } \|x - y\|_2 \leq R_m, \end{cases} \quad (3.5)$$



holds for  $R_m, L_m \geq 0, K_m > 0$ , then, for any inverse temperature  $\gamma > L^*/K_m$ , the diffusion with drift and diffusion coefficients  $b_\gamma = -\frac{1}{2}m\nabla f + \frac{1}{2\gamma}\langle \nabla, m \rangle$  and  $\sigma_\gamma = \frac{1}{\sqrt{\gamma}}\sigma$  has stationary density  $p_\gamma(x) \propto e^{-\gamma f(x)}$  and satisfies (3.4) with  $s = \frac{s_0}{\sqrt{\gamma}}, K = \frac{\gamma K_m - L^*}{s_0^2}, L = \frac{\gamma L_m + L^*}{s_0^2}$ , and  $R = R_m$ .

## 4 Explicit Bounds on Optimization Error

To convert our integration error bounds into bounds on optimization error, we now turn our attention to bounding the expected suboptimality term of (2.4). To characterize the expected suboptimality of sampling from a measure with modes matching the minima of  $f$ , we generalize a result due to Raginsky et al. [26]. The original result [26, Prop. 3.4] was designed to analyze the Gibbs measure (2.3) and demanded that  $\log p_\gamma$  be smooth, in the sense that  $\mu_2(\log p_\gamma) < \infty$ . Our next proposition, proved in Appendix D, is designed for more general measures  $p$  and importantly relaxes the smoothness requirements on  $\log p$ .

**Proposition 4.1** (Expected suboptimality: Sampling yields near-optima). *Suppose  $p$  is the stationary density of an  $(\alpha, \beta)$ -dissipative diffusion (Condition 2) with global maximizer  $x^*$ . Fix  $C > 0$  and  $\theta \in (0, 1]$ . If  $\log p(x^*) - \log p(x) \leq C\|x - x^*\|_2^{2\theta}$  for all  $x$ , then*

$$-p(\log p) + \log p(x^*) \leq \frac{d}{2\theta} \log\left(\frac{2C}{d}\right) + \frac{d}{2} \log\left(\frac{e\beta}{\alpha}\right).$$

If this  $p$  takes the generalized Gibbs form  $p_{\gamma,\theta}(x) \propto \exp(-\gamma(f(x) - f(x^*)))^\theta$  for  $\gamma > 0$ , we have

$$p_{\gamma,\theta}(f(x)) - f(x^*) \leq \sqrt{\frac{d}{2\gamma} \left( \frac{1}{\theta} \log\left(\frac{2\gamma}{d}\right) + \log\left(\frac{e\beta\mu_2(f)}{2\alpha}\right) \right)}. \quad (4.1)$$

When  $\theta = 1$ ,  $p_{\gamma,\theta}$  is the Gibbs measure, and the bound (4.1) exactly recovers [26, Prop. 3.4]. The generalized Gibbs measures with  $\theta < 1$  allow for improved dependence on the inverse temperature when  $\gamma \gg d/(2\theta)$ . Note however that, for  $\theta < 1$ , the distributions  $p_{\gamma,\theta}$  also require knowledge of the optimal value  $f(x^*)$ . In certain practical settings, such as neural network optimization, it is common to have  $f(x^*) = 0$ . When  $f(x^*)$  is unknown, a similar analysis can be carried out by replacing  $f(x^*)$  with an estimate, and the bound (4.1) still holds up to a controllable error factor.

By combining Prop. 4.1 with Theorem 3.1, we obtain a complete bound controlling the global optimization error of the best Markov chain iterate.

**Corollary 4.2** (Optimization error of discretized diffusions). *Instantiate the assumptions and notation of Theorem 3.1 and Prop. 4.1. If the diffusion has the generalized Gibbs stationary density  $p_{\gamma,\theta}(x) \propto \exp(-\gamma(f(x) - f(x^*)))^\theta$ , then*

$$\begin{aligned} \min_{m=1,\dots,M} \mathbb{E}[f(X_m)] - f(x^*) &\leq \left( c_1 \frac{1}{\eta M} + (c_2 + c_3)\eta \right) (\kappa_r(n_e) + \mathbb{E}[\|X_0\|_2^{n_e}]) \\ &\quad + \sqrt{\frac{d}{2\gamma} \left( \frac{1}{\theta} \log\left(\frac{2\gamma}{d}\right) + \log\left(\frac{e\beta\mu_2(f)}{2\alpha}\right) \right)}. \end{aligned} \quad (4.2)$$

Finally, we demonstrate that, for quadratic functions, the generalized Gibbs expected suboptimality bound (4.1) can be further refined to remove the  $\log(\gamma/d)^{1/\theta}$  dependence.

**Proposition 4.3** (Expected suboptimality bound Quadratic  $f$ ). *Let  $f(x) = \langle x - b, A(x - b) \rangle$  for a positive semidefinite  $A \in \mathbb{R}^{d \times d}$  and  $b \in \mathbb{R}^d$ . Then for  $p_{\gamma,\theta}(x) \propto \exp(-\gamma(f(x) - f(x^*)))^\theta$  with  $\theta > 0$ , and for each positive integer  $k$ , we have*

$$p_{\gamma,1/k}(f) - f(x^*) \leq \left( \frac{k(1+\frac{d}{2})-1}{\gamma} \right)^k. \quad (4.3)$$

The bound (4.3) applies to any  $f$  with level set (i.e.,  $\{x : f(x) = \rho\}$ ) volume proportional to  $\rho^{d-1}$ .

## 5 Applications to Non-convex Optimization

We next provide detailed examples of verifying that a given diffusion is appropriate for optimizing a given objective, using either uniform dissipativity (Prop. 3.3) or our user-friendly distant dissipativity conditions (Prop. 3.5). When the Gibbs measure Langevin diffusion is used, our results yield global optimization when  $f$  is strongly convex (condition (3.3) with  $b = -\nabla f$  and  $\sigma \equiv \sqrt{2/\gamma}I$ ) or has strongly convex tails (condition (3.5) with  $m \equiv I$ ). To highlight the value of non-constant diffusion coefficients, we will focus on ‘‘heavy-tailed’’ examples that are not covered by the Langevin theory.

## 5.1 A simple example with sublinear growth

We begin with a pedagogical example of selecting an appropriate diffusion and verifying our global optimization conditions. Fix  $c > \frac{d+3}{2}$  and consider  $f(x) = c \log(1 + \|x\|_2^2)$ , a simple non-convex objective which exhibits sublinear growth in  $\|x\|_2$  and hence does not satisfy dissipativity (Condition 2) when paired with the Gibbs measure Langevin diffusion ( $b = -\nabla f, \sigma = \sqrt{2/\gamma}I$ ). To target the Gibbs measure (2.3) with inverse temperature  $\gamma \geq 1$ , we choose the diffusion with coefficients  $b_\gamma(x) = -\frac{1}{2}a(x)\nabla f(x) + \frac{1}{2\gamma}(\nabla, a(x))$  and  $\sigma_\gamma(x) = \frac{1}{\sqrt{\gamma}}\sigma(x)$  for  $\sigma(x) \triangleq \sqrt{1 + \frac{1}{2}\|x\|_2^2}I$  and  $a(x) = \sigma(x)\sigma(x)^\top$ . This choice satisfies Condition 1 with  $\lambda_b = \mathcal{O}(1)$ ,  $\lambda_\sigma = \mathcal{O}(\gamma^{-1/2})$ , and  $\lambda_a = \mathcal{O}(\gamma^{-1})$  with respect to  $\gamma$  and Condition 2 with  $\alpha = c - \frac{d+3}{2\gamma}$  and  $\beta = d/\gamma$ . In fact, this diffusion satisfies uniform dissipativity,

$$\begin{aligned} & 2\langle b_\gamma(x) - b_\gamma(y), x - y \rangle + \|\sigma_\gamma(x) - \sigma_\gamma(y)\|_F^2, \\ & = -(c - \frac{1}{\gamma})\|x - y\|_2^2 + \frac{d}{\gamma} \left( \sqrt{1 + \frac{1}{2}\|x\|_2^2} - \sqrt{1 + \frac{1}{2}\|y\|_2^2} \right)^2 \leq -\alpha\|x - y\|_2^2, \end{aligned}$$

yielding  $L_1$  and  $L_2$ -Wasserstein rates  $\varrho_1(t) = \varrho_2(t) = e^{-t\alpha/2}$  by Prop. 3.3 and the relative rate  $\tilde{\varrho}_2(t) = 0$ . Hence, the  $i$ -th Stein factor in Theorem 3.2 satisfies  $\zeta_i = \mathcal{O}(\gamma^{(i-1)/2})$ . This implies that the coefficients  $c_i$  in Corollary 4.2 scale with  $\mathcal{O}\left(\frac{1}{M\eta} + \sum_{i=1}^3 \eta^i \gamma^{i/2} + \frac{1}{\gamma}\right)$  and the final optimization error bound (4.2) can be made of order  $\epsilon$  by choosing the inverse temperature  $\gamma = \mathcal{O}(\epsilon^{-1})$ , the step size  $\eta = \mathcal{O}(\epsilon^{1.5})$ , and the number of iterations  $M = \mathcal{O}(\epsilon^{-2.5})$ .

## 5.2 Non-convex learning with linear growth

Next consider the canonical learning problem of regularized loss minimization with

$$f(x) = \mathcal{L}(x) + \mathcal{R}(x)$$

for  $\mathcal{L}(x) \triangleq \frac{1}{L} \sum_{l=1}^L \psi_l(\langle x, v_l \rangle)$ ,  $\psi_l$  a datapoint-specific loss function,  $v_l \in \mathbb{R}^d$  the  $l$ -th datapoint covariate vector, and  $\mathcal{R}(x) = \rho(\frac{1}{2}\|x\|_2^2)$  a regularizer with concave  $\rho$  satisfying  $\delta_3 \rho'(z) \geq \sqrt{\max(0, -\rho'(0)z\rho'''(z))}$  and  $\frac{4g_1'(z)^2}{\delta_2} \leq \frac{g_1(z)}{z} \leq \delta_1$  for  $g_s(z) \triangleq \frac{\rho'(0)}{\rho'(\frac{1}{2}z)} - s$ , some  $\delta_1, \delta_2, \delta_3 > 0$ , and all  $z, s \in \mathbb{R}$ . Our aim is to select diffusion and stream coefficients that satisfy the Wasserstein decay preconditions of Prop. 3.5. To achieve this, we set  $c \equiv 0$  and choose  $\sigma$  with  $\mu_0(\sigma^{-1}) < \infty$  so that the regularization component of the drift is *one-sided Lipschitz*, i.e.,

$$-\langle a(x)\nabla\mathcal{R}(x) - a(y)\nabla\mathcal{R}(y), x - y \rangle \leq -K_a\|x - y\|_2^2 \quad \text{for some } K_a > 0. \quad (5.1)$$

We then show that  $L^*$  from Prop. 3.5 is bounded and that, for suitable loss choices,  $a(x)\nabla\mathcal{L}(x)$  is bounded and Lipschitz so that (3.5) holds with  $K_m = \frac{K_a}{2}$  and  $L_m, R_m$  sufficiently large.

Fix any  $x$ , let  $r = \|x\|_2$ , and define  $\tilde{\sigma}^{(s)}(x) = \sqrt{1-s}(I - \frac{xx^\top}{r^2}) + \frac{xx^\top}{r^2} \sqrt{g_s(r^2)}$  for all  $s \in [0, 1]$ . We choose  $\sigma = \tilde{\sigma}^{(0)}$  so that  $a(x)\nabla\mathcal{R}(x) = \rho'(0)x$  and (5.1) holds with  $K_a = \rho'(0)$ . Our constraints on  $\rho$  ensure that  $a(x) = I + \frac{xx^\top}{r^2}g_1(r^2)$  is positive definite, that  $\mu_0(\sigma^{-1}) \leq 1$ , and that  $\sigma$  and  $a$  have at most linear and quadratic growth respectively, in satisfaction of Condition 1. Moreover,

$$\begin{aligned} \nabla\langle \nabla, a(x) \rangle &= I \left( \frac{(d-1)g_1(r^2)}{r^2} + 2g_1'(r^2) \right) + 2\frac{xx^\top}{r^2} \left( (d-1)(g_1'(r^2) - \frac{g_1(r^2)}{r^2}) + 2r^2g_1''(r^2) \right), \text{ and} \\ \lambda_{\max}(\nabla\langle \nabla, a(x) \rangle) &= \max\left( \frac{(d-1)g_1(r^2)}{r^2} + 2g_1'(r^2), -\frac{(d-1)g_1(r^2)}{r^2} + 2dg_1'(r^2) + 4r^2g_1''(r^2) \right), \end{aligned}$$

so that  $\lambda_{\max}(\nabla\langle \nabla, a(x) \rangle) \leq \max((d-1)\delta_1 + \sqrt{\delta_1\delta_2}, d\sqrt{\delta_1\delta_2} + 2\delta_3)$ . For any  $s_0 \in (0, 1)$ , we have

$$\nabla\tilde{\sigma}^{(s_0)}(x)[v] = \left( I \frac{\langle x, v \rangle}{r} + \frac{xv^\top}{r} - 2\frac{xx^\top}{r^3} \langle x, v \rangle \right) \frac{\sqrt{g_{s_0}(r^2)} - \sqrt{1-s_0}}{r} + 2\frac{xx^\top}{r^3} \langle x, v \rangle \frac{r g'_{s_0}(r^2)}{\sqrt{g_{s_0}(r^2)}}$$

for each  $v \in \mathbb{R}^d$ , so, as  $|\sqrt{g_{s_0}(r^2)} - \sqrt{1-s_0}| \leq \sqrt{g_1(r^2)}$ ,  $\phi_1(\tilde{\sigma}) \leq d\sqrt{\delta_1} + \sqrt{\delta_2}$  for  $\tilde{\sigma} = \tilde{\sigma}^{(s_0)}$ .

Finally, to satisfy (3.5), it suffices to verify that  $a(x)\nabla\mathcal{L}(x)$  is bounded and Lipschitz. For example, in the case of a ridge regularizer,  $\mathcal{R}(x) = \frac{\lambda}{2}\|x\|_2^2$  for  $\lambda > 0$ , the coefficient  $a(x) = I$ , and it suffices



to check that  $\mathcal{L}$  is Lipschitz with Lipschitz gradient. This strongly convex regularizer satisfies our assumptions, but strong convexity is by no means necessary. Consider instead the pseudo-Huber function,  $\mathcal{R}(x) = \lambda(\sqrt{1 + \frac{1}{2}\|x\|_2^2} - 1)$ , popularized in computer vision [17]. This convex but non-strongly convex regularizer satisfies all of our criteria and yields a diffusion with  $a(x) = I + \frac{xx^\top}{r^2} \frac{\mathcal{R}(x)}{\lambda}$ . Moreover, since  $\nabla\mathcal{L}(x) = \frac{1}{L} \sum_l v_l \psi'_l(\langle x, v_l \rangle)$  and  $\nabla^2\mathcal{L}(x) = \frac{1}{L} \sum_l v_l v_l^\top \psi''_l(\langle x, v_l \rangle)$ ,  $a(x)\nabla\mathcal{L}(x)$  is bounded and Lipschitz whenever  $|\psi'_l(r)| \leq \frac{\delta_4}{1+r}$  and  $|\psi''_l(r)| \leq \frac{\delta_5}{1+r}$  for some  $\delta_4, \delta_5 > 0$ . Hence, Prop. 3.5 guarantees exponential Wasserstein decay for a variety of non-convex  $\mathcal{L}$  based on datapoint outcomes  $y_l$ , including the sigmoid ( $\psi(r) = \tanh((r - y_l)^2)$  for  $y_l \in \mathbb{R}$  or  $\psi(r) = 1 - \tanh(y_l r)$  for  $y_l \in \{\pm 1\}$ ) [1], the Student's t negative log likelihood ( $\psi_l(r) = \log(1 + (r - y_l)^2)$ ), and the Blake-Zisserman ( $\psi(r) = -\log(e^{-(r - y_l)^2} + \epsilon)$ ,  $\epsilon > 0$ ) [17]. The reader can verify that all of these examples also satisfy the remaining global optimization pre-conditions of Corollary 4.2 and Theorem 3.2. In contrast, these linear-growth examples do not satisfy dissipativity (Condition 2) when paired with the Gibbs measure Langevin diffusion.

## 6 Conclusion

In this paper, we showed that the Euler discretization of any smooth and dissipative diffusion can be used for global non-convex optimization. We established non-asymptotic bounds on global optimization error and integration error with convergence governed by Stein factors obtained from the solution of the Poisson equation. We further provided explicit bounds on Stein factors for large classes of convex and non-convex objective functions, based on computable properties of the objective and the diffusion. Using this flexibility, we designed suitable diffusions for optimizing non-convex functions not covered by the existing Langevin theory. We also demonstrated that targeting distributions other than the Gibbs measure can give rise to improved optimization guarantees.

## References

- [1] P. L. Bartlett, M. I. Jordan, and J. McAuliffe. Convexity, classification, and risk bounds. *Journal of the American Statistical Association*, 101:138–156, 2006.
- [2] J.-M. Bismut. Large deviation and malliavin calculus. *Progress in Mathematics*, 45, 1984.
- [3] P. Cattiaux and A. Guillin. Semi log-concave Markov diffusions. In *Séminaire de Probabilités XLVI*, volume 2123 of *Lecture Notes in Math.*, pages 231–292. Springer, Cham, 2014. doi: 10.1007/978-3-319-11970-0\_9.
- [4] S. Cerrai. *Second order PDE's in finite and infinite dimension: a probabilistic approach*, volume 1762. Springer Science & Business Media, 2001.
- [5] C. Chen, N. Ding, and L. Carin. On the convergence of stochastic gradient mcmc algorithms with high-order integrators. In *Advances in Neural Information Processing Systems*, pages 2278–2286, 2015.
- [6] X. Cheng, N. S. Chatterji, Y. Abbasi-Yadkori, P. L. Bartlett, and M. I. Jordan. Sharp convergence rates for langevin dynamics in the nonconvex setting. *arXiv preprint arXiv:1805.01648*, 2018.
- [7] A. S. Dalalyan. Further and stronger analogy between sampling and optimization: Langevin monte carlo and gradient descent. *arXiv preprint arXiv:1704.04752*, 2017.
- [8] A. S. Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79(3):651–676, 2017.
- [9] A. S. Dalalyan and A. Tsybakov. Sparse regression learning by aggregation and langevin monte-carlo. *Journal of Computer and System Sciences*, 78:1423–1443, 2012.
- [10] A. Durmus, E. Moulines, et al. Nonasymptotic convergence analysis for the unadjusted langevin algorithm. *The Annals of Applied Probability*, 27(3):1551–1587, 2017.
- [11] R. Dwivedi, Y. Chen, M. J. Wainwright, and B. Yu. Log-concave sampling: Metropolis-hastings algorithms are fast! *arXiv preprint arXiv:1801.02309*, 2018.
- [12] A. Eberle. Reflection couplings and contraction rates for diffusions. *Probability theory and related fields*, 166(3-4):851–886, 2016.
- [13] K. D. Elworthy and X.-M. Li. Formulae for the derivatives of heat semigroups. *Journal of Functional Analysis*, 125(1):252–286, 1994.
- [14] S. B. Gelfand and S. K. Mitter. Recursive stochastic algorithms for global optimization in  $\mathbb{R}^d$ . *SIAM Journal on Control and Optimization*, 29(5):999–1018, 1991.
- [15] J. Gorham, A. B. Duncan, S. J. Vollmer, and L. Mackey. Measuring sample quality with diffusions. *arXiv preprint arXiv:1611.06972*, 2016.

- [16] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Academic press, 2014.
- [17] R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*. Cambridge University Press, ISBN: 0521540518, second edition, 2004.
- [18] G. J. O. Jameson. Inequalities for gamma function ratios. *The American Mathematical Monthly*, 120(10):936–940, 2013. ISSN 00029890, 19300972. URL <http://www.jstor.org/stable/10.4169/amer.math.monthly.120.10.936>.
- [19] R. Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.
- [20] Y.-A. Ma, T. Chen, and E. Fox. A complete recipe for stochastic gradient mcmc. In *Advances in Neural Information Processing Systems*, pages 2917–2925, 2015.
- [21] A. Mathai and S. Provost. *Quadratic forms in random variables: Theory and applications*. 1992.
- [22] J. C. Mattingly, A. M. Stuart, and D. J. Higham. Ergodicity for sdes and approximations: locally lipschitz vector fields and degenerate noise. *Stochastic processes and their applications*, 101(2):185–232, 2002.
- [23] J. C. Mattingly, A. M. Stuart, and M. V. Tretyakov. Convergence of numerical time-averaging and stationary measures via poisson equations. *SIAM Journal on Numerical Analysis*, 48(2):552–577, 2010.
- [24] B. Øksendal. *Stochastic differential equations*. pages 65–84. Springer, 2003.
- [25] E. Pardoux and A. Veretennikov. On the Poisson equation and diffusion approximation. i. *Ann. Probab.*, pages 1061–1085, 2001.
- [26] M. Raginsky, A. Rakhlin, and M. Telgarsky. Non-convex learning via stochastic gradient langevin dynamics: a nonasymptotic analysis. *arXiv preprint arXiv:1702.03849*, 2017.
- [27] S. J. Vollmer, K. C. Zygalakis, and Y. W. Teh. Exploration of the (non-) asymptotic bias and variance of stochastic gradient langevin dynamics. *Journal of Machine Learning Research* 17, pages 1–48, 2016.
- [28] F. Wang. Exponential Contraction in Wasserstein Distances for Diffusion Semigroups with Negative Curvature. *arXiv:1608.04471*, Mar. 2016.
- [29] M. Welling and Y. W. Teh. Bayesian learning via stochastic gradient langevin dynamics. In *Proceedings of the 28th International Conference on Machine Learning (ICML-11)*, pages 681–688, 2011.
- [30] P. Xu, J. Chen, and Q. Gu. Global convergence of langevin dynamics based algorithms for nonconvex optimization. *arXiv preprint arXiv:1707.06618*, 2017.

## A Proof of Theorem 3.1: Integration error of discretized diffusions

*Proof of Theorem 3.1.* Denoting by  $\Delta X_m = X_{m+1} - X_m$  and using the integral form Taylor's theorem on  $u_f(X_{m+1})$  around the previous iterate  $X_m$ , and taking expectations, we obtain

$$\begin{aligned} \mathbb{E}[u_f(X_{m+1}) - u_f(X_m)] &= \mathbb{E}[\langle \nabla u_f(X_m), \Delta X_m \rangle] + \frac{1}{2} \mathbb{E}[\langle \Delta X_m, \nabla^2 u_f(X_m) \Delta X_m \rangle] \\ &\quad + \frac{1}{6} \mathbb{E}[\langle \Delta X_m, \nabla^3 u_f(X_m) [\Delta X_m, \Delta X_m] \rangle] \\ &\quad + \frac{1}{6} \int_0^1 (1-\tau)^3 \mathbb{E}[\langle \Delta X_m, \nabla^4 u_f(X_m + \tau \Delta X_m) [\Delta X_m, \Delta X_m, \Delta X_m] \rangle] d\tau. \end{aligned} \quad (\text{A.1})$$

The first term on the right hand side can be written as

$$\begin{aligned} \mathbb{E}[\langle \nabla u_f(X_m), \Delta X_m \rangle] &= \mathbb{E}[\langle \nabla u_f(X_m), \eta b(X_m) + \sqrt{\eta} \sigma(X_m) Z_m \rangle], \\ &= \eta \mathbb{E}[\langle \nabla u_f(X_m), b(X_m) \rangle] + \sqrt{\eta} \mathbb{E}[\langle \nabla u_f(X_m), \sigma(X_m) Z_m \rangle], \\ &= \eta \mathbb{E}[\langle \nabla u_f(X_m), b(X_m) \rangle], \end{aligned}$$

where in the last step, we used the fact that  $Z_m$  is independent from  $X_m$  and odd moments of  $Z_m$  is 0. Similarly for the second and the third terms, we obtain respectively

$$\begin{aligned} &\frac{1}{2} \mathbb{E}[\langle \Delta X_m, \nabla^2 u_f(X_m) \Delta X_m \rangle] \\ &= \frac{\eta^2}{2} \mathbb{E}[\langle b(X_m), \nabla^2 u_f(X_m) b(X_m) \rangle] + \frac{\eta^2}{2} \mathbb{E}[\langle \nabla^2 u_f(X_m), \sigma \sigma^\top(X_m) \rangle], \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{6} \mathbb{E}[\langle \Delta X_m, \nabla^3 u_f(X_m) [\Delta X_m, \Delta X_m] \rangle] \\ &= \frac{\eta^3}{6} \mathbb{E}[\langle b(X_m), \nabla^3 u_f(X_m) [b(X_m), b(X_m)] \rangle] + \frac{\eta^3}{2} \mathbb{E}[\langle \nabla^3 u_f(X_m) [b(X_m)], \sigma \sigma^\top(X_m) \rangle]. \end{aligned}$$

Combining these with (3.2), we obtain (A.1) can be written as,

$$\begin{aligned} \mathbb{E}[u_f(X_{m+1}) - u_f(X_m)] &= \eta \{ \mathbb{E}[f(X_m)] - p(f) \} + \frac{\eta^2}{2} \mathbb{E}[\langle b(X_m), \nabla^2 u_f(X_m) b(X_m) \rangle] \\ &\quad + \frac{\eta^3}{6} \mathbb{E}[\langle b(X_m), \nabla^3 u_f(X_m) [b(X_m), b(X_m)] \rangle] \\ &\quad + \frac{\eta^2}{2} \mathbb{E}[\langle \nabla^3 u_f(X_m) [b(X_m)], \sigma \sigma^\top(X_m) \rangle] \\ &\quad + \frac{1}{6} \int_0^1 (1-\tau)^3 \mathbb{E}[\langle \Delta X_m, \nabla^4 u_f(X_m + \tau \Delta X_m) [\Delta X_m, \Delta X_m, \Delta X_m] \rangle] d\tau. \end{aligned}$$

Finally, dividing each term by  $\eta$ , averaging over  $m$ , and using triangle inequalities, we reach the following bound

$$\begin{aligned} &\left| \frac{1}{M} \sum_{m=1}^M \mathbb{E}[f(X_m)] - p(f) \right| \\ &\leq \frac{1}{M\eta} \left| \sum_{m=1}^M \mathbb{E}[u_f(X_{m+1}) - u_f(X_m)] \right| + \frac{\eta}{2M} \left| \sum_{m=1}^M \mathbb{E}[\langle b(X_m), \nabla^2 u_f(X_m) b(X_m) \rangle] \right| \\ &\quad + \frac{\eta^2}{6M} \left| \sum_{m=1}^M \mathbb{E}[\langle b(X_m), \nabla^3 u_f(X_m) [b(X_m), b(X_m)] \rangle] \right| \\ &\quad + \frac{\eta}{2M} \left| \sum_{m=1}^M \mathbb{E}[\langle \nabla^3 u_f(X_m) [b(X_m)], \sigma \sigma^\top(X_m) \rangle] \right| \\ &\quad + \frac{1}{6M\eta} \left| \sum_{m=1}^M \int_0^1 (1-\tau)^3 \mathbb{E}[\langle \Delta X_m, \nabla^4 u_f(X_m + \tau \Delta X_m) [\Delta X_m, \Delta X_m, \Delta X_m] \rangle] d\tau \right| \end{aligned} \quad (\text{A.2})$$

For the first term on the right hand side, using Condition 3 and Lemma A.2, we can write

$$\begin{aligned} \left| \sum_{m=1}^M \mathbb{E}[u_f(X_{m+1}) - u_f(X_m)] \right| &= |\mathbb{E}[u_f(X_{M+1}) - u_f(X_1)]| \\ &\leq \tilde{\mu}_{1,n}(u_f) \mathbb{E}[(1 + \|X_{M+1}\|_2^n + \|X_1\|_2^n) \|X_{M+1} - X_1\|_2], \\ &\leq \tilde{\mu}_{1,n}(u_f) \mathbb{E}[2 + 3\|X_{M+1}\|_2^{n+1} + 3\|X_1\|_2^{n+1}], \\ &\leq 6\tilde{\mu}_{1,n}(u_f) \left( 2 + \frac{2\beta_{r,n\epsilon}}{\alpha} + \|x\|_2^{n\epsilon} \right). \end{aligned} \quad (\text{A.3})$$

where we used Young's inequality in the second step, and Lemma A.2 in the last step.

The second term in the above inequality can be bounded by

$$\begin{aligned}
\frac{\eta}{2M} \left| \sum_{m=1}^M \mathbb{E}[\langle b(X_m), \nabla^2 u_f(X_m) b(X_m) \rangle] \right| &\leq \frac{\eta}{2M} \sum_{m=1}^M \mathbb{E}[\langle b(X_m), \nabla^2 u_f(X_m) b(X_m) \rangle], \\
&\leq \frac{\eta}{2M} \sum_{m=1}^M \mathbb{E}[\|\nabla^2 u_f(X_m)\|_{\text{op}} \|b(X_m)\|_2^2], \\
&\leq \frac{\eta \lambda_b^2}{32M} \sum_{m=1}^M \mathbb{E}[\zeta_2 (1 + \|X_m\|_2^n) (1 + \|X_m\|_2^2)^2], \\
&\leq \frac{\eta \lambda_b^2 \zeta_2}{8M} \sum_{m=1}^M \mathbb{E}[1 + \|X_m\|_2^{n+2}], \\
&\leq \frac{\eta \lambda_b^2 \zeta_2}{8} \left( 2 + \frac{2\beta_{r,n_e}}{\alpha} + \|x\|_2^{n_e} \right),
\end{aligned} \tag{A.4}$$

where in the last step, we used Lemma A.2.

Similarly, the third and the fourth terms in the inequality (A.2) can be bounded by

$$\begin{aligned}
\frac{\eta^2}{6M} \left| \sum_{m=1}^M \mathbb{E}[\langle b(X_m), \nabla^3 u_f(X_m) [b(X_m)] b(X_m) \rangle] \right| &\leq \frac{\eta^2 \lambda_b^3 \zeta_3}{48M} \sum_{m=1}^M \mathbb{E}[1 + \|X_m\|_2^{n+3}], \\
&\leq \frac{\eta^2 \lambda_b^3 \zeta_3}{48} \left( 2 + \frac{2\beta_{r,n_e}}{\alpha} + \|x\|_2^{n_e} \right),
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
\frac{\eta}{2M} \left| \sum_{m=1}^M \mathbb{E}[\langle \nabla^3 u_f(X_m) [b(X_m)], \sigma \sigma^\top(X_m) \rangle] \right| &\leq \frac{\eta \lambda_b \lambda_\sigma^2 \zeta_3}{16M} \sum_{m=1}^M \mathbb{E}[1 + \|X_m\|_2^{n+3}], \\
&\leq \frac{\eta \lambda_b \lambda_\sigma^2 \zeta_3}{16} \left( 2 + \frac{2\beta_{r,n_e}}{\alpha} + \|x\|_2^{n_e} \right).
\end{aligned} \tag{A.6}$$

For the last term, we write

$$\begin{aligned}
\frac{1}{6\eta M} \left| \sum_{m=1}^M \mathbb{E} \left[ \int_0^1 (1-\tau)^3 \langle \Delta X_m, \nabla^4 u_f(X_m + \tau \Delta X_m) [\Delta X_m, \Delta X_m] \Delta X_m \rangle d\tau \right] \right|, \\
\leq \frac{1}{6\eta M} \sum_{m=1}^M \int_0^1 (1-\tau)^3 \zeta_4 \mathbb{E}[(1 + \|X_m + \tau \Delta X_m\|_2^n) \|\Delta X_m\|_2^4] d\tau.
\end{aligned}$$

We first bound the expectation in the above integral

$$\begin{aligned}
\mathbb{E}[(1 + \|X_m + \tau \Delta X_m\|_2^n) \|\Delta X_m\|_2^4] &\leq \mathbb{E} \left[ 8(\eta^4 \|b(X_m)\|_2^4 + \eta^2 \|\sigma(X_m) W_m\|_2^4) \right. \\
&\quad \times \left. (1 + 3^{n-1} \|X_m\|_2^n + 3^{n-1} \tau^n \eta^n \|b(X_m)\|_2^n + 3^{n-1} \tau^n \eta^{n/2} \|\sigma(X_m) W_m\|_2^n) \right], \\
&= A + \tau^n B, \quad \text{where} \\
A &\triangleq 8\mathbb{E}[(1 + 3^{n-1} \|X_m\|_2^n) (\eta^4 \|b(X_m)\|_2^4 + \eta^2 \|\sigma(X_m) W_m\|_2^4)] \\
B &\triangleq 8 \cdot 3^{n-1} \mathbb{E}[(\eta^n \|b(X_m)\|_2^n + \eta^{n/2} \|\sigma(X_m) W_m\|_2^n) (\eta^4 \|b(X_m)\|_2^4 + \eta^2 \|\sigma(X_m) W_m\|_2^4)].
\end{aligned} \tag{A.7}$$

Using Condition 1, Lemma F.1 and  $\eta < 1$ , we obtain

$$\begin{aligned}
A &\leq 8 \left\{ \eta^4 \left[ \frac{\lambda_b^4}{32} \mathbb{E}[1 + \|X_m\|_2^4] + 3^{n-1} \frac{\lambda_b^4}{16} \mathbb{E}[1 + \|X_m\|_2^{n+4}] \right] \right. \\
&\quad \left. + \eta^2 \left[ \frac{3\lambda_\sigma^4}{32} \mathbb{E}[1 + \|X_m\|_2^4] + 3^n \frac{\lambda_\sigma^4}{16} \mathbb{E}[1 + \|X_m\|_2^{n+4}] \right] \right\} \\
&\leq \frac{1+3^{n-1}}{2} (\eta^4 \lambda_b^4 + 3\eta^2 \lambda_\sigma^4) \mathbb{E}[1 + \|X_m\|_2^{n+4}], \\
B &\leq 8 \cdot 3^{n-1} \eta^{2+n/2} \mathbb{E} \left[ \eta^{2+n/2} \|b(X_m)\|_2^{n+4} + \eta^2 \|b(X_m)\|_2^4 \|\sigma(X_m) W_m\|_2^n \right. \\
&\quad \left. + 3 \|b(X_m)\|_2^n \|\sigma(X_m)\|_F^4 + \|\sigma(X_m) W_m\|_2^{n+4} \right], \\
&\leq \eta^{2+n/2} \frac{3^{n-1}}{2^{n+2}} (\eta^2 \lambda_b^4 + (n+4)(n+2) \lambda_\sigma^4) (\eta^{n/2} \lambda_b^n + n!! \lambda_\sigma^n) \mathbb{E}[1 + \|X_m\|_2^{n+4}], \\
&\leq \eta^{2+n/2} \frac{1}{12} 1.5^n (\lambda_b^4 + n_e^2 \lambda_\sigma^4) (\lambda_b^n + n!! \lambda_\sigma^n) \mathbb{E}[1 + \|X_m\|_2^{n+4}].
\end{aligned}$$

Plugging this in (A.7), we obtain

$$\begin{aligned}
&\mathbb{E}[(1 + \|X_m + \tau \Delta X_m\|_2^n) \|\Delta X_m\|_2^4] \\
&\leq \mathbb{E}[1 + \|X_m\|_2^{n+4}] \left[ \frac{1+3^{n-1}}{2} (\eta^4 \lambda_b^4 + 3\eta^2 \lambda_\sigma^4) + \tau^n \eta^{2+n/2} \frac{1}{12} 1.5^n (\lambda_b^4 + n_e^2 \lambda_\sigma^4) (\lambda_b^n + n!! \lambda_\sigma^n) \right].
\end{aligned}$$

Therefore, the last term in (A.2) can be bounded by

$$\begin{aligned} & \frac{1}{6\eta M} \left| \sum_{m=1}^M \mathbb{E} \left[ \int_0^1 (1-\tau)^3 \langle \Delta X_m, \nabla^4 u_f(X_m + \tau \Delta X_m) [\Delta X_m, \Delta X_m] \Delta X_m \rangle d\tau \right] \right|, \\ & \leq \frac{\zeta_4 \eta}{6} \frac{1}{M} \sum_{m=1}^M \mathbb{E} [1 + \|X_m\|_2^{n+4}] \\ & \times \int_0^1 (1-\tau)^3 \left( \frac{1+3^{n-1}}{2} (\eta^4 \lambda_b^4 + 3\eta^2 \lambda_\sigma^4) + \tau^n \eta^{2+n/2} \frac{1}{12} 1.5^n (\lambda_b^4 + n_e^2 \lambda_\sigma^4) (\lambda_b^n + n!! \lambda_\sigma^n) \right) d\tau. \end{aligned} \quad (\text{A.8})$$

Using Lemma A.2 and

$$\int_0^1 (1-\tau)^3 \tau^n d\tau \leq \frac{6}{n^4} \quad \text{and} \quad \int_0^1 (1-\tau)^3 d\tau = \frac{1}{4},$$

the right hand side of (A.8) can be bounded by

$$\frac{\zeta_4 \eta}{6} \left( \frac{1+3^{n-1}}{8} (\eta^2 \lambda_b^4 + 3\lambda_\sigma^4) + \eta^{n/2} \frac{1}{2n^4} 1.5^n (\lambda_b^4 + n_e^2 \lambda_\sigma^4) (\lambda_b^n + n!! \lambda_\sigma^n) \right) \left( 2 + \frac{2\beta_{r,n_e}}{\alpha} + \|x\|_2^{n_e} \right). \quad (\text{A.9})$$

Combining the above bounds in (A.3), (A.4), (A.5), (A.6) and (A.9) and applying them on (A.2), we reach the final bound

$$\left| \frac{1}{M} \sum_{m=1}^M \mathbb{E}[f(X_m)] - p(f) \right| \leq \left( c_1 \frac{1}{\eta M} + c_2 \eta + c_3 \eta^{1+|1 \wedge n/2|} \right) (\kappa_r + \|x\|_2^{n_e})$$

where

$$\begin{aligned} c_1 &= 6\zeta_1, \\ c_2 &= \frac{1}{16} \left[ \zeta_2 2\lambda_b^2 + \zeta_3 \lambda_b \lambda_\sigma^2 + \zeta_4 (1 + 3^{n-1}) \lambda_\sigma^4 \right], \\ c_3 &= \frac{1}{48} \left[ \zeta_3 \lambda_b^3 + \zeta_4 \lambda_b^4 (1 + 3^{n-1}) + \zeta_4 4 \frac{1.5^n}{n^4} (\lambda_b^4 + n_e^2 \lambda_\sigma^4) (\lambda_b^n + n!! \lambda_\sigma^n) \right], \\ \kappa_r &= 2 + \frac{2\beta}{\alpha} + \frac{n_e \lambda_a}{4\alpha} + \frac{\tilde{\alpha}_r}{\alpha} \left( \frac{n_e \lambda_a + 6r\beta}{2r\tilde{\alpha}_r} \right)^{n_e}, \end{aligned}$$

where  $\tilde{\alpha}_1 = \alpha$  and  $\tilde{\alpha}_2 = [\alpha - n_e \lambda_a / 4]_+$ . □

### A.1 Dissipativity for higher order moments

It is well known that the dissipativity condition on the second moment carries directly to the higher order moments [22]. The following lemma will be useful when we bound the higher order moments of the discretized diffusion.

**Lemma A.1.** *For  $n \geq k \geq 2$ , we have the following relation*

$$\mathcal{A} \|x\|_2^n = \frac{n}{k} \|x\|_2^{n-k} \mathcal{A} \|x\|_2^k + \frac{1}{2} n(n-k) \|x\|_2^{n-4} \|\sigma^\top(x)x\|_2^2.$$

Further, assume that Conditions 1 and 2 hold, and  $n \geq 3$ . Then,

$$\mathcal{A} \|x\|_2^n \leq -\alpha \|x\|_2^n + \beta_{r,n}$$

where

$$\beta_{r,n} = \beta + \frac{n\lambda_a}{8} + \frac{\tilde{\alpha}_r}{2} \left( \frac{n\lambda_a + 6r\beta}{2r\tilde{\alpha}_r} \right)^n,$$

with  $\tilde{\alpha}_2 = [\alpha - n\lambda_a/4]_+$  and  $\tilde{\alpha}_1 = \alpha$ .

*Proof.* The proof for the first statement easily follows from the following expression,

$$\mathcal{A} \|x\|_2^n = n \|x\|_2^{n-2} \langle x, b(x) \rangle + \frac{1}{2} n(n-2) \|x\|_2^{n-4} \langle x x^\top, \sigma \sigma^\top(x) \rangle + \frac{1}{2} n \|x\|_2^{n-2} \|\sigma(x)\|_F^2.$$

For second statement, we use the first statement with  $k = 2$  and Conditions 1 and 2. First, we consider the case  $r = 1$  and write

$$\begin{aligned} \mathcal{A} \|x\|_2^n &= \frac{1}{2} n \|x\|_2^{n-2} \mathcal{A} \|x\|_2^2 + \frac{1}{2} n(n-2) \|x\|_2^{n-4} \langle x x^\top, \sigma \sigma^\top(x) \rangle, \\ &\leq -\frac{1}{2} \alpha n \|x\|_2^n + \frac{1}{2} \beta n \|x\|_2^{n-2} + \frac{\lambda_a}{8} n(n-2) (\|x\|_2^{n-1} + \|x\|_2^{n-2}), \\ &= -\frac{1}{2} \alpha n \|x\|_2^n + \frac{\lambda_a}{8} n(n-2) \|x\|_2^{n-1} + \left\{ \frac{1}{2} \beta n + \frac{\lambda_a}{8} n(n-2) \right\} \|x\|_2^{n-2}. \end{aligned}$$

Using the inequality given in Lemma F.3 twice, we obtain

$$\begin{aligned} \mathcal{A} \|x\|_2^n &\leq -\frac{1}{2} \alpha n \|x\|_2^n + \left\{ \frac{\lambda_a}{4} n(n-2) + \frac{1}{2} \beta n \right\} \|x\|_2^{n-1} + \beta + \frac{\lambda_a n}{8}, \\ &\leq -\alpha \|x\|_2^n + \frac{\alpha(n-2)}{2n} \left( \frac{n\lambda_a}{2\alpha} + \frac{\beta n}{\alpha(n-2)} \right)^n + \frac{n(n-2)\lambda_a}{8(n-1)} + \frac{n\beta}{2(n-1)}. \end{aligned}$$

Same calculation yields a similar expression for the case  $r = 2$ . Generalizing, we obtain the following formula,

$$\begin{aligned} \mathcal{A} \|x\|_2^n &\leq -\alpha \|x\|_2^n + \frac{\alpha_r(n-2)}{2n} \left( \frac{n\lambda_a}{2r\alpha_r} + \frac{n\beta}{(n-2)\alpha_r} \right)^n + \frac{n(n-2)\lambda_a}{8(n-1)} + \frac{n\beta}{2(n-1)}, \\ &\leq -\alpha \|x\|_2^n + \frac{\alpha_r}{2} \left( \frac{n\lambda_a + 6r\beta}{2r\alpha_r} \right)^n + \frac{n\lambda_a}{8} + \beta. \end{aligned}$$

□

## A.2 Proof of Lemma A.2: Markov Chain Moment Bounds

**Lemma A.2.** *Let the Conditions 1 and 2 hold. For  $n \geq 1$ , denote by  $n_e$  an even integer satisfying  $n_e \geq n$ . If the step size satisfies*

$$\eta < 1 \wedge \frac{\alpha}{2(n_e-1)!!(1+\lambda_b/2+\lambda_\sigma/2)^{n_e}},$$

then we have

$$\begin{aligned} \mathbb{E}[\|X_m\|_2^n] &\leq \|x\|_2^{n_e} + 1 + \frac{2\beta_{r,n_e}}{\alpha}, \\ \frac{1}{M} \sum_{m=1}^M \mathbb{E}[\|X_m\|_2^n] &\leq \|x\|_2^{n_e} + 1 + \frac{2\beta_{r,n_e}}{\alpha}. \end{aligned}$$

*Proof of Lemma A.2.* First, we handle the even moments. For  $n \geq 1$ , we write

$$\begin{aligned} \mathbb{E}[\|X_m + \eta b(X_m) + \sqrt{\eta}\sigma(X_m)W_m\|_2^{2n}] &= \mathbb{E}\left[\left(\|X_m\|_2^2 + \eta^2\|b(X_m)\|_2^2 + \eta\|\sigma(X_m)W_m\|_2^2\right. \right. \\ &\quad \left. \left. + 2\eta\langle X_m, b(X_m)\rangle + 2\eta^{0.5}\langle X_m, \sigma(X_m)W_m\rangle + 2\eta^{1.5}\langle b(X_m), \sigma(X_m)W_m\rangle\right)^n\right] \\ &\stackrel{1}{=} \sum_{k_1+k_2+\dots+k_6=n} \binom{n}{k_1, k_2, \dots, k_6} \eta^{2k_2+k_3+k_4+k_5/2+3k_6/2} 2^{k_4+k_5+k_6} \\ \mathbb{E}\left[\|X_m\|_2^{2k_1}\|b(X_m)\|_2^{2k_2}\|\sigma(X_m)W_m\|_2^{2k_3}\langle X_m, b(X_m)\rangle^{k_4}\langle X_m, \sigma(X_m)W_m\rangle^{k_5}\langle b(X_m), \sigma(X_m)W_m\rangle^{k_6}\right], \\ &\stackrel{2}{\leq} \mathbb{E}[\|X_m\|_2^{2n}] + \eta \mathbb{E}[\mathcal{A}\|X_m\|_2^{2n}] + \sum_{\substack{k_1+k_2+\dots+k_6=n \\ k_5+k_6 \text{ is even} \\ 2k_2+k_3+k_4+k_5/2+3k_6/2 > 1}} \binom{n}{k_1, k_2, \dots, k_6} \eta^{2k_2+k_3+k_4+k_5/2+3k_6/2} 2^{k_4+k_5+k_6} \\ \mathbb{E}\left[\|X_m\|_2^{2k_1+k_4+k_5}\|b(X_m)\|_2^{2k_2+k_4+k_6}\|\sigma(X_m)W_m\|_2^{2k_3+k_5+k_6}\right] \\ &\stackrel{3}{\leq} \sum_{\substack{k_1+k_2+\dots+k_6=n \\ k_5+k_6 \text{ is even} \\ 2k_2+k_3+k_4+k_5/2+3k_6/2 > 1}} \binom{n}{k_1, k_2, \dots, k_6} (2k_3+k_5+k_6-1)!! \eta^{2k_2+k_3+k_4+k_5/2+3k_6/2} 2^{k_4+k_5+k_6} \\ \mathbb{E}\left[\|X_m\|_2^{2k_1+k_4+k_5}\|b(X_m)\|_2^{2k_2+k_4+k_6}\|\sigma(X_m)\|_F^{2k_3+k_5+k_6}\right] + (1-\eta\alpha)\mathbb{E}[\|X_m\|_2^{2n}] + \eta\beta_{r,2n} \\ &\stackrel{4}{\leq} (1-\eta\alpha + \eta^2 2\rho_n)\mathbb{E}[\|X_m\|_2^{2n}] + \eta\beta_{r,2n} + \eta^2\rho_n \end{aligned}$$

where

$$\rho_n = \frac{1}{2} \sum_{\substack{k_1+k_2+\dots+k_6=n \\ k_5+k_6 \text{ is even} \\ 2k_2+k_3+k_4+k_5/2+3k_6/2 > 1}} \binom{n}{k_1, k_2, \dots, k_6} (2k_3+k_5+k_6-1)!! \frac{\lambda_b^{2k_2+k_4+k_6} \lambda_\sigma^{2k_3+k_5+k_6}}{2^{2k_2+2k_3+k_6}}.$$

In the above derivation, step (1) follows from multinomial expansion theorem, step (2) follows from that the odd moments of a Gaussian random variable is 0, and that the terms with coefficient  $\eta$  add up to  $\mathbb{E}[\mathcal{A}\|X_m\|_2^{2n}]$ . Step (3) follows from Cauchy-Schwartz, Lemma F.1, and Condition 2, and finally step (4) uses Condition 1 and the fact that  $\eta < 1$ .

A compact and more interpretable estimate for  $\rho_n$  can be obtained as follows,

$$\begin{aligned} \rho_n &\leq \frac{1}{2} (2n-1)!! \sum_{k_1+k_2+\dots+k_6=n} \binom{n}{k_1, k_2, \dots, k_6} \frac{\lambda_b^{2k_2+k_4+k_6} \lambda_\sigma^{2k_3+k_5+k_6}}{2^{2k_2+2k_3+k_6}} \\ &= \frac{1}{2} (2n-1)!! \left(1 + \frac{\lambda_b}{2} + \frac{\lambda_\sigma}{2}\right)^{2n}. \end{aligned}$$

The above result reads

$$\mathbb{E}[\|X_{m+1}\|_2^{2n}] \leq \tau_n(\eta)\mathbb{E}[\|X_m\|_2^{2n}] + \tilde{\gamma}_n(\eta)$$

where  $\tau_n(\eta) = 1 - \eta\alpha + \eta^2 2\rho_n$ , and  $\tilde{\gamma}_n(\eta) = \eta\beta_{r,2n} + \eta^2\rho_n$ . Notice that  $\tau_n(0) = 1$  and  $\tau'_n(0) = -\alpha$  is negative. Therefore, we may obtain  $\tau_n(\eta) < 1$  by choosing  $\eta$  small. More specifically, we have  $\tau_n(\eta) < 1$  when  $\eta < \alpha/2\rho_n$ , but by choosing  $\eta < \alpha/(4\rho_n)$  we have control over the second term as well. That is, by Lemma F.2, we immediately obtain

$$\begin{aligned} \mathbb{E}[\|X_m\|_2^{2n}] &\leq \tau_n(\eta)^m \|x\|_2^{2n} + \frac{\tilde{\gamma}_n(\eta)}{1-\tau_n(\eta)} \\ &\leq \tau_n(\eta)^m \|x\|_2^{2n} + \frac{2\beta_{r,2n} + \alpha/2}{\alpha} \\ &\leq \|x\|_2^{2n} + \frac{2\beta_{r,2n} + \alpha}{\alpha} \end{aligned}$$



and

$$\frac{1}{M} \sum_{m=1}^M \mathbb{E}[\|X_m\|_2^{2n}] \leq \|x\|^{2n} + \frac{2\beta_{r,2n+\alpha}}{\alpha}$$

where we use a looser bound to ensure that the right hand side is larger than 1.

The above analysis only covers the even moments so far. For any integer  $n$ , denote by  $n_e$  an even integer that is not smaller than  $n$ . Then, by the Hölder's inequality we write

$$\begin{aligned} \mathbb{E}[\|X_m\|_2^n] &\leq \mathbb{E}[\|X_m\|_2^{n_e}]^{n/n_e} \leq \left( \|x\|^{n_e} + \frac{2\beta_{r,n_e+\alpha}}{\alpha} \right)^{n/n_e} \\ &\leq \|x\|^{n_e} + \frac{2\beta_{r,n_e}}{\alpha} + 1, \end{aligned}$$

which concludes the proof.  $\square$

## B Directional Derivative Flow Moment Bounds

In this section, we provide high order moment bounds for the first four directional derivative flows. These moment bounds will be used to bound the semigroup derivatives.

- The first directional derivative flow in the direction of  $v$  solves the first variation equation

$$dV_t^{x,v} = \nabla b(Z_t^x) V_t^{x,v} dt + \nabla \sigma(Z_t^x) V_t^{x,v} dB_t \quad \text{with } V_0^{x,v} = v. \quad (\text{B.1})$$

- The second derivative flow in the directions  $v, u$  solves the second variation equation

$$\begin{aligned} dU_t^{x,vu} &= (\nabla^2 b(Z_t^x) [V_t^{x,u}, V_t^{x,v}] + \nabla b(Z_t^x) U_t^{x,vu}) dt \\ &\quad + (\nabla^2 \sigma(Z_t^x) [V_t^{x,u}, V_t^{x,v}] + \nabla \sigma(Z_t^x) U_t^{x,vu}) dB_t \quad \text{with } U_0^{x,vu} = 0. \end{aligned} \quad (\text{B.2})$$

- The third derivative flow in the directions  $v, u, w$  solves the third variation equation

$$\begin{aligned} dW_t^{x,vuw} &= \left( \nabla^3 b(Z_t^x) [V_t^{x,v}, V_t^{x,u}, V_t^{x,w}] + \nabla^2 b(Z_t^x) [U_t^{x,vu}, V_t^{x,w}] \right. \\ &\quad \left. + \nabla^2 b(Z_t^x) [U_t^{x,uw}, V_t^{x,v}] + \nabla^2 b(Z_t^x) [U_t^{x,vw}, V_t^{x,u}] + \nabla b(Z_t^x) W_t^{x,vuw} \right) dt \\ &\quad + \left( \nabla^3 \sigma(Z_t^x) [V_t^{x,v}, V_t^{x,u}, V_t^{x,w}] + \nabla^2 \sigma(Z_t^x) [U_t^{x,vu}, V_t^{x,w}] + \nabla^2 \sigma(Z_t^x) [U_t^{x,uw}, V_t^{x,v}] \right. \\ &\quad \left. + \nabla^2 \sigma(Z_t^x) [U_t^{x,vw}, V_t^{x,u}] + \nabla \sigma(Z_t^x) W_t^{x,vuw} \right) dB_t \quad \text{with } W_0^{x,vuw} = 0. \end{aligned} \quad (\text{B.3})$$

- The fourth derivative flow in the directions  $v, u, w, y$  solves the fourth variation equation

$$\begin{aligned} dY_t^{x,vuwy} &= \left( \nabla^4 b(Z_t^x) [V_t^{x,v}, V_t^{x,u}, V_t^{x,w}, V_t^{x,y}] + \nabla^3 b(Z_t^x) [U_t^{x,vu}, V_t^{x,w}, V_t^{x,y}] \right. \\ &\quad + \nabla^3 b(Z_t^x) [U_t^{x,uw}, V_t^{x,v}, V_t^{x,y}] + \nabla^3 b(Z_t^x) [U_t^{x,vy}, V_t^{x,u}, V_t^{x,w}] \\ &\quad + \nabla^3 b(Z_t^x) [U_t^{x,vw}, V_t^{x,u}, V_t^{x,y}] + \nabla^3 b(Z_t^x) [U_t^{x,wy}, V_t^{x,v}, V_t^{x,u}] \\ &\quad + \nabla^3 b(Z_t^x) [U_t^{x,uy}, V_t^{x,v}, V_t^{x,w}] + \nabla^2 b(Z_t^x) [U_t^{x,vu}, U_t^{x,wy}] \\ &\quad + \nabla^2 b(Z_t^x) [U_t^{x,uw}, U_t^{x,vy}] + \nabla^2 b(Z_t^x) [U_t^{x,uy}, U_t^{x,vw}] \\ &\quad + \nabla^2 b(Z_t^x) [W_t^{x,vuy}, V_t^{x,w}] + \nabla^2 b(Z_t^x) [W_t^{x,vuw}, V_t^{x,y}] \\ &\quad \left. + \nabla^2 b(Z_t^x) [W_t^{x,uwy}, V_t^{x,v}] + \nabla^2 b(Z_t^x) [W_t^{x,vwy}, V_t^{x,u}] + \nabla b(Z_t^x) Y_t^{x,vuwy} \right) dt \\ &\quad + \left( \nabla^4 \sigma(Z_t^x) [V_t^{x,v}, V_t^{x,u}, V_t^{x,w}, V_t^{x,y}] + \nabla^3 \sigma(Z_t^x) [U_t^{x,vu}, V_t^{x,w}, V_t^{x,y}] \right. \\ &\quad + \nabla^3 \sigma(Z_t^x) [U_t^{x,uy}, V_t^{x,v}, V_t^{x,w}] + \nabla^3 \sigma(Z_t^x) [U_t^{x,wy}, V_t^{x,v}, V_t^{x,u}] \\ &\quad + \nabla^3 \sigma(Z_t^x) [U_t^{x,vy}, V_t^{x,u}, V_t^{x,w}] + \nabla^3 \sigma(Z_t^x) [U_t^{x,uw}, V_t^{x,v}, V_t^{x,y}] \\ &\quad + \nabla^3 \sigma(Z_t^x) [U_t^{x,vw}, V_t^{x,u}, V_t^{x,y}] + \nabla^2 \sigma(Z_t^x) [U_t^{x,vu}, U_t^{x,wy}] \\ &\quad + \nabla^2 \sigma(Z_t^x) [U_t^{x,uw}, U_t^{x,vy}] + \nabla^2 \sigma(Z_t^x) [U_t^{x,uy}, U_t^{x,vw}] + \nabla^2 \sigma(Z_t^x) [W_t^{x,vuy}, V_t^{x,w}] \\ &\quad + \nabla^2 \sigma(Z_t^x) [W_t^{x,vwy}, V_t^{x,u}] + \nabla^2 \sigma(Z_t^x) [W_t^{x,vuw}, V_t^{x,y}] \\ &\quad \left. + \nabla^2 \sigma(Z_t^x) [W_t^{x,uwy}, V_t^{x,v}] + \nabla \sigma(Z_t^x) Y_t^{x,vuwy} \right) dB_t \quad \text{with } Y_0^{x,vuwy} = 0. \end{aligned} \quad (\text{B.4})$$

In the following, we will rely on Dynkin's lemma [24] to find an upper bound on the  $n$ -th moment of the first four derivative flows. The following function will be used to characterize the rates of moments of the derivative flows of a diffusion process with drift  $b$ , and diffusion  $\sigma$ ,

$$\varphi_{i,n}(b, \sigma) \triangleq \mu_i(b) + n\mu_i(\sigma)^2 + \phi_i(\sigma)^2,$$

where we use the following notation for the Lipschitz coefficients of a differentiable function  $g$  and a matrix  $m$ ,

$$\begin{aligned} \mu_0(g) &= \sup_{x \in \mathbb{R}^d} \|g(x)\|_{\text{op}}, \quad \text{and} \quad \mu_i(g) = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{\|\nabla^{i-1}g(x) - \nabla^{i-1}g(y)\|_{\text{op}}}{\|x - y\|_2}, \\ \phi_i(m) &= \sup_{x \in \mathbb{R}^d, \|v_1\|_2 = \dots = \|v_i\|_2 = 1} \|\nabla^i m(x)[v_1, \dots, v_i]\|_{\text{F}}. \end{aligned}$$

**First derivative flow moments.** In the case of first derivative flow (B.1), the Dynkin's lemma applied to the function  $x \rightarrow \|x\|_2^n$  yields

$$\begin{aligned} \mathbb{E}[\|V_t^{x,v}\|_2^n] &= \|v\|_2^n + n \int_0^t \mathbb{E}[\|V_s^{x,v}\|_2^{n-2} \langle \nabla b(Z_s^x) V_s^{x,v}, V_s^{x,v} \rangle] ds \\ &\quad + \frac{1}{2} n(n-2) \int_0^t \mathbb{E}[\|V_s^{x,v}\|_2^{n-4} \langle \nabla \sigma(Z_s^x) [V_s^{x,v}] \{ \nabla \sigma(Z_s^x) [V_s^{x,v}] \}^\top, V_s^{x,v} [V_s^{x,v}]^\top \rangle] ds \\ &\quad + \frac{1}{2} n \int_0^t \mathbb{E}[\|V_s^{x,v}\|_2^{n-2} \langle \nabla \sigma(Z_s^x) [V_s^{x,v}] \{ \nabla \sigma(Z_s^x) [V_s^{x,v}] \}^\top, I \rangle] ds, \\ &\leq \|v\|_2^n + \{ n\mu_1(b) + \frac{1}{2} n(n-2)\mu_1(\sigma)^2 + \frac{1}{2} n\phi_1(\sigma)^2 \} \int_0^t \mathbb{E}[\|V_s^{x,v}\|_2^n] ds. \end{aligned}$$

Finally, applying Grönwall's inequality, we obtain the desired bound on the moments, i.e.,

$$\mathbb{E}[\|V_t^{x,v}\|_2^n] \leq e^{t\theta_{1,n}} \|v\|_2^n, \quad (\text{B.5})$$

where

$$\begin{aligned} \theta_{1,n} &\triangleq n\varphi_{1,n-2}(b, \sigma) \\ &\geq n\mu_1(b) + \frac{1}{2} n(n-2)\mu_1(\sigma)^2 + \frac{1}{2} n\phi_1(\sigma)^2. \end{aligned} \quad (\text{B.6})$$

**Second derivative flow moments.** Applying Dynkin's lemma to the function  $x \rightarrow \|x\|_2^n$  for the diffusion (B.2) yields,

$$\begin{aligned} \mathbb{E}[\|U_t^{x,vu}\|_2^n] &= n \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-2} \langle \nabla b(Z_s^x) U_s^{x,vu} + \nabla^2 b(Z_s^x) [V_s^{x,u}, V_s^{x,v}], U_s^{x,vu} \rangle] ds \\ &\quad + \frac{n(n-2)}{2} \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-4} \|(\nabla \sigma(Z_s^x) [U_s^{x,vu}] + \nabla^2 \sigma(Z_s^x) [V_s^{x,u}, V_s^{x,v}])^\top U_s^{x,vu}\|_2^2] ds \\ &\quad + \frac{n}{2} \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-2} \|\nabla \sigma(Z_s^x) [U_s^{x,vu}] + \nabla^2 \sigma(Z_s^x) [V_s^{x,u}, V_s^{x,v}]\|_{\text{F}}^2] ds, \\ &\leq n\mu_1(b) \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^n] ds + n\mu_2(b) \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-1} \|V_s^{x,u}\|_2 \|V_s^{x,v}\|_2] ds \\ &\quad + n(n-2) \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-4} \{ \|\nabla \sigma(Z_s^x) [U_s^{x,vu}, U_s^{x,vu}]\|_2^2 + \|\nabla^2 \sigma(Z_s^x) [V_s^{x,u}, V_s^{x,v}, U_s^{x,vu}]\|_2^2 \}] ds \\ &\quad + n \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-2} \{ \|\nabla \sigma(Z_s^x) [U_s^{x,vu}]\|_{\text{F}}^2 + \|\nabla^2 \sigma(Z_s^x) [V_s^{x,u}, V_s^{x,v}]\|_{\text{F}}^2 \}] ds, \\ &\leq \{ n\mu_1(b) + n(n-2)\mu_1(\sigma)^2 + n\phi_1(\sigma)^2 \} \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^n] ds \\ &\quad + n\mu_2(b) \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-1} \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2] ds \\ &\quad + \{ n(n-2)\mu_2(\sigma)^2 + n\phi_2(\sigma)^2 \} \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^{n-2} \|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2] ds. \end{aligned} \quad (\text{B.7})$$

By the Young's inequality, we have

$$\begin{aligned} \|U_s^{x,vu}\|_2^{n-1} \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 &\leq \frac{n-1}{n} \|U_s^{x,vu}\|_2^n + \frac{1}{n} \|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n, \\ \|U_s^{x,vu}\|_2^{n-2} \|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2 &\leq \frac{n-2}{n} \|U_s^{x,vu}\|_2^n + \frac{2}{n} \|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n. \end{aligned}$$

Using the above inequalities on the last result in (B.7), we obtain

$$\begin{aligned} \mathbb{E}[\|U_t^{x,vu}\|_2^n] &\leq \left\{ n\mu_1(b) + n(n-2)\mu_1(\sigma)^2 + n\phi_1(\sigma)^2 + (n-1)\mu_2(b) + (n-2)^2\mu_2(\sigma)^2 \right. \\ &\quad \left. + (n-2)\phi_2(\sigma)^2 \right\} \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^n] ds \\ &\quad + \left\{ \mu_2(b) + 2(n-2)\mu_2(\sigma)^2 + 2\phi_2(\sigma)^2 \right\} \int_0^t \mathbb{E}[\|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n] ds, \\ &\leq \left\{ n\mu_1(b) + n(n-2)\mu_1(\sigma)^2 + n\phi_1(\sigma)^2 + (n-1)\mu_2(b) + (n-2)^2\mu_2(\sigma)^2 \right. \\ &\quad \left. + (n-2)\phi_2(\sigma)^2 \right\} \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^n] ds \\ &\quad + \left\{ \mu_2(b) + 2(n-2)\mu_2(\sigma)^2 + 2\phi_2(\sigma)^2 \right\} \|v\|_2^n \|u\|_2^n \int_0^t e^{s\theta_{1,2n}} ds, \end{aligned}$$

where in the last line, we used Cauchy-Schwartz inequality on the first derivative flow moments together with (B.5). Finally, by the Grönwall inequality, we obtain

$$\mathbb{E}[\|U_t^{x,vu}\|_2^n] \leq \gamma_{2,n} e^{t\theta_{2,n}} \|v\|_2^n \|u\|_2^n,$$

where – using (B.6) – we have

$$\begin{aligned} \gamma_{2,n} &= \frac{\varphi_{2,n-2}(b,\sigma)}{n\varphi_{1,2n-2}(b,\sigma)} \geq \frac{1}{\theta_{1,2n}} \left\{ \mu_2(b) + 2(n-2)\mu_2(\sigma)^2 + 2\phi_2(\sigma)^2 \right\}, \\ \theta_{2,n} &= 3n\varphi_{1,2n-2}(b,\sigma) + n\varphi_{2,n-2}(b,\sigma), \\ &\geq \theta_{1,2n} + n\mu_1(b) + n(n-2)\mu_1(\sigma)^2 + n\phi_1(\sigma)^2 \\ &\quad + (n-1)\mu_2(b) + (n-2)^2\mu_2(\sigma)^2 + (n-2)\phi_2(\sigma)^2. \end{aligned} \tag{B.8}$$

**Third derivative flow moments.** Dynkin's lemma applied to the function  $x \rightarrow \|x\|_2^n$  for the diffusion (B.3) yields

$$\begin{aligned} \mathbb{E}[\|W_t^{x,vuw}\|_2^n] &= n \int_0^t \mathbb{E} \left[ \|W_s^{x,vuw}\|_2^{n-2} \langle \nabla^3 b(Z_s^x)[V_s^{x,v}, V_s^{x,u}, V_s^{x,w}] + \nabla^2 b(Z_s^x)[U_s^{x,vu}, V_s^{x,w}] \right. \\ &\quad \left. + \nabla^2 b(Z_s^x)[U_s^{x,uw}, V_s^{x,v}] + \nabla^2 b(Z_s^x)[U_s^{x,vw}, V_s^{x,u}] + \nabla b(Z_s^x)W_s^{x,vuw}, W_s^{x,vuw} \rangle \right] ds \\ &+ \frac{1}{2} n \int_0^t \mathbb{E} \left[ \|W_s^{x,vuw}\|_2^{n-2} \left\| \nabla^3 \sigma(Z_s^x)[V_s^{x,v}, V_s^{x,u}, V_s^{x,w}] + \nabla^2 \sigma(Z_s^x)[U_s^{x,vu}, V_s^{x,w}] \right. \right. \\ &\quad \left. \left. + \nabla^2 \sigma(Z_s^x)[U_s^{x,uw}, V_s^{x,v}] + \nabla^2 \sigma(Z_s^x)[U_s^{x,vw}, V_s^{x,u}] + \nabla \sigma(Z_s^x)W_s^{x,vuw} \right\|_F^2 \right] ds \\ &+ \frac{1}{2} n(n-2) \int_0^t \mathbb{E} \left[ \|W_s^{x,vuw}\|_2^{n-4} \left\| (\nabla^3 \sigma(Z_s^x)[V_s^{x,v}, V_s^{x,u}, V_s^{x,w}] \right. \right. \\ &\quad \left. \left. + \nabla^2 \sigma(Z_s^x)[U_s^{x,vu}, V_s^{x,w}] + \nabla^2 \sigma(Z_s^x)[U_s^{x,uw}, V_s^{x,v}] \right. \right. \\ &\quad \left. \left. + \nabla^2 \sigma(Z_s^x)[U_s^{x,vw}, V_s^{x,u}] + \nabla \sigma(Z_s^x)W_s^{x,vuw} \right\|_F^2 \right] ds, \\ &\leq n \int_0^t \mathbb{E} \left[ \|W_s^{x,vuw}\|_2^{n-1} \left\{ \mu_3(b) \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,w}\|_2 + \mu_2(b) \|U_s^{x,vu}\|_2 \|V_s^{x,w}\|_2 \right. \right. \\ &\quad \left. \left. + \mu_2(b) \|U_s^{x,uw}\|_2 \|V_s^{x,v}\|_2 + \mu_2(b) \|U_s^{x,vw}\|_2 \|V_s^{x,u}\|_2 + \mu_1(b) \|W_s^{x,vuw}\|_2 \right\} \right] ds \\ &+ \frac{5}{2} n \int_0^t \mathbb{E} \left[ \|W_s^{x,vuw}\|_2^{n-2} \left\{ \phi_3(\sigma)^2 \|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2 \|V_s^{x,w}\|_2^2 + \phi_2(\sigma)^2 \|U_s^{x,vu}\|_2^2 \|V_s^{x,w}\|_2^2 \right. \right. \\ &\quad \left. \left. + \phi_2(\sigma)^2 \|U_s^{x,uw}\|_2^2 \|V_s^{x,v}\|_2^2 + \phi_2(\sigma)^2 \|U_s^{x,vw}\|_2^2 \|V_s^{x,u}\|_2^2 + \phi_1(\sigma)^2 \|W_s^{x,vuw}\|_2^2 \right\} \right] ds \\ &+ \frac{5}{2} n(n-2) \int_0^t \mathbb{E} \left[ \|W_s^{x,vuw}\|_2^{n-2} \left\{ \mu_3(\sigma)^2 \|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2 \|V_s^{x,w}\|_2^2 \right. \right. \\ &\quad \left. \left. + \mu_2(\sigma)^2 \|U_s^{x,vu}\|_2^2 \|V_s^{x,w}\|_2^2 + \mu_2(\sigma)^2 \|U_s^{x,uw}\|_2^2 \|V_s^{x,v}\|_2^2 \right. \right. \\ &\quad \left. \left. + \mu_2(\sigma)^2 \|U_s^{x,vw}\|_2^2 \|V_s^{x,u}\|_2^2 + \mu_1(\sigma)^2 \|W_s^{x,vuw}\|_2^2 \right\} \right] ds. \end{aligned} \tag{B.9}$$

Using the Young's inequality, we obtain the following inequalities

$$\begin{aligned} \|W_s^{x,vuw}\|_2^{n-1} \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,w}\|_2 &\leq \frac{n-1}{n} \|W_s^{x,vuw}\|_2^n + \frac{1}{n} \|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n \|V_s^{x,w}\|_2^n, \\ \|W_s^{x,vuw}\|_2^{n-2} \|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2 \|V_s^{x,w}\|_2^2 &\leq \frac{n-2}{n} \|W_s^{x,vuw}\|_2^n + \frac{2}{n} \|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n \|V_s^{x,w}\|_2^n, \\ \|W_s^{x,vuw}\|_2^{n-1} \|U_s^{x,vu}\|_2 \|V_s^{x,w}\|_2 &\leq \frac{n-1}{n} \|W_s^{x,vuw}\|_2^n + \frac{1}{n} \|U_s^{x,vu}\|_2^n \|V_s^{x,w}\|_2^n, \\ \|W_s^{x,vuw}\|_2^{n-2} \|U_s^{x,vu}\|_2^2 \|V_s^{x,w}\|_2^2 &\leq \frac{n-2}{n} \|W_s^{x,vuw}\|_2^n + \frac{2}{n} \|U_s^{x,vu}\|_2^n \|V_s^{x,w}\|_2^n. \end{aligned}$$

Using the above inequalities and reorganizing the terms in (B.9), we obtain the following

$$\begin{aligned} \mathbb{E}[\|W_t^{x,vuw}\|_2^n] &\leq \left( n\mu_1(b) + \frac{5}{2}n(n-2)\mu_1(\sigma)^2 + \frac{5}{2}n\phi_1(\sigma)^2 \right. \\ &\quad \left. + 3(n-1)\mu_2(b) + \frac{15}{2}(n-2)^2\mu_2(\sigma)^2 + \frac{15}{2}(n-2)\phi_2(\sigma)^2 \right. \\ &\quad \left. + (n-1)\mu_3(b) + \frac{5}{2}(n-2)^2\mu_3(\sigma)^2 + \frac{5}{2}(n-2)\phi_3(\sigma)^2 \right) \mathbb{E}[\|W_s^{x,vuw}\|_2^n] ds \\ &+ (\mu_3(b) + 5(n-2)\mu_3(\sigma)^2 + 5\phi_3(\sigma)^2) \int_0^t \mathbb{E}[\|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n \|V_s^{x,w}\|_2^n] ds \\ &+ (\mu_2(b) + 5(n-2)\mu_2(\sigma)^2 + 5\phi_2(\sigma)^2) \int_0^t \mathbb{E}[\|U_s^{x,vu}\|_2^n \|V_s^{x,w}\|_2^n] ds \\ &+ (\mu_2(b) + 5(n-2)\mu_2(\sigma)^2 + 5\phi_2(\sigma)^2) \int_0^t \mathbb{E}[\|U_s^{x,vw}\|_2^n \|V_s^{x,u}\|_2^n] ds \\ &+ (\mu_2(b) + 5(n-2)\mu_2(\sigma)^2 + 5\phi_2(\sigma)^2) \int_0^t \mathbb{E}[\|U_s^{x,uw}\|_2^n \|V_s^{x,v}\|_2^n] ds. \end{aligned}$$

Using the moment bounds for the first and the second derivative flows derived previously in this section, we can write

$$\begin{aligned} \int_0^t \mathbb{E} [\|V_s^{x,v}\|_2^n \|V_s^{x,u}\|_2^n \|V_s^{x,w}\|_2^n] ds &\leq \int_0^t e^{s\theta_{1,3n}} ds \|v\|_2^n \|u\|_2^n \|w\|_2^n, \\ \int_0^t \mathbb{E} [\|U_s^{x,uv}\|_2^n \|V_s^{x,w}\|_2^n] ds &\leq \gamma_{2,2n}^{1/2} \int_0^t e^{s\theta_{2,2n}/2 + s\theta_{1,2n}/2} ds \|v\|_2^n \|u\|_2^n \|w\|_2^n, \end{aligned}$$

where we used (B.6) and (B.8). Therefore, we obtain that

$$\begin{aligned} \mathbb{E} [\|W_t^{x,vuw}\|_2^n] &\leq \frac{5}{2} n \left( \varphi_{1,n-2}(b, \sigma) + 3\varphi_{2,n-2}(b, \sigma) + \varphi_{3,n-2}(b, \sigma) \right) \mathbb{E} [\|W_s^{x,vuw}\|_2^n] ds \\ &\quad + 5\varphi_{3,n-2}(b, \sigma) \int_0^t e^{s\theta_{1,3n}} ds \|v\|_2^n \|u\|_2^n \|w\|_2^n \\ &\quad + 15\gamma_{2,2n}^{1/2} \varphi_{2,n-2}(b, \sigma) \int_0^t e^{s\{\theta_{2,2n}/2 + \theta_{1,2n}/2\}} \|v\|_2^n \|u\|_2^n \|w\|_2^n ds. \end{aligned}$$

We notice that both  $\theta_{2,2n}/2 + \theta_{1,2n}/2$  and  $\theta_{1,3n}$  can be upper bounded by  $4n\varphi_{1,4n-2}(b, \sigma) + n\varphi_{2,2n-2}(b, \sigma)$ . Using these bounds in the second and the third integrals, and using  $\gamma_{2,2n}^{1/2} \leq 1 + \gamma_{2,2n}$ , and finally applying the Grönwall's inequality, we obtain

$$\mathbb{E} [\|W_t^{x,vuw}\|_2^n] \leq \gamma_{3,n} e^{t\theta_{3,n}} \|v\|_2^n \|u\|_2^n \|w\|_2^n$$

where

$$\begin{aligned} \gamma_{3,n} &= \frac{15\varphi_{2,n-2}(b, \sigma) + 5\varphi_{3,n-2}(b, \sigma)}{4n\varphi_{1,4n-2}(b, \sigma)} \tag{B.10} \\ &\geq \frac{15\varphi_{2,n-2}(b, \sigma)}{4n\varphi_{1,4n-2}(b, \sigma)} + \frac{5\varphi_{3,n-2}(b, \sigma)}{4n\varphi_{1,4n-2}(b, \sigma) + n\varphi_{2,2n-2}(b, \sigma)} \\ &\geq \frac{15\varphi_{2,n-2}(b, \sigma) \{2n\varphi_{1,4n-2}(b, \sigma) + \varphi_{2,2n-2}(b, \sigma)\} + 10n\varphi_{1,4n-2}(b, \sigma)\varphi_{3,n-2}(b, \sigma)}{2n^2\varphi_{1,4n-2}(b, \sigma)\{4\varphi_{1,4n-2}(b, \sigma) + \varphi_{2,2n-2}(b, \sigma)\}} \\ &\geq \frac{5}{n} \frac{\varphi_{3,n-2}(b, \sigma) + 3\gamma_{2,2n}^{1/2}\varphi_{2,n-2}(b, \sigma)}{4\varphi_{1,4n-2}(b, \sigma) + \varphi_{2,2n-2}(b, \sigma)} \\ \theta_{3,n} &= 7n\varphi_{1,3n-2}(b, \sigma) + 10n\varphi_{2,n-2}(b, \sigma) + 3n\varphi_{3,n-2}(b, \sigma). \end{aligned}$$

**Fourth derivative flow moments.** In order to bound the fourth derivative of the semigroup, we only need to consider the second moment of the fourth derivative flow. Dynkin's lemma applied to the function  $x \rightarrow \|x\|_2^2$  for the diffusion (B.4) yields

$$\begin{aligned} \mathbb{E} [\|Y_t^{x,vuwy}\|_2^2] &= 2 \int_0^t \mathbb{E} \left[ \langle \nabla^4 b(Z_s^x) [V_s^{x,v}, V_s^{x,u}, V_s^{x,w}, V_s^{x,y}] + \nabla^3 b(Z_s^x) [U_s^{x,vu}, V_s^{x,w}, V_s^{x,y}] \right. \\ &\quad + \nabla^3 b(Z_s^x) [U_s^{x,uw}, V_s^{x,v}, V_s^{x,y}] + \nabla^3 b(Z_s^x) [U_s^{x,vy}, V_s^{x,u}, V_s^{x,w}] \\ &\quad + \nabla^3 b(Z_s^x) [U_s^{x,vw}, V_s^{x,u}, V_s^{x,y}] + \nabla^3 b(Z_s^x) [U_s^{x,wy}, V_s^{x,v}, V_s^{x,u}] \\ &\quad + \nabla^3 b(Z_s^x) [U_s^{x,uy}, V_s^{x,v}, V_s^{x,w}] + \nabla^2 b(Z_s^x) [U_s^{x,vu}, U_s^{x,wy}] \\ &\quad + \nabla^2 b(Z_s^x) [U_s^{x,uw}, U_s^{x,vy}] + \nabla^2 b(Z_s^x) [U_s^{x,uy}, U_s^{x,vw}] + \nabla^2 b(Z_s^x) [W_s^{x,vuy}, V_s^{x,w}] \\ &\quad + \nabla^2 b(Z_s^x) [W_s^{x,vuw}, V_s^{x,y}] + \nabla^2 b(Z_s^x) [W_s^{x,uwy}, V_s^{x,v}] \\ &\quad \left. + \nabla^2 b(Z_s^x) [W_s^{x,vwy}, V_s^{x,u}] + \nabla b(Z_s^x) Y_s^{x,vuwy}, Y_s^{x,vuwy} \right] ds \\ &+ \int_0^t \mathbb{E} \left[ \|\nabla^4 \sigma(Z_s^x) [V_s^{x,v}, V_s^{x,u}, V_s^{x,w}, V_s^{x,y}] + \nabla^3 \sigma(Z_s^x) [U_s^{x,vu}, V_s^{x,w}, V_s^{x,y}] \right. \\ &\quad + \nabla^3 \sigma(Z_s^x) [U_s^{x,uy}, V_s^{x,v}, V_s^{x,w}] + \nabla^3 \sigma(Z_s^x) [U_s^{x,wy}, V_s^{x,v}, V_s^{x,u}] \\ &\quad + \nabla^3 \sigma(Z_s^x) [U_s^{x,vy}, V_s^{x,u}, V_s^{x,w}] + \nabla^3 \sigma(Z_s^x) [U_s^{x,uw}, V_s^{x,v}, V_s^{x,y}] \\ &\quad + \nabla^3 \sigma(Z_s^x) [U_s^{x,vw}, V_s^{x,u}, V_s^{x,y}] + \nabla^2 \sigma(Z_s^x) [U_s^{x,vu}, U_s^{x,wy}] \\ &\quad + \nabla^2 \sigma(Z_s^x) [U_s^{x,uw}, U_s^{x,vy}] + \nabla^2 \sigma(Z_s^x) [U_s^{x,uy}, U_s^{x,vw}] + \nabla^2 \sigma(Z_s^x) [W_s^{x,vuy}, V_s^{x,w}] \\ &\quad + \nabla^2 \sigma(Z_s^x) [W_s^{x,vwy}, V_s^{x,u}] + \nabla^2 \sigma(Z_s^x) [W_s^{x,vuw}, V_s^{x,y}] \\ &\quad \left. + \nabla^2 \sigma(Z_s^x) [W_s^{x,uwy}, V_s^{x,v}] + \nabla \sigma(Z_s^x) Y_s^{x,vuwy} \right]_{\mathbb{F}}^2 ds. \end{aligned} \tag{B.11}$$

Once again we will use the Young's inequality to obtain a Grönwall form. The following inequalities will be useful

$$\begin{aligned} \|Y_s^{x,vuwy}\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2 &\leq \frac{1}{2} \|Y_s^{x,vuwy}\|_2^2 + \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2^2, \\ \|Y_s^{x,vuwy}\|_2 \|U_s^{x,vu}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2 &\leq \frac{1}{2} \|Y_s^{x,vuwy}\|_2^2 + \|U_s^{x,vu}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2^2, \\ \|Y_s^{x,vuwy}\|_2 \|U_s^{x,vu}\|_2 \|U_s^{x,wy}\|_2 &\leq \frac{1}{2} \|Y_s^{x,vuwy}\|_2^2 + \|U_s^{x,vu}\|_2 \|U_s^{x,wy}\|_2^2, \\ \|Y_s^{x,vuwy}\|_2 \|W_s^{x,vuy}\|_2 \|V_s^{x,w}\|_2 &\leq \frac{1}{2} \|Y_s^{x,vuwy}\|_2^2 + \|W_s^{x,vuy}\|_2 \|V_s^{x,w}\|_2^2. \end{aligned} \tag{B.12}$$

The expression in (B.11) can be upper bounded as follows.

$$\begin{aligned}
\mathbb{E}[\|Y_t^{x,vuwy}\|_2^2] &\leq 2 \int_0^t \left( \mathbb{E} \left[ \|Y_s^{x,vuwy}\|_2 \left\{ \mu_4(b) \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2 \right. \right. \right. \\
&\quad + \mu_3(b) \|U_s^{x,vu}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2 + \mu_3(b) \|U_s^{x,uw}\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,y}\|_2 \\
&\quad + \mu_3(b) \|U_s^{x,vy}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,w}\|_2 + \mu_3(b) \|U_s^{x,vw}\|_2 \|V_s^{x,u}\|_2 \|V_s^{x,y}\|_2 \\
&\quad + \mu_3(b) \|U_s^{x,wy}\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2 + \mu_3(b) \|U_s^{x,uy}\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,w}\|_2 \\
&\quad + \mu_2(b) \|U_s^{x,vu}\|_2 \|U_s^{x,wy}\|_2 + \mu_2(b) \|U_s^{x,uw}\|_2 \|U_s^{x,vy}\|_2 + \mu_2(b) \|U_s^{x,uy}\|_2 \|U_s^{x,vw}\|_2 \\
&\quad + \mu_2(b) \|W_s^{x,vuy}\|_2 \|V_s^{x,w}\|_2 + \mu_2(b) \|W_s^{x,vuw}\|_2 \|V_s^{x,y}\|_2 + \mu_2(b) \|W_s^{x,uwy}\|_2 \|V_s^{x,v}\|_2 \\
&\quad \left. \left. \left. + \mu_2(b) \|W_s^{x,vwy}\|_2 \|V_s^{x,u}\|_2 + \mu_1(b) \|Y_s^{x,vuwy}\|_2 \right\} \right] \right) ds \\
&+ 15 \int_0^t \left( \phi_4(\sigma)^2 \mathbb{E}[\|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2 \|V_s^{x,w}\|_2^2 \|V_s^{x,y}\|_2^2] \right. \\
&\quad + \phi_3(\sigma)^2 \mathbb{E}[\|U_s^{x,uy}\|_2^2 \|V_s^{x,v}\|_2^2 \|V_s^{x,w}\|_2^2] + \phi_3(\sigma)^2 \mathbb{E}[\|U_s^{x,wy}\|_2^2 \|V_s^{x,v}\|_2^2 \|V_s^{x,u}\|_2^2] \\
&\quad + \phi_3(\sigma)^2 \mathbb{E}[\|U_s^{x,vy}\|_2^2 \|V_s^{x,u}\|_2^2 \|V_s^{x,w}\|_2^2] + \phi_3(\sigma)^2 \mathbb{E}[\|U_s^{x,uw}\|_2^2 \|V_s^{x,v}\|_2^2 \|V_s^{x,y}\|_2^2] \\
&\quad + \phi_3(\sigma)^2 \mathbb{E}[\|U_s^{x,vw}\|_2^2 \|V_s^{x,u}\|_2^2 \|V_s^{x,y}\|_2^2] + \phi_3(\sigma)^2 \mathbb{E}[\|U_s^{x,vu}\|_2^2 \|V_s^{x,w}\|_2^2 \|V_s^{x,y}\|_2^2] \\
&\quad + \phi_2(\sigma)^2 \mathbb{E}[\|U_s^{x,uw}\|_2^2 \|U_s^{x,vu}\|_2^2] + \phi_2(\sigma)^2 \mathbb{E}[\|U_s^{x,vu}\|_2^2 \|U_s^{x,wy}\|_2^2] \\
&\quad + \phi_2(\sigma)^2 \mathbb{E}[\|U_s^{x,uy}\|_2^2 \|U_s^{x,vw}\|_2^2] + \phi_2(\sigma)^2 \mathbb{E}[\|W_s^{x,vuy}\|_2^2 \|V_s^{x,w}\|_2^2] \\
&\quad + \phi_2(\sigma)^2 \mathbb{E}[\|W_s^{x,vwy}\|_2^2 \|V_s^{x,u}\|_2^2] + \phi_2(\sigma)^2 \mathbb{E}[\|W_s^{x,vuw}\|_2^2 \|V_s^{x,y}\|_2^2] \\
&\quad \left. \left. + \phi_2(\sigma)^2 \mathbb{E}[\|W_s^{x,uwy}\|_2^2 \|V_s^{x,v}\|_2^2] + \phi_1(\sigma)^2 \mathbb{E}[\|Y_s^{x,vuwy}\|_2^2] \right) \right) ds.
\end{aligned} \tag{B.13}$$

Using the above Young's inequalities (B.12) in the integrals in (B.13), we obtain

$$\begin{aligned}
\mathbb{E}[\|Y_t^{x,vuwy}\|_2^2] &\leq \left\{ \mu_4(b) + 6\mu_3(b) + 7\mu_2(b) + 2\mu_1(b) + 15\phi_1(\sigma)^2 \right\} \int_0^t \mathbb{E}[\|Y_s^{x,vuwy}\|_2^2] ds \\
&\quad + \int_0^t \left\{ (\mu_4(b) + 15\phi_4(\sigma)^2) e^{s\theta_{1,s}} + (6\mu_3(b) + 90\phi_3(\sigma)^2) \gamma_{2,4}^{1/2} e^{s(\theta_{2,4} + \theta_{1,s})/2} \right. \\
&\quad + (4\mu_2(b) + 60\phi_2(\sigma)^2) \gamma_{3,4}^{1/2} e^{s(\theta_{3,4} + \theta_{1,4})/2} \\
&\quad \left. + (3\mu_2(b) + 45\phi_2(\sigma)^2) \gamma_{2,4} e^{s\theta_{2,4}} \right\} ds \|v\|_2^2 \|u\|_2^2 \|w\|_2^2 \|y\|_2^2.
\end{aligned} \tag{B.14}$$

Using the expressions derived earlier in (B.6), (B.8), and (B.10), we notice that all the exponents in the second integral in (B.14) can be upper bounded by  $16\varphi_{1,9}(b, \sigma) + 20\varphi_{2,2}(b, \sigma) + 6\varphi_{3,2}(b, \sigma)$ . Integrating the resulting upper bound, and finally, applying the Grönwall's lemma to the last expression, we obtain that

$$\mathbb{E}[\|Y_s^{x,vuwy}\|_2^2] \leq \gamma_{4,2} e^{t\theta_{4,2}} \|v\|_2^2 \|u\|_2^2 \|w\|_2^2 \|y\|_2^2$$

where

$$\begin{aligned}
\gamma_{4,2} &= \frac{\varphi_{4,0}(b, \sigma) + 6\varphi_{3,0}(b, \sigma) + 5\varphi_{2,0}(b, \sigma)}{16\varphi_{1,6}(b, \sigma)} \\
&\geq 15 \frac{\varphi_{4,0}(b, \sigma) + 6\gamma_{2,4}^{1/2} \varphi_{3,0}(b, \sigma) + (4\gamma_{3,4}^{1/2} + 3\gamma_{2,4}) \varphi_{2,0}(b, \sigma)}{16\varphi_{1,9}(b, \sigma) + 20\varphi_{2,2}(b, \sigma) + 6\varphi_{3,2}(b, \sigma)} \\
\theta_{4,2} &= 31\varphi_{1,5}(b, \sigma) + 27\varphi_{2,2}(b, \sigma) + 12\varphi_{3,1}(b, \sigma) + \varphi_{4,0}(b, \sigma) \\
&\geq \mu_4(b) + 6\mu_3(b) + 7\mu_2(b) + 2\mu_1(b) + 15\phi_1(\sigma)^2 \\
&\quad + 16\varphi_{1,9}(b, \sigma) + 20\varphi_{2,2}(b, \sigma) + 6\varphi_{3,2}(b, \sigma).
\end{aligned}$$

The following lemma collects the results derived in this section.

**Lemma B.1.** *Given the function*

$$\varphi_{i,n}(b, \sigma) \triangleq \mu_i(b) + n\mu_i(\sigma)^2 + \phi_i(\sigma)^2,$$

*we have the following moment bounds for the first four derivative flows*

$$\begin{aligned}
\mathbb{E}[\|V_t^{x,v}\|_2^n] &\leq \gamma_{1,n} e^{t\theta_{1,n}} \|v\|_2^n, \\
\mathbb{E}[\|U_t^{x,vu}\|_2^n] &\leq \gamma_{2,n} e^{t\theta_{2,n}} \|v\|_2^n \|u\|_2^n, \\
\mathbb{E}[\|W_t^{x,vuw}\|_2^n] &\leq \gamma_{3,n} e^{t\theta_{3,n}} \|v\|_2^n \|u\|_2^n \|w\|_2^n, \\
\mathbb{E}[\|Y_t^{x,vuwy}\|_2^n] &\leq \gamma_{4,n} e^{t\theta_{4,n}} \|v\|_2^n \|u\|_2^n \|w\|_2^n \|y\|_2^n,
\end{aligned}$$

where,

$$\begin{aligned}
\gamma_{1,n} &= 1, \\
\theta_{1,n} &= n\varphi_{1,n-2}(b, \sigma), \\
\gamma_{2,n} &= \frac{\varphi_{2,n-2}(b, \sigma)}{n\varphi_{1,2n-2}(b, \sigma)}, \\
\theta_{2,n} &= 3n\varphi_{1,2n-2}(b, \sigma) + n\varphi_{2,n-2}(b, \sigma), \\
\gamma_{3,n} &= \frac{15\varphi_{2,n-2}(b, \sigma) + 5\varphi_{3,n-2}(b, \sigma)}{4n\varphi_{1,4n-2}(b, \sigma)}, \\
\theta_{3,n} &= 7n\varphi_{1,3n-2}(b, \sigma) + 10n\varphi_{2,n-2}(b, \sigma) + 3n\varphi_{3,n-2}(b, \sigma), \\
\gamma_{4,2} &= \frac{\varphi_{4,0}(b, \sigma) + 6\varphi_{3,0}(b, \sigma) + 5\varphi_{2,0}(b, \sigma)}{16\varphi_{1,6}(b, \sigma)}, \\
\theta_{4,2} &= 31\varphi_{1,5}(b, \sigma) + 27\varphi_{2,2}(b, \sigma) + 12\varphi_{3,1}(b, \sigma) + \varphi_{4,0}(b, \sigma).
\end{aligned}$$

## C Bounds on the Semigroup Derivatives

In this section, we establish the polynomial growth of semigroup derivatives of up to fourth order. We state a few preliminary results that will be helpful throughout this section. Our first lemma is a standard application of Dynkin's formula [22], and establishes a moment bound for the diffusion process.

**Lemma C.1.** *If the diffusion satisfies Condition 2, then we have*

$$\mathbb{E}[\|Z_t^x\|_2^n] \leq e^{-\alpha t} \|x\|_2^n + \frac{\beta_{r,n}}{\alpha} (1 - e^{-\alpha t}) \leq \|x\|_2^n + \frac{\beta_{r,n}}{\alpha},$$

where  $\beta_{r,2} = \beta$  and for  $n \geq 3$ ,  $\beta_{r,n}$  is as in Lemma A.1.

*Proof.* Applying Dynkin's formula to the function  $(t, x) \rightarrow e^{\alpha t} \|x\|_2^n$ , we obtain

$$\begin{aligned}
e^{\alpha t} \mathbb{E}[\|Z_t^x\|_2^n] &= \|x\|_2^n + \int_0^t \mathbb{E}[\alpha e^{\alpha s} \|Z_s^x\|_2^n + e^{\alpha s} \mathcal{A} \|Z_s^x\|_2^n] ds \\
&\leq \|x\|_2^n + \int_0^t \mathbb{E}[\alpha e^{\alpha s} \|Z_s^x\|_2^n - \alpha e^{\alpha s} \|Z_s^x\|_2^n + \beta_{r,n} e^{\alpha s}] ds \\
&= \|x\|_2^n + \frac{\beta_{r,n}}{\alpha} (e^{\alpha t} - 1),
\end{aligned}$$

where the second line follows from Lemma A.1. Multiplying both sides with  $e^{-\alpha t}$  provides the desired result.  $\square$

We will use the polynomial coefficient of a function  $f$  defined as in (2.6). We observe that for any  $m \in \mathbb{N}$ , we can write

$$\|\nabla^i f(x)\|_{\text{op}} \leq \inf_k \tilde{\pi}_{i,k}(f) (1 + \|x\|_2^k) \leq \tilde{\pi}_{i,m}(f) (1 + \|x\|_2^m). \quad (\text{C.1})$$

If  $f$  has a higher degree polynomial growth than  $m$ , we have  $\tilde{\pi}_{i,m} = \infty$ , and the above inequality is still satisfied. In order to simplify our bounds, we define the following function  $\nu_{i,j}(g) = 1 \vee \max_{i \leq k \leq j} \mu_k(g)$  for  $j > i$ .

**Lemma C.2.** *Let  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  be a pseudo-Lipschitz continuous function of order  $n$  as in 2.5. Then, the following items hold:*

- $g$  has polynomial growth of degree at most  $n + 1$ ,

$$|g(x)| \leq \|g(0)\|_2 + \tilde{\mu}_{1,n}(g) (1 + 2\|x\|_2^{n+1}).$$

- The gradient of  $g$  has polynomial growth of degree at most  $n$ ,

$$\|\nabla g(x)\|_2 \leq 2\tilde{\mu}_{1,n}(g) (1 + \|x\|_2^n), \text{ i.e. } \tilde{\pi}_{1,n}(g) \leq 2\tilde{\mu}_{1,n}(g).$$

*Proof.* The first result follows from (2.5) by letting  $y = 0$ , and applying the triangle inequality. For the second result, for  $u \in \mathbb{R}^d$  with  $\|u\|_2 = 1$  and we write

$$\begin{aligned}
\langle \nabla g(x), u \rangle &= \lim_{\epsilon \downarrow 0} \frac{g(x + \epsilon u) - g(x)}{\epsilon}, \\
&\leq \lim_{\epsilon \downarrow 0} \tilde{\mu}_{1,n}(g) (1 + \|x + \epsilon u\|_2^n + \|x\|_2^n), \\
&= \tilde{\mu}_{1,n}(g) (1 + 2\|x\|_2^n).
\end{aligned}$$

Then the result follows by taking the supremum over  $u$  on the left hand side, and using the relation  $\|\nabla g(x)\|_2 = \sup_{\|u\|_2=1} \langle \nabla g(x), u \rangle$ .  $\square$



**Lemma C.3.** For a function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  with  $i$ -th derivative  $\nabla^i g$ , and any  $n, m, l \geq 1$ , we have

$$\mathbb{E}[\|\nabla^i g(Z_t^x)\|_{\text{op}}^l]^{1/l} \leq 2\tilde{\pi}_{i,n}(g) \{1 + (\beta_{r,lmn}/\alpha)^{1/lm}\} (1 + \|x\|_2^n).$$

**Remark 1.** Note that the above lemma still holds in the trivial case where  $g$  cannot be bounded with a degree- $n$  polynomial. In this case,  $\tilde{\pi}_{i,n} = \infty$  enforces the inequality.

*Proof.* We have,

$$\begin{aligned} \mathbb{E}[\|\nabla^i g(Z_t^x)\|_{\text{op}}^l]^{1/l} &\leq \tilde{\pi}_{i,n}(g) \mathbb{E}[(1 + \|Z_t^x\|_2^n)^l]^{1/l}, \\ &\leq 2\tilde{\pi}_{i,n}(g) \{1 + \mathbb{E}[\|Z_t^x\|_2^{ln}]\}^{1/l}, \\ &\leq 2\tilde{\pi}_{i,n}(g) \{1 + \mathbb{E}[\|Z_t^x\|_2^{lmn}]^{1/lm}\}, \\ &\leq 2\tilde{\pi}_{i,n}(g) \{1 + \|x\|_2^n + (\beta_{r,lmn}/\alpha)^{1/lm}\}, \\ &\leq 2\tilde{\pi}_{i,n}(g) \{1 + (\beta_{r,lmn}/\alpha)^{1/lm}\} (1 + \|x\|_2^n), \end{aligned}$$

where the first step follows from the polynomial growth assumption (C.1), and the last step follows from Lemma C.1.  $\square$

The semigroup derivatives can be obtained by taking the derivative of  $P_t f(x) = \mathbb{E}[f(Z_t^x)]$  with respect to  $x$ . This provides us with the following expressions.

- The first derivative of the semigroup in the direction  $v$  is given as

$$\langle v, \nabla(P_t f)(x) \rangle = \mathbb{E}[\langle \nabla f(Z_t^x), V_t^{x,v} \rangle].$$

- The second derivative of the semigroup in the directions  $v$  and  $u$  is given as

$$\langle v, \nabla^2(P_t f)(x)u \rangle = \mathbb{E}[\langle \nabla^2 f(Z_t^x) V_t^{x,u}, V_t^{x,v} \rangle] + \mathbb{E}[\langle \nabla f(Z_t^x), U_t^{x,vu} \rangle].$$

- The third derivative of the semigroup in the directions  $v, u$ , and  $w$  is given as

$$\begin{aligned} \nabla^3(P_t f)(x)[v, u, w] &= \mathbb{E}[\nabla^3 f(Z_t^x)[V_t^{x,v}, V_t^{x,u}, V_t^{x,w}]] + \mathbb{E}[\langle \nabla^2 f(Z_t^x) U_t^{x,uw}, V_t^{x,v} \rangle] \\ &\quad + \mathbb{E}[\langle \nabla^2 f(Z_t^x) V_t^{x,u}, U_t^{x,vw} \rangle] + \mathbb{E}[\langle \nabla^2 f(Z_t^x) V_t^{x,w}, U_t^{x,vu} \rangle] \\ &\quad + \mathbb{E}[\langle \nabla f(Z_t^x), W_t^{x,vuw} \rangle]. \end{aligned} \tag{C.2}$$

- The fourth derivative of the semigroup in the directions  $v, u, w$ , and  $y$  is given as

$$\begin{aligned} \nabla^4(P_t f)(x)[v, u, w, y] &= \mathbb{E}[\nabla^4 f(Z_t^x)[V_t^{x,v}, V_t^{x,u}, V_t^{x,w}, V_t^{x,y}]] \\ &\quad + \mathbb{E}[\nabla^3 f(Z_t^x)[U_t^{x,vy}, V_t^{x,u}, V_t^{x,w}]] + \mathbb{E}[\nabla^3 f(Z_t^x)[U_t^{x,uy}, V_t^{x,v}, V_t^{x,w}]] \\ &\quad + \mathbb{E}[\nabla^3 f(Z_t^x)[U_t^{x,wy}, V_t^{x,v}, V_t^{x,u}]] + \mathbb{E}[\nabla^3 f(Z_t^x)[U_t^{x,uw}, V_t^{x,y}, V_t^{x,v}]] \\ &\quad + \mathbb{E}[\nabla^3 f(Z_t^x)[U_t^{x,vw}, V_t^{x,u}, V_t^{x,y}]] + \mathbb{E}[\nabla^3 f(Z_t^x)[U_t^{x,vu}, V_t^{x,w}, V_t^{x,y}]] \\ &\quad + \mathbb{E}[\langle \nabla^2 f(Z_t^x) U_t^{x,uw}, U_t^{x,vy} \rangle] + \mathbb{E}[\langle \nabla^2 f(Z_t^x) U_t^{x,uy}, U_t^{x,vw} \rangle] \\ &\quad + \mathbb{E}[\langle \nabla^2 f(Z_t^x) U_t^{x,wy}, U_t^{x,vu} \rangle] + \mathbb{E}[\langle \nabla^2 f(Z_t^x) W_t^{x,uwy}, V_t^{x,v} \rangle] \\ &\quad + \mathbb{E}[\langle \nabla^2 f(Z_t^x) W_t^{x,vwy}, V_t^{x,u} \rangle] + \mathbb{E}[\langle \nabla^2 f(Z_t^x) W_t^{x,vuy}, V_t^{x,w} \rangle] \\ &\quad + \mathbb{E}[\langle \nabla^2 f(Z_t^x) W_t^{x,vuw}, V_t^{x,y} \rangle] + \mathbb{E}[\langle \nabla f(Z_t^x), Y_t^{x,vuwy} \rangle]. \end{aligned} \tag{C.3}$$

The above expressions will be useful when we derive bounds on the semigroup derivatives when  $t$  is small, say  $t < 1$ . When  $t > 1$ , we will appeal to a Bismut-Elworthy-Li (BEL) type equality to obtain upper bounds. BEL equality and related expressions will be provided in Section C.2.

### C.1 First derivative of the semigroup

The following lemma establishes the polynomial growth of the first derivative of the semigroup.

**Lemma C.4.** Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a pseudo-Lipschitz continuous function of order  $n$ , and the diffusion  $Z_t^x$  satisfies Condition 1 and 2, and has  $L_1$ - and  $L_2$ -Wasserstein rates  $\varrho_1$  and  $\varrho_2$ , respectively. Then, the following items hold:

- For any  $m, l \geq 1$ , we have

$$\mathbb{E}[\|\nabla f(Z_t^x)\|_2^l]^{1/l} \leq 4\tilde{\mu}_{1,n}(f) \{1 + (\beta_{r,lmn}/\alpha)^{1/lm}\} (1 + \|x\|_2^n).$$

- Let  $P_t f(x)$  denote the transition semigroup of an Itô diffusion. Define  $\tilde{\varrho}_1(t) = \log(\varrho_2/\varrho_1(t))$  and  $\tilde{\varrho}_2(t) = [\log(\varrho_1/\varrho_2(t)/\varrho_1(0))]/\log(\varrho_1(t)/\varrho_1(0))$ , and when  $r = 2$  assume that  $\alpha > n\lambda_a/2\tilde{\varrho}_2(t) \forall t \geq 0$ . Then for any  $m \geq 1$ ,  $P_t f(x)$  is pseudo-Lipschitz continuous of order  $n$  with constant

$$\tilde{\mu}_{1,n}(P_t f) \leq \tilde{\mu}_{1,n}(f)\varrho_1(t)\omega_r(t), \quad (\text{C.4})$$

and consequently  $\tilde{\pi}_{1,n}(P_t f) \leq 2\tilde{\mu}_{1,n}(f)\varrho_1(t)\omega_r(t)$  where

$$\omega_r(t) = 1 + 4\varrho_1(t)^{1-1/r}\varrho_1(0)^{1/2} \left(1 + 2 \left[ \frac{[1 \vee \tilde{\varrho}_r(t)]2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r} \right]^n \right),$$

where  $\tilde{\alpha}_1 = \alpha$  and  $\tilde{\alpha}_2 \geq \inf_t [\alpha - n\lambda_a(1 \vee \tilde{\varrho}_2)]_+$ .

- Let  $u_f$  denote the function that solves the corresponding Poisson equation. Then for any  $m \geq 1$ ,  $u_f$  is pseudo-Lipschitz continuous of order  $n$  with constant

$$\tilde{\mu}_{1,n}(u_f) \leq \tilde{\mu}_{1,n}(f) \int_0^\infty \varrho_1(t)\omega_r(t)dt.$$

**Remark 2.** We would like to highlight two important cases when  $r = 1$ .

- When we have  $L_2$ -Wasserstein exponential decay, i.e.,  $\varrho_1(t) = \varrho_2(t) = e^{-\kappa t}$  then we obtain

$$w_1(t) = 5 + 8([2\lambda_a n + 3\beta]/\alpha)^n,$$

which is independent of time  $t$ . In this case, we obtain that  $\tilde{\mu}_{1,n}(P_t f) = \mathcal{O}(e^{-\kappa t})$  which decays exponentially with time.

- When we have  $L_1$ -Wasserstein exponential decay and  $L_2$ -Wasserstein exponential growth, i.e., for  $\kappa_1, \kappa_2 > 0$   $\varrho_1(t) = e^{-\kappa_1 t}$  and  $\varrho_2(t) = e^{\kappa_2 t}$ , then we obtain

$$\omega_1(t) = 5 + 8(\{[1 \vee [\kappa_1 + \kappa_2]t]\}2\lambda_a n + 3\beta/\alpha)^n = \mathcal{O}(t^n),$$

which grows polynomially with time. In this case, we obtain  $\tilde{\mu}_{1,n}(P_t f) = \mathcal{O}(t^n e^{-\kappa t})$  which still decays exponentially with time.

*Proof.* For the first item, by Lemma C.2 we have a bound on the polynomial growth of the gradient, i.e.,  $\tilde{\pi}_{1,n}(f) \leq 2\tilde{\mu}_{1,n}(f)$ . Next, applying Lemma C.3 for some  $l, m$ , we immediately obtain

$$\mathbb{E}[\|\nabla f(Z_t^x)\|_2^l]^{1/l} \leq 4\tilde{\mu}_{1,n}(f)\{1 + (\beta_{r,lmn}/\alpha)^{1/lm}\}(1 + \|x\|_2^n).$$

For the second item, we write

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\leq \mathbb{E}[|f(Z_t^x) - f(Z_t^y)|], \\ &\leq \tilde{\mu}_{1,n}(f)\mathbb{E}[(1 + \|Z_t^x\|^n + \|Z_t^y\|^n)\|Z_t^x - Z_t^y\|], \\ &\leq \tilde{\mu}_{1,n}(f)\{\mathbb{E}[\|Z_t^x - Z_t^y\|] + \mathbb{E}[\|Z_t^x\|^n\|Z_t^x - Z_t^y\|] + \mathbb{E}[\|Z_t^y\|^n\|Z_t^x - Z_t^y\|]\}. \end{aligned}$$

The first term can be bounded by the  $L_1$ -Wasserstein rate, i.e.,  $\mathbb{E}[\|Z_t^x - Z_t^y\|] \leq \varrho_1(t)\|x - y\|_2$ . For the second and the third terms, we use the following lemma.

**Lemma C.5.** Assume that Condition 1 holds. We have,

$$\begin{aligned} \mathbb{E}[\|Z_t^x\|_2^n\|Z_t^x - Z_t^y\|_2] &\leq (e \vee \varrho_1(0))^{1/2}\varrho_1(t)^{1/r} \left[ \|x\|_2^n + 1 + 2 \left( \frac{[1 \vee \tilde{\varrho}_r]2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r} \right)^n \right] \|x - y\|_2, \\ \mathbb{E}[\|Z_t^y\|_2^n\|Z_t^x - Z_t^y\|_2] &\leq (e \vee \varrho_1(0))^{1/2}\varrho_1(t)^{1/r} \left[ \|y\|_2^n + 1 + 2 \left( \frac{[1 \vee \tilde{\varrho}_r]2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r} \right)^n \right] \|x - y\|_2. \end{aligned}$$

where  $\tilde{\varrho}_1(t) = \log(\varrho_2/\varrho_1(t))$  and  $\tilde{\varrho}_2(t) = [\log(\varrho_1/\varrho_2(t)/r_1(0))]/\log(\varrho_1(t)/r_1(0))$ .

*Proof.* First, we prove the result for  $r = 2$ . We choose  $\epsilon_t = \frac{2}{3} \wedge \frac{1}{2 \log(\varrho_2/\varrho_1(t))}$ , and write

$$\begin{aligned} \mathbb{E}[\|Z_t^x\|_2^n\|Z_t^x - Z_t^y\|_2] &= \mathbb{E}[\|Z_t^x\|_2^n\|Z_t^x - Z_t^y\|_2^{\epsilon_t}\|Z_t^x - Z_t^y\|_2^{1-\epsilon_t}], \\ &\leq \mathbb{E}[\|Z_t^x - Z_t^y\|_2]^{1-\epsilon_t} \mathbb{E}[\|Z_t^x\|_2^{\frac{n}{\epsilon_t}}\|Z_t^x - Z_t^y\|_2]^{\epsilon_t}, \\ &\leq \mathbb{E}[\|Z_t^x - Z_t^y\|_2]^{1-\epsilon_t} \mathbb{E}[\|Z_t^x\|_2^{\frac{2n}{\epsilon_t}}]^{\frac{\epsilon_t}{2}} \mathbb{E}[\|Z_t^x - Z_t^y\|_2^2]^{\frac{\epsilon_t}{2}}. \end{aligned} \quad (\text{C.5})$$

The first term above is bounded by the  $L_1$ -Wasserstein rate. The second term can be bounded by Lemma C.1. Since we have  $\tau = 2/\epsilon_t \geq 3$ , we can write

$$\mathbb{E}[\|Z_t^x\|_2^{\frac{2n}{\epsilon_t}}] \leq \|x\|_2^n + (\beta_{r,2n/\epsilon_t}/\alpha)^{\epsilon_t/2}. \quad (\text{C.6})$$

Below, we use the definition of  $\beta_{r,\tau n}$  given in Lemma A.1 and that  $1 + x^n > x^{1/\tau}$  for any  $x > 0$ , and that  $1/2^{1/\tau} \leq 1$  and  $\tilde{\alpha}_r \leq \alpha$  in order to derive a bound on  $(\beta_{r,\tau n}/\alpha)^{1/\tau}$ . For  $\tau \geq 3$ , we have

$$\begin{aligned} \left(\frac{\beta_{r,\tau n}}{\alpha}\right)^{1/\tau} &\leq \frac{1}{2^{1/\tau}} \left(\frac{\lambda_a \tau n + 6r\beta}{2r\tilde{\alpha}_r}\right)^n + \left(\frac{\lambda_a \tau n + 4 + 2\beta}{2\alpha}\right)^{1/\tau}, \\ &\leq 1 + 2 \left(\frac{\lambda_a \tau n + 6r\beta}{2r\tilde{\alpha}_r}\right)^n. \end{aligned}$$

Combining the last line above with (C.6), we obtain

$$\mathbb{E}[\|Z_t^x\|_2^{2n/\epsilon_t}]^{\epsilon_t/2} \leq \|x\|_2^n + 1 + 2 \left(\frac{\lambda_a n/\epsilon_t + 3r\beta}{r\tilde{\alpha}_r}\right)^n. \quad (\text{C.7})$$

In the sequel, we let  $r = 1$ . For the last term in (C.5), we use  $L_2$ -Wasserstein rate and obtain  $\mathbb{E}[\|Z_t^x - Z_t^y\|_2^2]^{\epsilon_t/2} \leq \varrho_2(t)^{\epsilon_t} \|x - y\|_2^{\epsilon_t}$ . Combining these bounds, we obtain

$$\begin{aligned} \mathbb{E}[\|Z_t^x\|_2^n \|Z_t^x - Z_t^y\|_2] &\leq \varrho_1(t)^{1-\epsilon_t} \varrho_2(t)^{\epsilon_t} (\|x\|_2^n + 1 + 2[(\lambda_a n/\epsilon_t + 3\beta)/\alpha]^n) \|x - y\|_2, \\ &\leq \varrho_1(t) e^{1/2} \left(\|x\|_2^n + 1 + 2 \left[\frac{\{3/2\sqrt{2} \log(\varrho_2/\varrho_1(t))\} \lambda_a n + 3\beta}{\alpha}\right]^n\right) \|x - y\|_2, \\ &\leq 2\varrho_1(t) \left(\|x\|_2^n + 1 + 2 \left[\frac{\{1 \vee \log(\varrho_2/\varrho_1(t))\} 2\lambda_a n + 3\beta}{\alpha}\right]^n\right) \|x - y\|_2. \end{aligned}$$

Next, we consider the case  $r = 2$ . Define  $\tilde{\varrho}_1(t) = \varrho_1(t)/\varrho_1(0)$ . We again use the inequalities (C.5) and (C.7), but this time we choose  $\epsilon_t = \frac{2}{3} \wedge \log(\tilde{\varrho}_1(t))/[2 \log(\tilde{\varrho}_1/\varrho_2(t))]$  which implies  $2n/\epsilon_t \geq 3$ . Hence, for  $\tilde{\alpha}_2 = [\alpha - n\lambda_a/(2\epsilon_t)]_+$  we get the same inequality in (C.7) for  $r = 2$ .

$$\begin{aligned} \mathbb{E}[\|Z_t^x\|_2^n \|Z_t^x - Z_t^y\|_2] &\leq \varrho_1(0)^{1/2} \varrho_1(t)^{1/2} \left[\|x\|_2^n + 1 + 2 \left(\frac{\lambda_a n/\epsilon_t + 3r\beta}{r\tilde{\alpha}_r}\right)^n\right] \|x - y\|_2, \\ &\leq \varrho_1(0)^{1/2} \varrho_1(t)^{1/2} \left[\|x\|_2^n + 1 + 2 \left(\frac{[1 \vee \log(\tilde{\varrho}_1/\varrho_2(t))]/\log(\tilde{\varrho}_1(t)) 2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r}\right)^n\right] \|x - y\|_2. \end{aligned}$$

Combining these bounds, we obtain

$$\begin{aligned} \mathbb{E}[\|Z_t^x\|_2^n \|Z_t^x - Z_t^y\|_2] &\leq (e \vee \varrho_1(0))^{1/2} \varrho_1(t)^{1/r} \left[\|x\|_2^n + 1 + 2 \left(\frac{[1 \vee \tilde{\varrho}_r] 2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r}\right)^n\right] \|x - y\|_2, \end{aligned}$$

where  $\tilde{\varrho}_1(t) = \log(\varrho_2/\varrho_1(t))$  and  $\tilde{\varrho}_2(t) = [\log(\varrho_1/\varrho_2(t)/\varrho_1(0))]/\log(\varrho_1(t)/\varrho_1(0))$ . □

Therefore, we obtain

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\leq \tilde{\mu}_{1,n}(f) \{ \mathbb{E}[\|Z_t^x - Z_t^y\|_2] + \mathbb{E}[\|Z_t^x\|_2^n \|Z_t^x - Z_t^y\|_2] + \mathbb{E}[\|Z_t^y\|_2^n \|Z_t^x - Z_t^y\|_2] \}, \\ &\leq \tilde{\mu}_{1,n}(f) \left[ \varrho_1(t) + \varrho_1(t)^{1/r} (e \vee \varrho_1(0))^{1/2} \left(\|x\|_2^n + \|y\|_2^n + 2 + 4 \left[\frac{[1 \vee \tilde{\varrho}_r] 2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r}\right]^n\right) \right] \|x - y\|_2, \\ &\leq \tilde{\mu}_{1,n}(f) \varrho_1(t) \left[ 1 + 2\varrho_1(t)^{1-1/r} (e \vee \varrho_1(0))^{1/2} \left(1 + 2 \left[\frac{[1 \vee \tilde{\varrho}_r(t)] 2\lambda_a n + 3r\beta}{r\tilde{\alpha}_r}\right]^n\right) \right] \\ &\quad \times (1 + \|x\|_2^n + \|y\|_2^n) \|x - y\|_2. \end{aligned}$$

Hence, we conclude that  $P_t f$  is pseudo-Lipschitz continuous of order  $n$ , with coefficient with

$$\begin{aligned} \tilde{\mu}_{1,n}(P_t f) &\leq \tilde{\mu}_{1,n}(f) \varrho_1(t) \omega_r(t) \quad \text{where} \\ \omega_r(t) &= 1 + 2\varrho_1(t)^{1-1/r} (e \vee \varrho_1(0))^{1/2} \left(1 + 2 \left[\frac{[1 \vee \tilde{\varrho}_r(t)] 2\lambda_b n + 3r\beta}{\tilde{\alpha}_r}\right]^n\right). \end{aligned}$$

For the last item, we write

$$\begin{aligned} |u_f(x) - u_f(y)| &\leq \int_0^\infty |P_t f(x) - P_t f(y)| dt, \\ &\leq \int_0^\infty \tilde{\mu}_{1,n}(P_t f) dt (1 + \|x\|_2^n + \|y\|_2^n) \|x - y\|_2, \end{aligned}$$

which concludes the proof. □

## C.2 Second derivative of the semigroup

For the higher order derivatives, our main tools will be the Markov property of the semigroup,

$$(P_t f)(x) = (P_s [P_{t-s} f])(x) \text{ for } s \in [0, t],$$

the following Bismut-Elworthy-Li-type formula [2, 4, 13],

$$\langle \nabla (P_t f)(x), v \rangle = \frac{1}{t} \mathbb{E} \left[ f(Z_t^x) \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle \right], \quad (\text{C.8})$$

and the following identity which is obtained by applying Itô's formula [24] to the function  $(s, x) \rightarrow P_{t-s} f(x)$  and used in the derivation of (C.8) [13]

$$f(Z_t^x) = (P_t f)(x) + \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), \sigma(Z_s^x) dB_s \rangle. \quad (\text{C.9})$$

Differentiating the expression given in (C.8), we obtain

$$\begin{aligned} \langle u, \nabla^2 (P_t f)(x) v \rangle &= \frac{1}{t} \mathbb{E} \left[ \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle \right], \\ &+ \frac{1}{t} \mathbb{E} \left[ f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}], dB_s \rangle \right], \\ &+ \frac{1}{t} \mathbb{E} \left[ f(Z_t^x) \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vu}, dB_s \rangle \right]. \end{aligned} \quad (\text{C.10})$$

The second and the third terms depend on  $f$  multiplied by an integrated Brownian motion, which helps us work directly with its gradient. To see this, we plug in the identity given in (C.9) and the second term becomes,

$$\begin{aligned} &\frac{1}{t} \mathbb{E} \left[ f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}], dB_s \rangle \right] \\ &= \frac{1}{t} \mathbb{E} \left[ \left\{ (P_t f)(x) + \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), \sigma(Z_s^x) dB_s \rangle \right\} \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}], dB_s \rangle \right], \\ &= \frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), \sigma(Z_s^x) \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}] \rangle ds \right], \\ &= -\frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), \nabla \sigma(Z_s^x) [V_s^{x,u}] \sigma^{-1}(Z_s^x) V_s^{x,v} \rangle ds \right], \end{aligned} \quad (\text{C.11})$$

where the second equality follows from Itô isometry, and for the last line, we used the identity given in Lemma F.4.

Similarly, for the third term we obtain

$$\begin{aligned} &\frac{1}{t} \mathbb{E} \left[ f(Z_t^x) \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vu}, dB_s \rangle \right] \\ &= \frac{1}{t} \mathbb{E} \left[ \left\{ (P_t f)(x) + \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), \sigma(Z_s^x) dB_s \rangle \right\} \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vu}, dB_s \rangle \right], \\ &= \frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), U_s^{x,vu} \rangle ds \right], \end{aligned} \quad (\text{C.12})$$

where the last line follows from the Itô isometry.

Plugging the expressions derived in (C.11) and (C.12) into (C.10), we obtain that

$$\begin{aligned} \langle u, \nabla^2 (P_t f)(x) v \rangle &= \frac{1}{t} \mathbb{E} \left[ \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle \right], \\ &- \frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), \nabla \sigma(Z_s^x) [V_s^{x,u}] \sigma^{-1}(Z_s^x) V_s^{x,v} \rangle ds \right], \\ &+ \frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \nabla (P_{t-s} f)(Z_s^x), U_s^{x,vu} \rangle ds \right]. \end{aligned} \quad (\text{C.13})$$

We will bound the absolute value of each term in (C.13) in turn. For the first term, we write

$$\begin{aligned} &\frac{1}{t} \left| \mathbb{E} \left[ \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle \right] \right| \\ &\leq \frac{1}{t} \mathbb{E} \left[ \|\nabla f(Z_t^x)\|_2 \|V_t^{x,u}\|_2 \left| \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle \right| \right], \\ &\leq \frac{1}{t} \mathbb{E} \left[ \|\nabla f(Z_t^x)\|_2^2 \|V_t^{x,u}\|_2^2 \right]^{1/2} \mathbb{E} \left[ \left| \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle \right|^2 \right]^{1/2}, \\ &\leq \frac{1}{t} \mathbb{E} \left[ \|\nabla f(Z_t^x)\|_2^4 \right]^{1/4} \mathbb{E} \left[ \|V_t^{x,u}\|_2^4 \right]^{1/4} \mathbb{E} \left[ \int_0^t \|\sigma^{-1}(Z_s^x) V_s^{x,v}\|_2^2 ds \right]^{1/2}, \\ &\leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t\theta_{1,4}/2} \|u\|_2 \|v\|_2, \end{aligned}$$

where we used Cauchy-Schwartz inequality in the first two steps, Itô isometry in the second step and Lemma C.4 with  $l = 4$  and  $m = 1.5$  to obtain

$$\mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \leq 4\tilde{\mu}_{1,n}(f)(1 + (\beta_{r,6n}/\alpha)^{1/6})(1 + \|x\|_2^n),$$

and Lemma B.1 to bound the derivative flow moments, i.e.,

$$\begin{aligned} \mathbb{E}[\|V_t^{x,u}\|_2^4]^{1/4} &\leq \|u\|_2 e^{t\theta_{1,4}/4}, \\ \mathbb{E}\left[\int_0^t \|\sigma^{-1}(Z_s^x)V_s^{x,v}\|_2^2 ds\right]^{1/2} &\leq \mu_0(\sigma^{-1})\|v\|_2 \sqrt{t} e^{t\theta_{1,4}/4}. \end{aligned}$$

For the second term in (C.13), we write

$$\begin{aligned} &\frac{1}{t} \left| \mathbb{E}\left[\int_0^t \langle \nabla(P_{t-s}f)(Z_s^x), \nabla\sigma(Z_s^x)[V_s^{x,u}]\sigma^{-1}(Z_s^x)V_s^{x,v} \rangle ds\right] \right| \\ &\leq \mu_1(\sigma)\mu_0(\sigma^{-1})\frac{1}{t} \int_0^t \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,u}\|_2] ds, \\ &\leq \mu_1(\sigma)\mu_0(\sigma^{-1})\frac{1}{t} \int_0^t \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^2]^{1/2} \mathbb{E}[\|V_s^{x,v}\|_2^4]^{1/4} \mathbb{E}[\|V_s^{x,u}\|_2^4]^{1/4} ds, \\ &\leq 4\mu_1(\sigma)\mu_0(\sigma^{-1})\tilde{\mu}_{1,n}(f)(1 + (\beta_{r,6n}/\alpha)^{1/6})\frac{1}{t} \int_0^t \varrho_1(t-s)\omega_r(t-s)e^{s\theta_{1,4}/2} ds (1 + \|x\|_2^n) \|u\|_2 \|v\|_2, \\ &\leq 4\mu_1(\sigma)\mu_0(\sigma^{-1})\tilde{\mu}_{1,n}(f)\varrho_1(0)\omega_r(t)e^{t\theta_{1,4}/2} (1 + (\beta_{r,6n}/\alpha)^{1/6})(1 + \|x\|_2^n) \|u\|_2 \|v\|_2, \end{aligned}$$

where we used Lemma C.4 with  $l = 2$  and  $m = 3$  to obtain

$$\begin{aligned} \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^2]^{1/2} &\leq 4\tilde{\mu}_{1,n}(P_{t-s}f)(1 + (\beta_{r,6n}/\alpha)^{1/6})(1 + \|x\|_2^n), \\ &\leq 4\tilde{\mu}_{1,n}(f)\varrho_1(t-s)\omega_r(t-s)(1 + (\beta_{r,6n}/\alpha)^{1/6})(1 + \|x\|_2^n), \end{aligned} \tag{C.14}$$

and Lemma B.1 to bound the derivative flow moments, and for  $s \in [0, t]$ ,

$$\varrho_1(t-s)\omega_r(t-s)e^{s\theta_{1,4}/2} \leq \varrho_1(0)\omega_r(t)e^{t\theta_{1,4}/2}.$$

For the third term in (C.13), we have

$$\begin{aligned} &\frac{1}{t} \left| \mathbb{E}\left[\int_0^t \langle \nabla(P_{t-s}f)(Z_s^x), U_s^{x,vu} \rangle ds\right] \right| \leq \frac{1}{t} \int_0^t \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^2]^{1/2} \mathbb{E}[\|U_s^{x,vu}\|_2^2]^{1/2} ds, \\ &\leq 4\tilde{\mu}_{1,n}(f)\gamma_{2,2}^{1/2}\frac{1}{t} \int_0^t \varrho_1(t-s)\omega_r(t-s)e^{s\theta_{2,2}/2} ds (1 + (\beta_{r,6n}/\alpha)^{1/6})(1 + \|x\|_2^n) \|u\|_2 \|v\|_2, \\ &\leq 4\tilde{\mu}_{1,n}(f)\varrho_1(0)\gamma_{2,2}^{1/2}\omega_r(t)e^{t\theta_{2,2}/2} (1 + (\beta_{r,6n}/\alpha)^{1/6})(1 + \|x\|_2^n) \|u\|_2 \|v\|_2, \end{aligned}$$

where we used (C.14), and Lemma B.1 to bound the moments of the second derivative flow, and  $\varrho_1(t-s)\omega_r(t-s)e^{s\theta_{2,2}/2} \leq \varrho_1(0)\omega_r(t)e^{t\theta_{2,2}/2}$  for  $s \in [0, t]$ .

Combining the above bounds, we obtain a bound of the form

$$\begin{aligned} \langle u, \nabla^2(P_t f)(x)v \rangle &\leq 4\bar{\xi}_2(t)(1 + (\beta_{r,6n}/\alpha)^{1/6})\tilde{\mu}_{1,n}(f)(1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \quad \text{where} \\ \bar{\xi}_2(t) &= \nu_0(\sigma^{-1})\varrho_1(0)\omega_r(t)e^{t\theta_{2,2}/2} \left[ \frac{1}{\sqrt{t}} + \gamma_{2,2}^{1/2} + \mu_1(\sigma) \right], \\ &\geq \mu_0(\sigma^{-1})\frac{1}{\sqrt{t}} e^{t\theta_{1,4}/2} + \mu_1(\sigma)\mu_0(\sigma^{-1})\varrho_1(0)\omega_r(t)e^{t\theta_{1,4}/2} + \varrho_1(0)\gamma_{2,2}^{1/2}\omega_r(t)e^{t\theta_{2,2}/2}. \end{aligned} \tag{C.15}$$

In order to obtain our final bound, we will appeal to an argument given in [4, 15]. Using the Markov property of the diffusion semigroup, we write

$$\begin{aligned} \langle u, \nabla^2(P_t f)(x)v \rangle &= \langle u, \nabla^2(P_s[P_{t-s}f])(x)v \rangle, \\ &\leq 4\bar{\xi}_2(s)(1 + (\beta_{r,6n}/\alpha)^{1/6})\tilde{\mu}_{1,n}(P_{t-s}f)(1 + \|x\|_2^n) \|u\|_2 \|v\|_2, \\ &\leq 4\bar{\xi}_2(s)(1 + (\beta_{r,6n}/\alpha)^{1/6})\tilde{\mu}_{1,n}(f)\varrho_1(t-s)\omega_r(t-s)(1 + \|x\|_2^n) \|u\|_2 \|v\|_2, \end{aligned} \tag{C.16}$$

for any  $s \in (0, t]$ . Note that we also used Lemma C.4 to get  $\tilde{\mu}_{1,n}(P_{t-s}f) \leq \tilde{\mu}_{1,n}(f)\varrho_1(t-s)\omega_r(t-s)$ . For simplicity, we choose  $s = 1 \wedge t$  in our calculations below, but it will be useful to only consider the case  $t \geq 1$  in the next two sections when we bound the third and the fourth derivative of the Poisson function, where the above bound reduces to

$$\begin{aligned} \tilde{\pi}_{2,n}(P_t f) &\leq \xi_2\varrho_1(t-1)\omega_r(t-1) \quad \text{for} \\ \xi_2 &= 4\tilde{\mu}_{1,n}(f)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}\varrho_1(0)\omega_r(1)\nu_0(\sigma^{-1})\left[1 + \gamma_{2,2}^{1/2} + \mu_1(\sigma)\right]e^{\theta_{2,2}/2}. \end{aligned}$$

Consequently, the Poisson function can be bounded as

$$\begin{aligned}\langle u, \nabla^2 u_f(x)v \rangle &= \int_0^\infty \langle u, \nabla^2(P_t f)(x)v \rangle dt \\ &\leq 4\bar{\rho}_2 \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \tilde{\mu}_{1,n}(f) (1 + \|x\|_2^n) \|u\|_2 \|v\|_2,\end{aligned}$$

where

$$\begin{aligned}\bar{\rho}_2 &= \int_0^\infty \bar{\xi}_2(1 \wedge t) \varrho_1(t-1 \wedge t) \omega_r(t-1 \wedge t) dt \\ &= \varrho_1(0) \omega_r(0) \int_0^1 \bar{\xi}_2(t) dt + \bar{\xi}_2(1) \int_1^\infty \varrho_1(t-1) \omega_r(t-1) dt, \\ &= \varrho_1(0) \omega_r(0) \nu_0(\sigma^{-1}) \varrho_1(0) \omega_r(1) e^{\theta_{2,2}/2} \left[2 + \gamma_{2,2}^{1/2} + \mu_1(\sigma)\right] + \bar{\xi}_2(1) \int_0^\infty \varrho_1(t) \omega_r(t) dt.\end{aligned}$$

The final bound is obtained by taking the supremum over  $u$  and  $v$ , i.e.,

$$\|\nabla^2 u_f(x)\|_{\text{op}} = \sup_{\|u\|_2=\|v\|_2=1} \langle u, \nabla^2 u_f(x)v \rangle \leq \zeta_2(1 + \|x\|_2^n),$$

where

$$\begin{aligned}\zeta_2 &\triangleq 2\xi_2 \varrho_1(0) \omega_r(0) + \xi_2 \int_0^\infty \varrho_1(t) \omega_r(t) dt, \quad \text{with} \\ \xi_2 &= 4\tilde{\mu}_{1,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \varrho_1(0) \omega_r(1) \nu_0(\sigma^{-1}) \left[1 + \gamma_{2,2}^{1/2} + \mu_1(\sigma)\right] e^{\theta_{2,2}/2}.\end{aligned}$$

### C.3 Third derivative of the semigroup

We differentiate (C.10) and obtain

$$\begin{aligned}\nabla^3(P_t f)(x)[u, v, w] &= \frac{1}{t} \sum_j \mathbb{E}[T_{3,j}] \quad \text{where} \tag{C.17} \\ T_{3,1} &= \nabla^2 f(Z_t^x) [V_t^{x,u}, V_t^{x,w}] \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\ T_{3,2} &= \langle \nabla f(Z_t^x), U_t^{x,uw} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\ T_{3,3} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_s^{x,w}], dB_s \rangle, \\ T_{3,4} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}], dB_s \rangle, \\ T_{3,5} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vw}, dB_s \rangle, \\ T_{3,6} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vu}, dB_s \rangle, \\ T_{3,7} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}, V_s^{x,w}], dB_s \rangle, \\ T_{3,8} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vu}, V_s^{x,w}], dB_s \rangle, \\ T_{3,9} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uw}, V_s^{x,v}], dB_s \rangle, \\ T_{3,10} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vw}, V_s^{x,u}], dB_s \rangle, \\ T_{3,11} &= f(Z_t^x) \int_0^t \langle \sigma^{-1}(Z_s^x) W_s^{x,vuw}, dB_s \rangle.\end{aligned}$$

We will bound each of the terms in (C.17) in turn. For the first term, we write

$$\begin{aligned}\frac{1}{t} |\mathbb{E}[T_{3,1}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^2 \|V_t^{x,w}\|_2^2 \|V_t^{x,u}\|_2^2]^{1/2} \mathbb{E}\left[\left|\int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle\right|^2\right]^{1/2}, \tag{C.18} \\ &\leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^2]^{1/6} \mathbb{E}[\|V_t^{x,w}\|_2^6]^{1/6} \mathbb{E}[\|V_t^{x,u}\|_2^6]^{1/6} \mu_0(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 ds\right]^{1/2}, \\ &\leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t\theta_{1,6}/2} \|u\|_2 \|w\|_2 \|v\|_2,\end{aligned}$$

where the first and the second inequalities follow from Cauchy-Schwartz and the Itô isometry, and the third one follows from Lemma C.3 for  $l = 6, m = 1$ , and Lemma B.1 with  $\theta_{1,6}/3 \geq \theta_{1,2}$ . We note that the inequality (C.18) holds for any  $n \in \mathbb{N}$ . When  $f$  has a larger polynomial growth than  $n$ , we let  $\tilde{\pi}_{2,n}(f)$  to be  $\infty$ .

For the second term, we write

$$\begin{aligned}\frac{1}{t} |\mathbb{E}[T_{3,2}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^6]^{1/6} \mathbb{E}[\|U_t^{x,uw}\|_2^3]^{1/3} \mathbb{E}\left[\left|\int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle\right|^2\right]^{1/2}, \\ &\leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,3}^{1/3} \frac{1}{\sqrt{t}} e^{t\theta_{2,3}/2} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2,\end{aligned}$$



where the first step follows from Cauchy-Schwartz and Hölder inequalities, and the second step follows from the Itô isometry, Lemma C.4 for  $l = 6, m = 1$ , as well as Lemma B.1 with  $\theta_{2,3}/3 \geq \theta_{1,2}$ .

For the terms involving  $T_{3,3}, T_{3,4}$  we write

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{3,3}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \mathbb{E}[\|V_t^{x,u}\|_2^4]^{1/4} \mathbb{E}\left[\left|\int_0^t \langle \nabla \sigma^{-1}(Z_s^x)[V_s^{x,w}, V_s^{x,v}], dB_s \rangle\right|^2\right]^{1/2}, \\ &\leq 4\tilde{\mu}_{1,n}(f)\mu_1(\sigma^{-1})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \frac{1}{\sqrt{t}} e^{t3\theta_{1,4}/4} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2, \\ \frac{1}{t} |\mathbb{E}[T_{3,4}]| &\leq 4\tilde{\mu}_{1,n}(f)\mu_1(\sigma^{-1})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \frac{1}{\sqrt{t}} e^{t3\theta_{1,4}/4} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2, \end{aligned}$$

where we use Cauchy-Schwartz, the Itô isometry, Lemma C.4 for  $l = 4, m = 1.5$ , as well as Lemma B.1.

Similar steps yield the following bound for the terms involving  $T_{3,5}, T_{3,6}$ ,

$$\begin{aligned} &\frac{1}{t} |\mathbb{E}[T_{3,5}]| \vee |\mathbb{E}[T_{3,6}]| \\ &\leq 4\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,2}^{1/2} \frac{1}{\sqrt{t}} e^{t2\theta_{2,2}/3} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2. \end{aligned}$$

For the terms  $T_{3,7}, T_{3,8}, T_{3,9}, T_{3,10}, T_{3,11}$ , we again invoke the Bismut-Elworthy-Li-type formula as given in (C.9) together with the Itô isometry and obtain

$$\begin{aligned} \mathbb{E}[T_{3,7}] &= \mathbb{E}\left[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x)[V_s^{x,u}, V_s^{x,v}, V_s^{x,v}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds\right], \\ \mathbb{E}[T_{3,8}] &= \mathbb{E}\left[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x)[U_s^{x,vu}, V_s^{x,w}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds\right], \\ \mathbb{E}[T_{3,9}] &= \mathbb{E}\left[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x)[U_s^{x,uw}, V_s^{x,v}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds\right], \\ \mathbb{E}[T_{3,10}] &= \mathbb{E}\left[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x)[U_s^{x,uw}, V_s^{x,v}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds\right], \\ \mathbb{E}[T_{3,11}] &= \mathbb{E}\left[\int_0^t \langle W_s^{x,vuw}, \nabla(P_{t-s}f)(Z_s^x) \rangle ds\right]. \end{aligned} \tag{C.19}$$

In the above expression, we use the chain rules given in Lemma F.4, and for the first term in (C.19) we obtain

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{3,7}]| &\leq \mu_0(\sigma^{-1})\{2\mu_1(\sigma)^2\mu_0(\sigma^{-1}) + \mu_2(\sigma)\} \\ &\quad \times \frac{1}{t} \int_0^t \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^4]^{1/4} \mathbb{E}[\|V_s^{x,u}\|_2^4]^{1/4} \mathbb{E}[\|V_s^{x,v}\|_2^4]^{1/4} \mathbb{E}[\|V_s^{x,v}\|_2^4]^{1/4} ds, \\ &\leq 4\mu_0(\sigma^{-1})\{2\mu_1(\sigma)^2\mu_0(\sigma^{-1}) + \mu_2(\sigma)\} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \\ &\quad \times \frac{1}{t} \int_0^t \tilde{\mu}_{1,n}(P_{t-s}f) e^{s3\theta_{1,4}/4} ds (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2, \\ &\leq 4\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\{2\mu_1(\sigma)^2\mu_0(\sigma^{-1}) + \mu_2(\sigma)\} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \\ &\quad \times \varrho_1(0)\omega_r(t) e^{t3\theta_{1,4}/4} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2, \end{aligned}$$

where we used Lemma C.4 and that  $\varrho_1(t)$  is non-increasing, and that  $\omega_r(t)$  is non-decreasing.

Similar steps yield the following bound for the terms involving  $T_{3,8}, T_{3,9}, T_{3,10}$ ,

$$\begin{aligned} &\frac{1}{t} |\mathbb{E}[T_{3,8}]| \vee |\mathbb{E}[T_{3,9}]| \vee |\mathbb{E}[T_{3,10}]| \\ &\leq 4\tilde{\mu}_{1,n}(f)\mu_1(\sigma)\mu_0(\sigma^{-1})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \varrho_1(0)\gamma_{2,2}^{1/2}\omega_r(t) e^{t2\theta_{2,2}/3} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2. \end{aligned}$$

Finally, for the last term we have

$$\frac{1}{t} |\mathbb{E}[T_{3,11}]| \leq 4\tilde{\mu}_{1,n}(f)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \varrho_1(0)\gamma_{3,2}^{1/2}\omega_r(t) e^{t\theta_{3,2}/2} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2.$$

Combining all the bounds on terms  $T_{3,i}$ , we obtain

$$\begin{aligned} &\nabla^3(P_t f)(x)[u, v, w] \\ &\leq 2\tilde{\pi}_{2,n}(f)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t\theta_{1,6}/2} \|u\|_2 \|w\|_2 \|v\|_2 \\ &\quad + 4\tilde{\mu}_{1,n}(f)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \left\{ \mu_0(\sigma^{-1}) \gamma_{2,3}^{1/3} \frac{1}{\sqrt{t}} e^{t\theta_{2,3}/2} + 2\mu_1(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t3\theta_{1,4}/4} \right. \\ &\quad \left. + 2\mu_0(\sigma^{-1}) \gamma_{2,2}^{1/2} \frac{1}{\sqrt{t}} e^{t2\theta_{2,2}/3} + \mu_0(\sigma) \mu_2(\sigma^{-1}) \varrho_1(0) \omega_r(t) e^{t3\theta_{1,4}/4} \right. \\ &\quad \left. + 3\mu_1(\sigma^{-1}) \mu_0(\sigma) \varrho_1(0) \gamma_{2,2}^{1/2} \omega_r(t) e^{t2\theta_{2,2}/3} + \varrho_1(0) \gamma_{3,2}^{1/2} \omega_r(t) e^{t\theta_{3,2}/2} \right\} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2, \\ &\leq 2\tilde{\pi}_{2,n}(f)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t\theta_{3,4}/4} \|u\|_2 \|w\|_2 \|v\|_2 \\ &\quad + 4\tilde{\mu}_{1,n}(f)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \left\{ \frac{1}{\sqrt{t}} \bar{\xi}_{3,1} + \bar{\xi}_{3,2} \omega_r(t) \right\} e^{t\theta_{3,4}/4} (1 + \|x\|_2^n) \|u\|_2 \|w\|_2 \|v\|_2, \end{aligned}$$

where

$$\begin{aligned}\bar{\xi}_{3,1} &= \mu_0(\sigma^{-1})\gamma_{2,3}^{1/3} + 2\mu_1(\sigma^{-1}) + 2\mu_0(\sigma^{-1})\gamma_{2,2}^{1/2} \\ \bar{\xi}_{3,2} &= [\mu_0(\sigma^{-1})\{2\mu_1(\sigma)^2\mu_0(\sigma^{-1}) + \mu_2(\sigma)\} + 3\mu_0(\sigma^{-1})\mu_1(\sigma)\gamma_{2,2}^{1/2} + \gamma_{3,2}^{1/2}]\varrho_1(0)\end{aligned}$$

The last inequality is obtained by bounding each of the exponential terms with  $e^{t\theta_{3,4}/4}$ , and rearranging.

Once again using the Markov property of the semigroup, we write for any  $r \in [0, t]$ ,

$$\begin{aligned}\nabla^3(P_t f)(x)[u, v, w] &= \nabla^3(P_r[P_{t-r}f])(x)[u, v, w] \\ &\leq \left| 2\tilde{\pi}_{2,n}(P_{t-r}f)\mu_0(\sigma^{-1})\frac{1}{\sqrt{r}} + 4\tilde{\mu}_{1,n}(P_{t-r}f)\left\{\frac{1}{\sqrt{r}}\bar{\xi}_{3,1} + \bar{\xi}_{3,2}\omega_r(r)\right\}\right| e^{r\theta_{3,4}/4} \\ &\quad \times \{1 + (\beta_{r,6n}/\alpha)^{1/6}\}(1 + \|x\|_2^n)\|u\|_2\|w\|_2\|v\|_2.\end{aligned}$$

Using the results of previous section – specifically (C.16) and (C.4) – we obtain for any  $r \in (0, t]$  and  $s \in (0, t - r]$ ,

$$\begin{aligned}\nabla^3(P_t f)(x)[u, v, w] &\leq \left(2\mu_0(\sigma^{-1})\xi_2(s)\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{r}}\bar{\xi}_{3,1} + \bar{\xi}_{3,2}\omega_r(r)\right)\varrho_1(t - r - s)\omega_r(t - r) \\ &\quad \times 4\tilde{\mu}_{1,n}(f)e^{r\theta_{3,4}/4}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^2(1 + \|x\|_2^n)\|u\|_2\|w\|_2\|v\|_2.\end{aligned}$$

Next, we plug in the definition of  $\xi_2(s)$  (C.15) from the previous section, and assuming  $t \geq 2$ , and choosing  $s = r = 1$ , we obtain

$$\begin{aligned}\nabla^3(P_t f)(x)[u, v, w] &\leq \left| 2\xi_2\mu_0(\sigma^{-1}) + 4\tilde{\mu}_{1,n}(f)\{\bar{\xi}_{3,1} + \bar{\xi}_{3,2}\omega_r(1)\} \right| \\ &\quad \times \varrho_1(t - 2)\omega_r(t - 1)e^{\theta_{3,4}/4}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}(1 + \|x\|_2^n)\|u\|_2\|w\|_2\|v\|_2 \\ &\leq 4\tilde{\mu}_{1,n}(f)\varrho_1(0)\omega_r(1)\nu_1(\sigma)\nu_{1:2}(\sigma)\nu_0(\sigma^{-1})\nu_{0:1}(\sigma^{-1})e^{\theta_{3,4}/2}(7 + 7\gamma_{2,2}^{1/2} + \gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2}) \\ &\quad \times \varrho_1(t - 2)\omega_r(t - 1)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^2(1 + \|x\|_2^n)\|u\|_2\|w\|_2\|v\|_2.\end{aligned}$$

Therefore, we obtain the following polynomial growth on the third derivative of the semigroup when  $t \geq 2$ ,

$$\begin{aligned}\tilde{\pi}_{3,n}(P_t f) &\leq \xi_3\varrho_1(t - 2)\omega_r(t - 1) \quad \text{where} \tag{C.20} \\ \xi_3 &= 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)\nu_{1:2}(\sigma)\nu_0(\sigma^{-1})\nu_{0:1}(\sigma^{-1})\varrho_1(0)\omega_r(1)e^{\theta_{3,4}/2} \\ &\quad \times (7 + 7\gamma_{2,2}^{1/2} + \gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^2.\end{aligned}$$

The technique used in the previous section does not yield a converging integral for bounding  $\|\nabla^3 u_f\|_{\text{op}}$ . Therefore, we split the problem into two cases and deal with them separately. The case  $t \geq 2$  is dealt by the above argument. For the case  $t < 2$ , we use (C.2) and Lemma C.3 and the polynomial growth assumption on the derivatives of  $f$  to write

$$\begin{aligned}\nabla^3(P_t f)(x)[v, u, w] &\leq \mathbb{E}[\|\nabla^3 f(Z_t^x)\|_{\text{op}}^4] \mathbb{E}[\|V_t^{x,v}\|_2^4] \mathbb{E}[\|V_t^{x,u}\|_2^4] \mathbb{E}[\|V_t^{x,w}\|_2^4]^{1/4} \\ &\quad + \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_{\text{op}}^3] \mathbb{E}[\|U_t^{x,uv}\|_2^3] \mathbb{E}[\|V_t^{x,v}\|_2^3]^{1/3} \\ &\quad + \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_{\text{op}}^3] \mathbb{E}[\|V_t^{x,u}\|_2^3] \mathbb{E}[\|U_t^{x,vw}\|_2^3]^{1/3} \\ &\quad + \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_{\text{op}}^3] \mathbb{E}[\|V_t^{x,w}\|_2^3] \mathbb{E}[\|U_t^{x,vu}\|_2^3]^{1/3} \\ &\quad + \mathbb{E}[\|\nabla f(Z_t^x)\|_2^2] \mathbb{E}[\|W_t^{x,vuw}\|_2^2]^{1/2}, \\ &\leq 2\tilde{\pi}_{1:3,n}(f)(1 + 3\gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2})e^{t\theta_{3,4}/4}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}(1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2.\end{aligned}$$

Combining this with (C.20), we obtain

$$\begin{aligned}\nabla^3 u_f(x)[v, u, w] &= \int_0^2 \nabla^3(P_t f)(x)[v, u, w]dt + \int_2^\infty \nabla^3(P_t f)(x)[v, u, w]dt \\ &\leq \left[ 4\tilde{\pi}_{1:3,n}(f)(1 + 3\gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2})e^{\theta_{3,4}/2}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \right. \\ &\quad \left. + \xi_3 \int_2^\infty \varrho_1(t - 2)\omega_r(t - 1)dt \right] (1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \|\nabla^3 u_f(x)\|_{\text{op}} &\leq \zeta_3(1 + \|x\|_2^n) \quad \text{where} \\ \zeta_3 &= 4\tilde{\pi}_{1:3,n}(f)(1 + 3\gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\} + \xi_3 \int_0^\infty \varrho_1(t)\omega_r(t+1)dt \quad \text{where} \\ \xi_3 &= 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)\nu_{1:2}(\sigma)\nu_0(\sigma^{-1})\nu_{0:1}(\sigma^{-1})\varrho_1(0)\omega_r(1)e^{\theta_{3,4}/2} \\ &\quad \times (7 + 7\gamma_{2,2}^{1/2} + \gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^2. \end{aligned}$$

#### C.4 Fourth derivative of the semigroup

We differentiate (C.17) and obtain that

$$\begin{aligned} \nabla^4(P_t f)(x)[u, v, w, y] &= \frac{1}{t} \sum_j \mathbb{E}[T_{4,j}] \quad \text{where} \tag{C.21} \\ T_{4,1} &= \nabla^3 f(Z_t^x)[V_t^{x,u}, V_t^{x,w}, V_t^{x,y}] \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\ T_{4,2} &= \nabla^2 f(Z_t^x)[V_t^{x,u}, V_t^{x,w}] \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vy}, dB_s \rangle, \\ T_{4,3} &= \nabla^2 f(Z_t^x)[V_t^{x,w}, V_t^{x,y}] \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vu}, dB_s \rangle, \\ T_{4,4} &= \nabla^2 f(Z_t^x)[V_t^{x,u}, V_t^{x,y}] \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vw}, dB_s \rangle, \\ T_{4,5} &= \nabla^2 f(Z_t^x)[V_t^{x,u}, V_t^{x,w}] \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_t^{x,y}], dB_s \rangle, \\ T_{4,6} &= \nabla^2 f(Z_t^x)[V_t^{x,u}, V_t^{x,y}] \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_s^{x,w}], dB_s \rangle, \\ T_{4,7} &= \nabla^2 f(Z_t^x)[V_t^{x,w}, V_t^{x,y}] \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}], dB_s \rangle, \\ T_{4,8} &= \nabla^2 f(Z_t^x)[U_t^{x,uy}, V_t^{x,w}] \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\ T_{4,9} &= \nabla^2 f(Z_t^x)[U_t^{x,wy}, V_t^{x,u}] \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\ T_{4,10} &= \nabla^2 f(Z_t^x)[U_t^{x,uw}, V_t^{x,y}] \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\ T_{4,11} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_s^{x,w}, V_t^{x,y}], dB_s \rangle, \\ T_{4,12} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}, V_t^{x,y}], dB_s \rangle, \\ T_{4,13} &= \langle \nabla f(Z_t^x), V_t^{x,y} \rangle \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}, V_s^{x,w}], dB_s \rangle, \\ T_{4,14} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vy}, V_s^{x,w}], dB_s \rangle, \\ T_{4,15} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,wy}, V_s^{x,v}], dB_s \rangle, \\ T_{4,16} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vw}, V_t^{x,y}], dB_s \rangle, \\ T_{4,17} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uy}, V_s^{x,v}], dB_s \rangle, \\ T_{4,18} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vy}, V_s^{x,u}], dB_s \rangle, \\ T_{4,19} &= \langle \nabla f(Z_t^x), V_t^{x,y} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uw}, V_s^{x,v}], dB_s \rangle, \\ T_{4,20} &= \langle \nabla f(Z_t^x), V_t^{x,y} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vw}, V_s^{x,u}], dB_s \rangle, \\ T_{4,21} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vu}, V_s^{x,y}], dB_s \rangle, \\ T_{4,22} &= \langle \nabla f(Z_t^x), V_t^{x,y} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vu}, V_s^{x,w}], dB_s \rangle, \\ T_{4,23} &= \langle \nabla f(Z_t^x), U_t^{x,uw} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_s^{x,y}], dB_s \rangle, \\ T_{4,24} &= \langle \nabla f(Z_t^x), U_t^{x,uy} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_s^{x,w}], dB_s \rangle, \\ T_{4,25} &= \langle \nabla f(Z_t^x), U_t^{x,wy} \rangle \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}], dB_s \rangle, \\ T_{4,26} &= \langle \nabla f(Z_t^x), U_t^{x,wy} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vu}, dB_s \rangle, \\ T_{4,27} &= \langle \nabla f(Z_t^x), U_t^{x,uy} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vw}, dB_s \rangle, \\ T_{4,28} &= \langle \nabla f(Z_t^x), U_t^{x,uw} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) U_s^{x,vy}, dB_s \rangle, \\ T_{4,29} &= \langle \nabla f(Z_t^x), V_t^{x,u} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) W_s^{x,vwy}, dB_s \rangle, \\ T_{4,30} &= \langle \nabla f(Z_t^x), V_t^{x,y} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) W_s^{x,vuw}, dB_s \rangle, \\ T_{4,31} &= \langle \nabla f(Z_t^x), V_t^{x,w} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) W_s^{x,vuy}, dB_s \rangle, \end{aligned}$$

$$\begin{aligned}
T_{4,32} &= \langle \nabla f(Z_t^x), W_t^{x,uvw} \rangle \int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle, \\
T_{4,33} &= f(Z_t^x) \int_0^t \langle \nabla^3 \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}, V_s^{x,w}, V_t^{x,y}], dB_s \rangle, \\
T_{4,34} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,uy}, V_s^{x,v}, V_s^{x,w}], dB_s \rangle, \\
T_{4,35} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,vy}, V_s^{x,u}, V_s^{x,w}], dB_s \rangle, \\
T_{4,36} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,wy}, V_s^{x,u}, V_s^{x,v}], dB_s \rangle, \\
T_{4,37} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,uw}, V_s^{x,v}, V_s^{x,y}], dB_s \rangle, \\
T_{4,38} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,vw}, V_s^{x,u}, V_s^{x,y}], dB_s \rangle, \\
T_{4,39} &= f(Z_t^x) \int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,vu}, V_s^{x,w}, V_s^{x,y}], dB_s \rangle, \\
T_{4,40} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uw}, U_s^{x,vy}], dB_s \rangle, \\
T_{4,41} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uy}, U_s^{x,vw}], dB_s \rangle, \\
T_{4,42} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vu}, U_s^{x,wy}], dB_s \rangle, \\
T_{4,43} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,uvw}, V_s^{x,v}], dB_s \rangle, \\
T_{4,44} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,vwy}, V_s^{x,u}], dB_s \rangle, \\
T_{4,45} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,vuy}, V_s^{x,w}], dB_s \rangle, \\
T_{4,46} &= f(Z_t^x) \int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,vuw}, V_s^{x,y}], dB_s \rangle, \\
T_{4,47} &= f(Z_t^x) \int_0^t \langle \sigma^{-1}(Z_s^x) Y_s^{x,vuwy}, dB_s \rangle.
\end{aligned}$$

We will bound each of the terms contributing to the summation in (C.21) in turn. For the first term above, we have

$$\begin{aligned}
\frac{1}{t} |\mathbb{E}[T_{4,1}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla^3 f(Z_t^x) [V_t^{x,u}, V_t^{x,w}, V_t^{x,y}]\|_2^2]^{1/2} \mathbb{E}\left[\left|\int_0^t \|\sigma^{-1}(Z_s^x) V_s^{x,v}\|_2^2 ds\right|^{1/2}\right], \\
&\leq \frac{1}{t} \mathbb{E}[\|\nabla^3 f(Z_t^x)\|_{\text{op}}^2 \|V_t^{x,u}\|_2^2 \|V_t^{x,w}\|_2^2 \|V_t^{x,y}\|_2^2]^{1/2} \mu_0(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 ds\right]^{1/2}, \\
&\leq \frac{1}{t} \mu_0(\sigma^{-1}) \mathbb{E}[\|\nabla^3 f(Z_t^x)\|_{\text{op}}^6]^{1/6} \left|\mathbb{E}[\|V_t^{x,u}\|_2^9] \mathbb{E}[\|V_t^{x,w}\|_2^9] \mathbb{E}[\|V_t^{x,y}\|_2^9]\right|^{1/9} \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 ds\right]^{1/2}, \\
&\leq 2\tilde{\pi}_{3,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t2\theta_{1,9}/3} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2,
\end{aligned}$$

where the second step follows from Cauchy-Schwartz, the third step follows from the Itô isometry, and the last step follows from Lemma C.3 for  $l = 6$ ,  $m = 1$ , and Lemma B.1.

For the term involving  $T_{4,2}$ , we write

$$\begin{aligned}
\frac{1}{t} |\mathbb{E}[T_{4,2}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x) [V_t^{x,u}, V_t^{x,w}]\|_2^2]^{1/2} \mathbb{E}\left[\int_0^t \|\sigma^{-1}(Z_s^x) U_s^{x,vy}\|_2^2 ds\right]^{1/2}, \\
&\leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^6]^{1/6} \mathbb{E}[\|V_t^{x,u}\|_2^6]^{1/6} \mathbb{E}[\|V_t^{x,w}\|_2^6]^{1/6} \mu_0(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|U_s^{x,vy}\|_2^2 ds\right]^{1/2}, \\
&\leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \gamma_{2,2}^{1/2} \frac{1}{\sqrt{t}} e^{t7\theta_{2,2}/6} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2,
\end{aligned}$$

where the above steps follow from Cauchy-Schwartz, the Itô isometry, Lemma C.4 with  $l = 6$ ,  $m = 1$ , and Lemma B.1.

Similarly for the terms involving  $T_{4,3}$ ,  $T_{4,4}$ , we obtain the same bound, i.e.,

$$\begin{aligned}
&\frac{1}{t} |\mathbb{E}[T_{4,3}]| \vee |\mathbb{E}[T_{4,4}]| \\
&\leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \gamma_{2,2}^{1/2} \frac{1}{\sqrt{t}} e^{t7\theta_{2,2}/6} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

By the same steps above, for the term involving  $T_{4,5}$ , we obtain

$$\begin{aligned}
&\frac{1}{t} |\mathbb{E}[T_{4,5}]| \\
&\leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^2 \|V_t^{x,u}\|_2^2 \|V_t^{x,w}\|_2^2]^{1/2} \mu_1(\sigma^{-1}) \mathbb{E}\left[\left|\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_t^{x,y}], dB_s \rangle\right|^2\right]^{1/2}, \\
&\leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^6] \mathbb{E}[\|V_t^{x,u}\|_2^6] \mathbb{E}[\|V_t^{x,w}\|_2^6]^{1/6} \mu_1(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 \|V_t^{x,y}\|_2^2 ds\right]^{1/2}, \\
&\leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_1(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t2\theta_{1,6}/3} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

The above steps applied to the terms  $T_{4,6}$  and  $T_{4,7}$  yield the following bounds,

$$\begin{aligned} & \frac{1}{t} |\mathbb{E}[T_{4,6}]| \vee |\mathbb{E}[T_{4,7}]| \\ & \leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_1(\sigma^{-1}) \frac{1}{\sqrt{t}} e^{t\theta_{1,6}/3} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term involving  $T_{4,8}$ , we have

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,8}]| & \leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^2 \|U_t^{x,uy}\|_2^2 \|V_t^{x,w}\|_2^2]^{1/2} \mathbb{E}\left[\left|\int_0^t \langle \sigma^{-1}(Z_s^x) V_s^{x,v}, dB_s \rangle\right|^2\right]^{1/2}, \\ & \leq \frac{1}{t} \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^6]^{1/6} \mathbb{E}[\|U_t^{x,uy}\|_2^6]^{1/6} \mathbb{E}[\|V_t^{x,w}\|_2^6]^{1/6} \mu_0(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 ds\right]^{1/2}, \\ & \leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \gamma_{2,6}^{1/6} \frac{1}{\sqrt{t}} e^{t\theta_{2,6}/3} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

The above steps applied to the terms  $T_{4,9}$  and  $T_{4,10}$  yield the same bounds,

$$\begin{aligned} & \frac{1}{t} |\mathbb{E}[T_{4,9}]| \vee |\mathbb{E}[T_{4,10}]| \\ & \leq 2\tilde{\pi}_{2,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n) \mu_0(\sigma^{-1}) \gamma_{2,6}^{1/6} \frac{1}{\sqrt{t}} e^{t\theta_{2,6}/3} \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term involving  $T_{4,11}$ , we write

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,11}]| & \leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^2 \|V_t^{x,u}\|_2^2]^{1/2} \mathbb{E}\left[\left|\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [V_s^{x,v}, V_s^{x,w}, V_s^{x,y}], dB_s \rangle\right|^2\right]^{1/2}, \\ & \leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \mathbb{E}[\|V_t^{x,u}\|_2^4]^{1/4} \mu_2(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 \|V_s^{x,w}\|_2^2 \|V_s^{x,y}\|_2^2 ds\right]^{1/2}, \\ & \leq 4\tilde{\mu}_{1,n}(f) \mu_2(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \frac{1}{\sqrt{t}} e^{t\theta_{1,6}/3} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2, \end{aligned}$$

where the last step follows from Lemma C.4 with  $l = 4, m = 3/2$ .

The above steps applied to the terms  $T_{4,12}$  and  $T_{4,13}$  yield the same bounds, i.e.,

$$\begin{aligned} & \frac{1}{t} |\mathbb{E}[T_{4,12}]| \vee |\mathbb{E}[T_{4,13}]| \\ & \leq 4\tilde{\mu}_{1,n}(f) \mu_2(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \frac{1}{\sqrt{t}} e^{t\theta_{1,6}/3} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term involving  $T_{4,14}$ , we have

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,14}]| & \leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \mathbb{E}[\|V_t^{x,u}\|_2^4]^{1/4} \mu_1(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|U_s^{x,vy}\|_2^2 \|V_s^{x,w}\|_2^2 ds\right]^{1/2}, \\ & \leq 4\tilde{\mu}_{1,n}(f) \mu_1(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,4}^{1/4} \frac{1}{\sqrt{t}} e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2, \end{aligned}$$

where the last step follows from Lemma C.4 with  $l = 4, m = 3/2$ .

The above steps applied to the terms from  $T_{4,15}$  to  $T_{4,22}$  yield the following bounds,

$$\begin{aligned} & \frac{1}{t} |\mathbb{E}[T_{4,15}]| \vee |\mathbb{E}[T_{4,16}]| \vee \dots \vee |\mathbb{E}[T_{4,22}]| \\ & \leq 4\tilde{\mu}_{1,n}(f) \mu_1(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,4}^{1/4} \frac{1}{\sqrt{t}} e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term involving  $T_{4,23}$ , we write

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,23}]| & \leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \mathbb{E}[\|U_t^{x,u}\|_2^4]^{1/4} \mu_1(\sigma^{-1}) \mathbb{E}\left[\int_0^t \|V_s^{x,v}\|_2^2 \|V_s^{x,y}\|_2^2 ds\right]^{1/2}, \\ & \leq 4\tilde{\mu}_{1,n}(f) \mu_1(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,4}^{1/4} \frac{1}{\sqrt{t}} e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2, \end{aligned}$$

where the last step follows from Lemma C.4 with  $l = 4, m = 3/2$ .

Similarly for the terms  $T_{4,24}$  and  $T_{4,25}$ , we obtain

$$\begin{aligned} & \frac{1}{t} |\mathbb{E}[T_{4,24}]| \vee |\mathbb{E}[T_{4,25}]| \\ & \leq 4\tilde{\mu}_{1,n}(f) \mu_1(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,4}^{1/4} \frac{1}{\sqrt{t}} e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term involving  $T_{4,26}$ , we write

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,26}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \mathbb{E}[\|U_t^{x,wy}\|_2^4]^{1/4} \mu_0(\sigma^{-1}) \mathbb{E}[\int_0^t \|U_s^{x,vu}\|_2^2 ds]^{1/2}, \\ &\leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,4}^{1/2} \frac{1}{\sqrt{t}} e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

Similarly, for the terms  $T_{4,27}$  to  $T_{4,28}$ , we have the same bounds,

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,27}]| \vee |\mathbb{E}[T_{4,28}]| \\ \leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{2,4}^{1/2} \frac{1}{\sqrt{t}} e^{t\theta_{2,4}/4} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term involving  $T_{4,29}$ , we write

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,29}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^4]^{1/4} \mathbb{E}[\|V_t^{x,u}\|_2^4]^{1/4} \mu_0(\sigma^{-1}) \mathbb{E}[\int_0^t \|W_s^{x,vwy}\|_2^2 ds]^{1/2}, \\ &\leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{3,2}^{1/2} \frac{1}{\sqrt{t}} e^{t\theta_{3,2}/3} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

The above steps applied to the terms  $T_{4,30}$  to  $T_{4,31}$  yield the following bounds,

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,30}]| \vee |\mathbb{E}[T_{4,31}]| \\ \leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{3,2}^{1/2} \frac{1}{\sqrt{t}} e^{t\theta_{3,2}/3} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the term  $T_{4,32}$  we write

$$\begin{aligned} \frac{1}{t} |\mathbb{E}[T_{4,32}]| &\leq \frac{1}{t} \mathbb{E}[\|\nabla f(Z_t^x)\|_2^6]^{1/6} \mathbb{E}[\|W_t^{x,uwy}\|_2^3]^{1/3} \mu_0(\sigma^{-1}) \mathbb{E}[\int_0^t \|V_s^{x,v}\|_2^2 ds]^{1/2}, \\ &\leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \gamma_{3,3}^{1/3} \frac{1}{\sqrt{t}} e^{t\theta_{3,3}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2. \end{aligned}$$

For the terms from  $T_{3,33}$ , to  $T_{3,47}$ , we again invoke the Bismut-Elworthy-Li type identity as given in (C.9) together with the Itô isometry and obtain

$$\begin{aligned} \mathbb{E}[T_{4,33}] &= \mathbb{E}[\int_0^t \langle \nabla^3 \sigma^{-1}(Z_s^x) [V_s^{x,u}, V_s^{x,v}, V_s^{x,w}, V_s^{x,y}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,34}] &= \mathbb{E}[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,uy}, V_s^{x,v}, V_s^{x,w}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,35}] &= \mathbb{E}[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,vy}, V_s^{x,u}, V_s^{x,w}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,36}] &= \mathbb{E}[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,wy}, V_s^{x,u}, V_s^{x,v}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,37}] &= \mathbb{E}[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,uw}, V_s^{x,v}, V_s^{x,y}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,38}] &= \mathbb{E}[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,vw}, V_s^{x,u}, V_s^{x,y}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,39}] &= \mathbb{E}[\int_0^t \langle \nabla^2 \sigma^{-1}(Z_s^x) [U_s^{x,vu}, V_s^{x,w}, V_s^{x,y}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,40}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uw}, U_s^{x,vy}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,41}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,uy}, U_s^{x,vw}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,42}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [U_s^{x,vu}, U_s^{x,wy}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,43}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,uwy}, V_s^{x,v}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,44}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,vwy}, V_s^{x,u}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,45}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,vuy}, V_s^{x,w}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,46}] &= \mathbb{E}[\int_0^t \langle \nabla \sigma^{-1}(Z_s^x) [W_s^{x,vuw}, V_s^{x,y}], \sigma(Z_s^x)^\top \nabla(P_{t-s}f)(Z_s^x) \rangle ds], \\ \mathbb{E}[T_{4,47}] &= \mathbb{E}[\int_0^t \langle Y_s^{x,vuwy}, \nabla(P_{t-s}f)(Z_s^x) \rangle ds]. \end{aligned}$$

We bound each of the above terms in turn. For the term involving  $T_{4,33}$  we write

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,33}]| \leq \frac{1}{t} \mu_0(\sigma^{-1}) \{5\mu_1(\sigma)^3 \mu_0(\sigma^{-1}) \\
& + 6\mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) + \mu_0(\sigma^{-1})^2 \mu_1(\sigma)^2 \mu_2(\sigma) + \mu_3(\sigma)\} \\
& \quad \times \mathbb{E}[\int_0^t \|V_s^{x,u}\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,w}\|_2 \|V_s^{x,y}\|_2 \|\nabla(P_{t-s}f)(Z_s^x)\|_2 ds], \\
& \leq \frac{1}{t} \mu_0(\sigma^{-1}) \{5\mu_1(\sigma)^3 \mu_0(\sigma^{-1}) + 6\mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) + \mu_0(\sigma^{-1})^2 \mu_1(\sigma)^2 \mu_2(\sigma) + \mu_3(\sigma)\} \\
& \quad \times \int_0^t \left| \mathbb{E}[\|V_s^{x,u}\|_2^5] \mathbb{E}[\|V_s^{x,v}\|_2^5] \mathbb{E}[\|V_s^{x,w}\|_2^5] \mathbb{E}[\|V_s^{x,y}\|_2^5] \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^5] \right|^{1/5} ds, \\
& \leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{5\mu_1(\sigma)^3 \mu_0(\sigma^{-1}) + 6\mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) + \mu_0(\sigma^{-1})^2 \mu_1(\sigma)^2 \mu_2(\sigma) + \mu_3(\sigma)\} \\
& \quad \times (1 + (\beta_{r,6n}/\alpha)^{1/6}) (1 + \|x\|_2^n) \frac{1}{t} \int_0^t e^{s\theta_{1,5}} \varrho_1(t-s) \omega_r(t-s) ds \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2, \\
& \leq 4\tilde{\mu}_{1,n}(f) \mu_0(\sigma^{-1}) \{5\mu_1(\sigma)^3 \mu_0(\sigma^{-1}) + 6\mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) + \mu_0(\sigma^{-1})^2 \mu_1(\sigma)^2 \mu_2(\sigma) + \mu_3(\sigma)\} \\
& \quad \times (1 + (\beta_{r,6n}/\alpha)^{1/6}) \varrho_1(0) \omega_r(t) e^{t\theta_{1,5}} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2,
\end{aligned}$$

where we used Lemma C.4 with  $l = 5$  and  $m = 6/5$ .

For the term involving  $T_{4,34}$  we have

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,34}]| \\
& \leq \frac{1}{t} \mu_0(\sigma^{-1}) \{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\} \mathbb{E}[\int_0^t \|U_s^{x,uy}\|_2 \|V_s^{x,v}\|_2 \|V_s^{x,w}\|_2 \|\nabla(P_{t-s}f)(Z_s^x)\|_2 ds], \\
& \leq \frac{1}{t} \mu_0(\sigma^{-1}) \{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\} \\
& \quad \times \int_0^t \mathbb{E}[\|U_s^{x,uy}\|_2^4]^{1/4} \mathbb{E}[\|V_s^{x,v}\|_2^4]^{1/4} \mathbb{E}[\|V_s^{x,w}\|_2^4]^{1/4} \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^4]^{1/4} ds, \\
& \leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) \mu_0(\sigma^{-1}) \{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\} \\
& \quad \times \varrho_1(0) \gamma_{2,4}^{1/4} \omega_r(t) e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

The above steps applied to the terms from  $T_{4,35}$  to  $T_{4,39}$  yield the following bounds,

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,35}]| \vee |\mathbb{E}[T_{4,36}]| \vee \dots \vee |\mathbb{E}[T_{4,39}]| \leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) \\
& \quad \times \mu_0(\sigma^{-1}) \{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\} \varrho_1(0) \gamma_{2,4}^{1/4} \omega_r(t) e^{t\theta_{2,4}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

For the term involving  $T_{4,40}$  we write

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,40}]| \leq \frac{1}{t} \mu_1(\sigma) \mu_0(\sigma^{-1}) \mathbb{E}[\int_0^t \|U_s^{x,uw}\|_2 \|U_s^{x,vy}\|_2 \|\nabla(P_{t-s}f)(Z_s^x)\|_2 ds], \\
& \leq \frac{1}{t} \mu_1(\sigma) \mu_0(\sigma^{-1}) \int_0^t \mathbb{E}[\|U_s^{x,uw}\|_2^3]^{1/3} \mathbb{E}[\|U_s^{x,vy}\|_2^3]^{1/3} \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^3]^{1/3} ds, \\
& \leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) \mu_1(\sigma) \mu_0(\sigma^{-1}) \varrho_1(0) \gamma_{2,3}^{2/3} \omega_r(t) e^{t\theta_{2,3}/3} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

Similar reasoning as above yields the same bound for the terms  $T_{4,41}$  and  $T_{4,42}$ ,

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,41}]| \vee |\mathbb{E}[T_{4,42}]| \\
& \leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) \mu_1(\sigma) \mu_0(\sigma^{-1}) \varrho_1(0) \gamma_{2,3}^{2/3} \omega_r(t) e^{t\theta_{2,3}/3} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

For the term involving  $T_{4,43}$  we write

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,43}]| \leq \frac{1}{t} \mu_1(\sigma^{-1}) \mu_0(\sigma) \mathbb{E}[\int_0^t \|W_s^{x,uwy}\|_2 \|V_s^{x,v}\|_2 \|\nabla(P_{t-s}f)(Z_s^x)\|_2 ds], \\
& \leq \frac{1}{t} \mu_1(\sigma) \mu_0(\sigma^{-1}) \int_0^t \mathbb{E}[\|W_s^{x,uwy}\|_2^3]^{1/3} \mathbb{E}[\|V_s^{x,v}\|_2^3]^{1/3} \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^3]^{1/3} ds, \\
& \leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) \mu_1(\sigma) \mu_0(\sigma^{-1}) \gamma_{3,3}^{1/3} \varrho_1(0) \omega_r(t) e^{t\theta_{3,3}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2,
\end{aligned}$$

The same reasoning yields the same bound for the terms  $T_{4,44}$ ,  $T_{4,45}$ , and  $T_{4,46}$ ,

$$\begin{aligned}
& \frac{1}{t} |\mathbb{E}[T_{4,44}]| \vee |\mathbb{E}[T_{4,45}]| \vee |\mathbb{E}[T_{4,46}]| \\
& \leq 4\tilde{\mu}_{1,n}(f) (1 + (\beta_{r,6n}/\alpha)^{1/6}) \mu_1(\sigma) \mu_0(\sigma^{-1}) \gamma_{3,3}^{1/3} \varrho_1(0) \omega_r(t) e^{t\theta_{3,3}/2} (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$



Finally, for the term involving  $T_{4,47}$  we write

$$\begin{aligned} \frac{1}{t} \mathbb{E}[T_{4,47}] &\leq \frac{1}{t} \int_0^t \mathbb{E}[\|Y_s^{x,vuwy}\|_2^2]^{1/2} \mathbb{E}[\|\nabla(P_{t-s}f)(Z_s^x)\|_2^2]^{1/2} ds, \\ &\leq 4\tilde{\mu}_{1,n}(f)(1 + (\beta_{r,6n}/\alpha)^{1/6})\gamma_{4,2}^{1/2}\varrho_1(0)\omega_r(t)e^{t\theta_{4,2}/2}(1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2\|y\|_2. \end{aligned}$$

Combining these bounds, we obtain

$$\begin{aligned} \nabla^4(P_t f)(x)[u, v, w, y] &= \frac{1}{t} \sum_j \mathbb{E}[T_{4,j}], \\ &\leq \left\{ 2\tilde{\pi}_{3,n}(f)\mu_0(\sigma^{-1})\frac{1}{\sqrt{t}} + 6\tilde{\pi}_{2,n}(f)\mu_0(\sigma^{-1})\gamma_{2,2}^{1/2}\frac{1}{\sqrt{t}} + 6\tilde{\pi}_{2,n}(f)\mu_1(\sigma^{-1})\frac{1}{\sqrt{t}} \right. \\ &\quad + 6\tilde{\pi}_{2,n}(f)\mu_0(\sigma^{-1})\gamma_{2,6}^{1/6}\frac{1}{\sqrt{t}} + 12\tilde{\mu}_{1,n}(f)\mu_2(\sigma^{-1})\frac{1}{\sqrt{t}} + 24\tilde{\mu}_{1,n}(f)\mu_1(\sigma^{-1})\gamma_{2,4}^{1/4}\frac{1}{\sqrt{t}} \\ &\quad + 12\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\gamma_{2,4}^{1/2}\frac{1}{\sqrt{t}} + 12\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\gamma_{3,2}^{1/2}\frac{1}{\sqrt{t}} + 4\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\gamma_{3,3}^{1/3}\frac{1}{\sqrt{t}} \\ &\quad + 4\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\{5\mu_1(\sigma)^3\mu_0(\sigma^{-1}) \\ &\quad + \mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma)[6 + \mu_0(\sigma^{-1})\mu_1(\sigma)] + \mu_3(\sigma)\}\varrho_1(0)\omega_r(t) \\ &\quad + 24\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\}\varrho_1(0)\gamma_{2,4}^{1/4}\omega_r(t) \\ &\quad + 12\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\mu_1(\sigma)\varrho_1(0)\gamma_{2,3}^{2/3}\omega_r(t) + 16\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\mu_1(\sigma)\gamma_{3,3}^{1/3}\varrho_1(0)\omega_r(t) \\ &\quad \left. + 4\tilde{\mu}_{1,n}(f)\gamma_{4,2}^{1/2}\varrho_1(0)\omega_r(t) \right\} e^{t2\theta_{4,2}/3} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2\|y\|_2. \end{aligned}$$

Using once again the Markov property of the semigroup, we obtain for any  $\tau \in (0, t]$

$$\begin{aligned} \nabla^4(P_t f)(x)[u, v, w, y] &= \nabla^4(P_\tau[P_{t-\tau}f])(x)[u, v, w, y] \\ &\leq \left\{ 2\tilde{\pi}_{3,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\frac{1}{\sqrt{\tau}} + 6\tilde{\pi}_{2,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\gamma_{2,2}^{1/2}\frac{1}{\sqrt{\tau}} + 6\tilde{\pi}_{2,n}(P_{t-\tau}f)\mu_1(\sigma^{-1})\frac{1}{\sqrt{\tau}} \right. \\ &\quad + 6\tilde{\pi}_{2,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\gamma_{2,6}^{1/6}\frac{1}{\sqrt{\tau}} + 12\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_2(\sigma^{-1})\frac{1}{\sqrt{\tau}} + 24\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_1(\sigma^{-1})\gamma_{2,4}^{1/4}\frac{1}{\sqrt{\tau}} \\ &\quad + 12\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\gamma_{2,4}^{1/2}\frac{1}{\sqrt{\tau}} + 12\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\gamma_{3,2}^{1/2}\frac{1}{\sqrt{\tau}} + 4\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\gamma_{3,3}^{1/3}\frac{1}{\sqrt{\tau}} \\ &\quad + 4\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\{5\mu_1(\sigma)^3\mu_0(\sigma^{-1}) + \mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) \\ &\quad \times [6 + \mu_0(\sigma^{-1})\mu_1(\sigma)] + \mu_3(\sigma)\}\varrho_1(0)\omega_r(\tau) \\ &\quad + 24\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\}\varrho_1(0)\gamma_{2,4}^{1/4}\omega_r(\tau) \\ &\quad + 12\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\mu_1(\sigma)\varrho_1(0)\gamma_{2,3}^{2/3}\omega_r(\tau) + 16\tilde{\mu}_{1,n}(P_{t-\tau}f)\mu_0(\sigma^{-1})\mu_1(\sigma)\gamma_{3,3}^{1/3}\varrho_1(0)\omega_r(\tau) \\ &\quad \left. + 4\tilde{\mu}_{1,n}(P_{t-\tau}f)\gamma_{4,2}^{1/2}\varrho_1(0)\omega_r(\tau) \right\} e^{\tau 2\theta_{4,2}/3} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} (1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2\|y\|_2. \end{aligned}$$

Next, we use the results of the previous section – specifically (C.16) and (C.20) – and choose  $\tau = 1$ . For  $t \geq 3$  we write

$$\begin{aligned}
& \nabla^4(P_t f)(x)[u, v, w, y] \\
& \leq \left\{ 2\tilde{\pi}_{3,n}(P_{t-1}f)\mu_0(\sigma^{-1}) + 6\tilde{\pi}_{2,n}(P_{t-1}f)\mu_0(\sigma^{-1})\gamma_{2,2}^{1/2} + 6\tilde{\pi}_{2,n}(P_{t-1}f)\mu_1(\sigma^{-1}) \right. \\
& \quad + 6\tilde{\pi}_{2,n}(P_{t-1}f)\mu_0(\sigma^{-1})\gamma_{2,6}^{1/6} + 12\tilde{\mu}_{1,n}(P_{t-1}f)\mu_2(\sigma^{-1}) + 24\tilde{\mu}_{1,n}(P_{t-1}f)\mu_1(\sigma^{-1})\gamma_{2,4}^{1/4} \\
& \quad + 12\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\gamma_{2,4}^{1/2} + 12\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\gamma_{3,2}^{1/2} + 4\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\gamma_{3,3}^{1/3} \\
& \quad + 4\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\{5\mu_1(\sigma)^3\mu_0(\sigma^{-1}) + \mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) \\
& \quad \times [6 + \mu_0(\sigma^{-1})\mu_1(\sigma)] + \mu_3(\sigma)\}\varrho_1(0)\omega_r(1) \\
& \quad + 24\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\}\varrho_1(0)\gamma_{2,4}^{1/4}\omega_r(1) \\
& \quad + 12\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\mu_1(\sigma)\varrho_1(0)\gamma_{2,3}^{2/3}\omega_r(1) + 16\tilde{\mu}_{1,n}(P_{t-1}f)\mu_0(\sigma^{-1})\mu_1(\sigma)\gamma_{3,3}^{1/3}\varrho_1(0)\omega_r(1) \\
& \quad \left. + 4\tilde{\mu}_{1,n}(P_{t-1}f)\gamma_{4,2}^{1/2}\varrho_1(0)\omega_r(1)\right\}e^{2\theta_{4,2}/3}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}(1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2\|y\|_2, \\
& \leq \varrho_1(t-3)\omega_r(t-1)\left\{ 2\xi_3\mu_0(\sigma^{-1}) + 6\xi_2\mu_0(\sigma^{-1})\gamma_{2,2}^{1/2} + 6\xi_2\mu_1(\sigma^{-1}) \right. \\
& \quad + 6\xi_2\mu_0(\sigma^{-1})\gamma_{2,6}^{1/6} + 12\tilde{\mu}_{1,n}(f)\mu_2(\sigma^{-1}) + 24\tilde{\mu}_{1,n}(f)\mu_1(\sigma^{-1})\gamma_{2,4}^{1/4} \\
& \quad + 12\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\gamma_{2,4}^{1/2} + 12\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\gamma_{3,2}^{1/2} + 4\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\gamma_{3,3}^{1/3} \\
& \quad + 4\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\{5\mu_1(\sigma)^3\mu_0(\sigma^{-1}) + \mu_0(\sigma^{-1})\mu_1(\sigma)\mu_2(\sigma) \\
& \quad \times [6 + \mu_0(\sigma^{-1})\mu_1(\sigma)] + \mu_3(\sigma)\}\varrho_1(0)\omega_r(1) \\
& \quad + 24\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\{2\mu_0(\sigma^{-1})\mu_1(\sigma)^2 + \mu_2(\sigma)\}\varrho_1(0)\gamma_{2,4}^{1/4}\omega_r(1) \\
& \quad + 12\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\mu_1(\sigma)\varrho_1(0)\gamma_{2,3}^{2/3}\omega_r(1) + 16\tilde{\mu}_{1,n}(f)\mu_0(\sigma^{-1})\mu_1(\sigma)\gamma_{3,3}^{1/3}\varrho_1(0)\omega_r(1) \\
& \quad \left. + 4\tilde{\mu}_{1,n}(f)\gamma_{4,2}^{1/2}\varrho_1(0)\omega_r(1)\right\}e^{2\theta_{4,2}/3}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}(1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2\|y\|_2 \\
& \leq 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)^2\nu_{1:3}(\sigma)\nu_0(\sigma^{-1})^2\nu_{0:2}(\sigma^{-1})e^{\theta_{4,2}}\varrho_1(0)\omega_r(1)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^3 \\
& \quad \times \left[ 42 + 32\gamma_{2,2}^{1/2} + 6\gamma_{2,2} + 2\gamma_{2,3}^{1/3} + 3\gamma_{2,3}^{2/3} + 24\gamma_{2,4}^{1/4} + 3\gamma_{2,4}^{1/2} \right. \\
& \quad \left. + 12\gamma_{2,6}^{1/6} + 5\gamma_{3,2}^{1/2} + 5\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2} + 6\gamma_{2,2}^{1/2}\gamma_{2,6}^{1/6} \right] \\
& \quad \times \varrho_1(t-3)\omega_r(t-1)(1 + \|x\|_2^n)\|u\|_2\|v\|_2\|w\|_2\|y\|_2.
\end{aligned}$$

Hence, obtain the following polynomial growth on the fourth derivative of the semigroup for  $t \geq 3$

$$\begin{aligned}
& \tilde{\pi}_{4,n}(P_t f) \leq \xi_4\varrho_1(t-3)\omega_r(t-1), \quad \text{where} \tag{C.22} \\
& \xi_4 = 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)^2\nu_{1:3}(\sigma)\nu_0(\sigma^{-1})^2\nu_{0:2}(\sigma^{-1})e^{\theta_{4,2}}\varrho_1(0)\omega_r(1)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^3 \\
& \quad \times \left[ 42 + 32\gamma_{2,2}^{1/2} + 6\gamma_{2,2} + 2\gamma_{2,3}^{1/3} + 3\gamma_{2,3}^{2/3} + 24\gamma_{2,4}^{1/4} + 3\gamma_{2,4}^{1/2} + 12\gamma_{2,6}^{1/6} \right. \\
& \quad \left. + 5\gamma_{3,2}^{1/2} + 5\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2} + 6\gamma_{2,2}^{1/2}\gamma_{2,6}^{1/6} \right].
\end{aligned}$$

For the case  $t < 3$ , we use (C.3) and a slight abuse of notation to write

$$\begin{aligned}
\nabla^4(P_t f)(x)[v, u, w, y] &\leq |J_1(v, u, w, y)|^{1/5} + |J_2(vy, w, y)|^{1/4} + |J_2(uy, w, y)|^{1/4} \\
&\quad + |J_2(wy, w, y)|^{1/4} + |J_2(uw, w, y)|^{1/4} + |J_2(vw, w, y)|^{1/4} + |J_2(vu, w, y)|^{1/4} \\
&\quad + |J_3(uvw, y)|^{1/3} + |J_3(uvy, w)|^{1/3} + |J_3(uwy, v)|^{1/3} + |J_3(vwy, u)|^{1/3} \\
&\quad + |J_4(uw, vy)|^{1/3} + |J_4(uy, vw)|^{1/3} + |J_4(wy, vu)|^{1/3} + |J_5(vuwy)|^{1/2}, \\
J_1(a, b, c, d) &= \mathbb{E}[\|\nabla^4 f(Z_t^x)\|_{\text{op}}^5] \mathbb{E}[\|V_t^{x,a}\|_2^5] \mathbb{E}[\|V_t^{x,b}\|_2^5] \mathbb{E}[\|V_t^{x,c}\|_2^5] \mathbb{E}[\|V_t^{x,d}\|_2^5], \\
J_2(a, b, c) &= \mathbb{E}[\|\nabla^3 f(Z_t^x)\|_{\text{op}}^4] \mathbb{E}[\|U_t^{x,a}\|_2^4] \mathbb{E}[\|V_t^{x,b}\|_2^4] \mathbb{E}[\|V_t^{x,c}\|_2^4], \\
J_3(a, b) &= \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_2^3] \mathbb{E}[\|W_t^{x,a}\|_2^3] \mathbb{E}[\|V_t^{x,b}\|_2^3], \\
J_4(a, b) &= \mathbb{E}[\|\nabla^2 f(Z_t^x)\|_{\text{op}}^3] \mathbb{E}[\|U_t^{x,a}\|_2^3] \mathbb{E}[\|U_t^{x,b}\|_2^3], \\
J_5(a) &= \mathbb{E}[\|\nabla f(Z_t^x)\|_2^2] \mathbb{E}[\|Y_t^{x,a}\|_2^2].
\end{aligned}$$

Using Lemmas B.1 and C.3, we obtain the following bound for  $t < 3$ ,

$$\begin{aligned}
\nabla^4(P_t f)(x)[v, u, w, y] &\leq 2\tilde{\pi}_{1:4,n}(f) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \\
&\quad \times \left[1 + 6\gamma_{2,4}^{1/4} + 4\gamma_{2,3} + 3\gamma_{2,3}^{2/3} + 4\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2}\right] (1 + \|x\|^n) e^{t\theta_{4,2}/2} \|v\|_2 \|u\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

Combining this with (C.22), we can write

$$\begin{aligned}
\nabla^4 u_f(x)[v, u, w, y] &= \int_0^3 \nabla^4(P_t f)(x)[v, u, w, y] dt + \int_3^\infty \nabla^4(P_t f)(x)[v, u, w, y] dt \\
&\leq \left[6\tilde{\pi}_{1:4,n}(f) \left[1 + 6\gamma_{2,4}^{1/4} + 4\gamma_{2,3} + 3\gamma_{2,3}^{2/3} + 4\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2}\right] e^{3\theta_{4,2}/2} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \right. \\
&\quad \left. + \xi_4 \int_3^\infty \varrho_1(t-3)\omega_r(t-1) dt\right] (1 + \|x\|_2^n) \|u\|_2 \|v\|_2 \|w\|_2 \|y\|_2.
\end{aligned}$$

The final result follows from taking a supremum over  $u, v, w, y$ ,

$$\begin{aligned}
\|\nabla^4 u_f(x)\|_{\text{op}} &\leq \zeta_4 (1 + \|x\|_2^n) \quad \text{where} \\
\zeta_4 &= 6\tilde{\pi}_{1:4,n}(f) \left[1 + 6\gamma_{2,4}^{1/4} + 4\gamma_{2,3} + 3\gamma_{2,3}^{2/3} + 4\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2}\right] e^{3\theta_{4,2}/2} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \\
&\quad + \xi_4 \int_0^\infty \varrho_1(t)\omega_r(t+2) dt,
\end{aligned}$$

where

$$\begin{aligned}
\xi_4 &= 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)^2\nu_{1:3}(\sigma)\nu_0(\sigma^{-1})^2\nu_{0:2}(\sigma^{-1})e^{\theta_{4,2}}\varrho_1(0)\omega_r(1)\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^3 \\
&\quad \times \left[42 + 32\gamma_{2,2}^{1/2} + 6\gamma_{2,2} + 2\gamma_{2,3}^{1/3} + 3\gamma_{2,3}^{2/3} + 24\gamma_{2,4}^{1/4} + 3\gamma_{2,4}^{1/2} + 12\gamma_{2,6}^{1/6} \right. \\
&\quad \left. + 5\gamma_{3,2}^{1/2} + 5\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2} + 6\gamma_{2,2}^{1/2}\gamma_{2,6}^{1/6}\right].
\end{aligned}$$

Collecting the results of this section, we reach Theorem C.6

**Theorem C.6.** Assume that  $f$  is pseudo-Lipschitz continuous of order  $n$ , and for  $i = 2, 3, 4$  its  $i$ -th derivative has at most degree- $n$  polynomial growth, i.e.,

$$\tilde{\mu}_{1,n}(f) \vee \max_{i=2,3,4} \tilde{\pi}_{i,n}(f) < \infty.$$

Then, we have

$$\|\nabla^i u_f(x)\|_{\text{op}} \leq \zeta_i (1 + \|x\|_2^n)$$

where

$$\begin{aligned}
\zeta_2 &= 2\xi_2\varrho_1(0)\omega_r(0) + \xi_2 \int_0^\infty \varrho_1(t)\omega_r(t) dt, \\
\zeta_3 &= 4\tilde{\pi}_{1:3,n}(f) (1 + 3\gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2}) \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} + \xi_3 \int_0^\infty \varrho_1(t)\omega_r(t+1) dt \\
\zeta_4 &= 6\tilde{\pi}_{1:4,n}(f) \left[1 + 6\gamma_{2,4}^{1/4} + 4\gamma_{2,3} + 3\gamma_{2,3}^{2/3} + 4\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2}\right] e^{3\theta_{4,2}/2} \{1 + (\beta_{r,6n}/\alpha)^{1/6}\} \\
&\quad + \xi_4 \int_0^\infty \varrho_1(t)\omega_r(t+2) dt,
\end{aligned}$$

with

$$\begin{aligned}
\xi_2 &= 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)\nu_0(\sigma^{-1})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}\varrho_1(0)\omega_r(1)[2 + \gamma_{2,2}^{1/2}]e^{\theta_{2,2}/2}, \\
\xi_3 &= 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)\nu_{1:2}(\sigma)\nu_0(\sigma^{-1})\nu_{0:1}(\sigma^{-1})\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^2 \\
&\quad \times \varrho_1(0)\omega_r(1)e^{\theta_{3,4}/2}[7 + 7\gamma_{2,2}^{1/2} + \gamma_{2,3}^{1/3} + \gamma_{3,2}^{1/2}], \\
\xi_4 &= 4\tilde{\mu}_{1,n}(f)\nu_1(\sigma)^2\nu_{1:3}(\sigma)\nu_0(\sigma^{-1})^2\nu_{0:2}(\sigma^{-1})e^{\theta_{4,2}}\{1 + (\beta_{r,6n}/\alpha)^{1/6}\}^3\varrho_1(0)\omega_r(1) \\
&\quad \times \left[42 + 32\gamma_{2,2}^{1/2} + 6\gamma_{2,2} + 2\gamma_{2,3}^{1/3} + 3\gamma_{2,3}^{2/3} + 24\gamma_{2,4}^{1/4} + 3\gamma_{2,4}^{1/2} + 12\gamma_{2,6}^{1/6} \right. \\
&\quad \left. + 5\gamma_{3,2}^{1/2} + 5\gamma_{3,3}^{1/3} + \gamma_{4,2}^{1/2} + 6\gamma_{2,2}^{1/2}\gamma_{2,6}^{1/6}\right],
\end{aligned}$$

where  $\gamma_{i,j}$  and  $\theta_{i,j}$  are as defined in Lemma B.1.

## D Proofs of Expected Suboptimality Bounds

*Proof of Prop. 4.1.* By Lemma C.1, our dissipativity assumption implies that  $p(\|\cdot\|_2) \leq \beta/\alpha$ . Moreover, as noted in the proof of [26, Prop. 3.4], the differential entropy is bounded by that of a multivariate Gaussian with the same second moments:

$$-p(\log p) \leq \frac{d}{2} \log\left(\frac{2\pi e p(\|\cdot\|_2^2)}{d}\right) \leq \frac{d}{2} \log\left(\frac{2\pi e \beta}{d\alpha}\right).$$

Meanwhile,  $\log p(x^*) = -\log \int p(x)/p(x^*)dx$ . Our smoothness assumption, a polar coordinate transform, and the integral identity of [16, 3.326 2] imply that

$$\begin{aligned}
\int p(x)/p(x^*)dx &= \int \exp(\log p(x) - \log p(x^*))dx \geq \int \exp(-C\|x - x^*\|_2^\theta)dx \\
&= \int_0^\infty S_{d-1}r^{d-1} \exp(-Cr^{2\theta})dr = S_{d-1} \frac{1}{2\theta} \Gamma\left(\frac{d}{2\theta}\right) C^{-d/(2\theta)}
\end{aligned} \tag{D.1}$$

where  $S_{d-1} = 2\frac{\pi^{d/2}}{\Gamma(d/2)}$  is the surface area of the unit sphere in  $\mathbb{R}^d$  and  $\Gamma(\cdot)$  is the Gamma function. Since, by [18, Thm. 2],  $\Gamma(x+y)/\Gamma(y) \geq x^y \frac{x}{x+y}$  for all  $x, y > 0$ ,

$$\begin{aligned}
\log p(x^*) &\leq \frac{d}{2\theta} \log(C) - \log\left(\frac{S_{d-1}}{2\theta} \Gamma\left(\frac{d}{2\theta}\right)\right) = \frac{d}{2\theta} \log(C) + \frac{d}{2} \log\left(\frac{1}{\pi}\right) - \log\left(\frac{1}{\theta} \frac{\Gamma(\frac{d}{2\theta})}{\Gamma(\frac{d}{2})}\right) \\
&\leq \frac{d}{2\theta} \log(C) + \frac{d}{2} \log\left(\frac{1}{\pi}\right) - \left(\frac{1}{\theta} - 1\right) \frac{d}{2} \log\left(\frac{d}{2}\right) \\
&\leq \frac{d}{2\theta} \log\left(\frac{2C}{d}\right) + \frac{d}{2} \log\left(\frac{d}{2\pi}\right).
\end{aligned} \tag{D.2}$$

The first result now follows by summing the estimates (D.1) and (D.2).

Now consider the case in which  $p = p_{\gamma,\theta}$ . By design,  $x^*$  is also a global minimizer of  $f$ , and hence  $\nabla f(x^*) = 0$ . Therefore, by Taylor's theorem, we have for each  $x$

$$\begin{aligned}
\log p_{\gamma,\theta}(x^*) - \log p_{\gamma,\theta}(x) &= \gamma(f(x) - f(x^*))^\theta \\
&= \gamma(\langle \nabla f(x^*), x - x^* \rangle + \frac{1}{2} \langle x - x^*, \nabla^2 f(z)(x - x^*) \rangle)^\theta \\
&\leq \frac{\gamma \mu_2^\theta(f)}{2^\theta} \|x - x^*\|_2^{2\theta}.
\end{aligned}$$

The result now follows from Jensen's inequality as  $p_{\gamma,\theta}(\gamma(f(x) - f(x^*))^\theta) \geq \gamma p_{\gamma,\theta}^\theta(f(x) - f(x^*))$  for  $\theta \in (0, 1]$ .  $\square$

*Proof of Prop. 4.3.* Let  $\alpha = 1/k$ . We have

$$\mathbb{E}_{x \sim p_{\gamma,\alpha}}[f(x)] - f^* = \frac{\int ((x-b)^\top A(x-b)) \exp(-\gamma((x-b)^\top A(x-b))^\alpha) dx}{\int \exp(-\gamma((x-b)^\top A(x-b))^\alpha) dx}.$$

Using the variable change  $y = A^{1/2}(x-b)$ ,  $dy = \det(A^{1/2})dx$ , the above equals

$$\frac{\int \|y\|^{2\alpha} \exp(-\gamma\|y\|^{2\alpha}) dy}{\int \exp(-\gamma\|y\|^{2\alpha}) dy} = \frac{\int_0^\infty S_{d-1} r^{d-1} r^{2\alpha} \exp(-\gamma r^{2\alpha}) dr}{\int_0^\infty S_{d-1} r^{d-1} \exp(-\gamma r^{2\alpha}) dr} = \frac{\int_0^\infty r^{d+1} \exp(-\gamma r^{2\alpha}) dr}{\int_0^\infty r^{d-1} \exp(-\gamma r^{2\alpha}) dr},$$

where  $S_{d-1}$  is the surface area of the unit sphere in  $\mathbb{R}^d$ . Substituting an explicit expression for these integrals we get

$$\frac{\Gamma(\frac{d+2}{2\alpha})/2\alpha\gamma^{(d+2)/2\alpha}}{\Gamma(\frac{d}{2\alpha})/2\alpha\gamma^{d/2\alpha}} = \frac{\Gamma(\frac{d}{2\alpha} + \frac{1}{\alpha})}{\Gamma(\frac{d}{2\alpha})} \gamma^{-1/\alpha},$$

where  $\Gamma(\cdot)$  is the Gamma function. Substituting back  $k = 1/\alpha$ , and noting that  $\Gamma(z + 1) = z\Gamma(z)$  for all  $z$ , we get that the above equals

$$\frac{\Gamma(\frac{dk}{2} + k)}{\Gamma(\frac{dk}{2})} \gamma^{-k} = \gamma^{-k} \prod_{i=0}^{k-1} (\frac{dk}{2} + i) \leq \gamma^{-k} (\frac{dk}{2} + k - 1)^k = \left( \frac{k(\frac{1}{2}d+1)-1}{\gamma} \right)^k.$$

□

## E Proof of Prop. 3.5: User-friendly Wasserstein decay for Gibbs measures

*Proof of Prop. 3.5.* Define  $\tilde{\sigma}_\gamma(x) = (\sigma_\gamma(x)\sigma_\gamma(x)^\top - s^2 I)^{1/2} = \frac{1}{\sqrt{\gamma}} \tilde{\sigma}(x)$ . Our assumptions imply

$$\begin{aligned} & \frac{\langle b_\gamma(x) - b_\gamma(y), x - y \rangle}{s^2 \|x - y\|_2^2 / 2} + \frac{\|\tilde{\sigma}_\gamma(x) - \tilde{\sigma}_\gamma(y)\|_F^2}{s^2 \|x - y\|_2^2} - \frac{\|(\tilde{\sigma}_\gamma(x) - \tilde{\sigma}_\gamma(y))^\top (x - y)\|_2^2}{s^2 \|x - y\|_2^4} \\ & \leq \frac{-\gamma \langle m(x) \nabla f(x) - m(y) \nabla f(y), x - y \rangle}{s_0^2 \|x - y\|_2^2} + \frac{\langle \nabla, m(x) - m(y) \rangle, x - y + \|\tilde{\sigma}(x) - \tilde{\sigma}(y)\|_F^2}{s_0^2 \|x - y\|_2^2} \\ & \leq \begin{cases} -\frac{\gamma K_{m-L^*}}{s_0^2} & \text{if } \|x - y\|_2 > R \\ \frac{\gamma L_{m+L^*}}{s_0^2} & \text{if } \|x - y\|_2 \leq R, \end{cases} \end{aligned}$$

as advertised. □

## F Auxiliary Lemmas

**Lemma F.1** (Quadratic form moment bounds). *For  $W_m \sim \mathcal{N}_d(0, I)$  which is independent from  $X_m$ , we have*

$$\mathbb{E}[\|\sigma(X_m)W_m\|_2^{2n}] \leq (2n - 1)!! \mathbb{E}[\|\sigma(X_m)\|_F^{2n}].$$

*Proof.* The exact expressions for the quadratic form moments can be found in [21]. We simply use the properties of Frobenius norm to obtain a compact upper bound. □

**Lemma F.2.** *For a sequence of real nonnegative numbers  $\{a_i\}_{i=0}^n$  satisfying  $a_{i+1} \leq \tau a_i + \gamma$  for  $\tau \in (0, 1)$  and  $\gamma \in \mathbb{R}$  we have*

$$\frac{1}{n} \sum_{i=1}^n a_i \leq a_0 + \frac{\gamma}{1-\tau}.$$

*Proof.* By the recursive inequality, we have

$$a_i \leq \tau^i a_0 + \gamma \frac{1-\tau^i}{1-\tau}.$$

Averaging over  $i$ , we obtain

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n a_i & \leq \frac{1}{n} \sum_{i=1}^n \left( a_0 \tau^i + \gamma \frac{1-\tau^i}{1-\tau} \right), \\ & \leq \frac{a_0 \tau}{n} \frac{1-\tau^n}{1-\tau} + \frac{\gamma}{1-\tau} \leq a_0 + \frac{\gamma}{1-\tau}, \end{aligned}$$

where in the last step, we used  $\tau \leq 1$  and the Bernoulli inequality

$$1 - \tau^n = 1 - (1 - (1 - \tau))^n \leq n(1 - \tau).$$

□

**Lemma F.3.** *For  $x, a, c > 0$  and  $m \geq 1$ , we have  $ax^m + a(c/a)^m/m \geq cx^{m-1}$ .*

*Proof.* The derivative of the polynomial  $p(x) = ax^m - cx^{m-1} + b$  has  $m - 2$  roots at 0, and a root at  $x_0 = c(m - 1)/(am)$ . Therefore,  $p(x)$  for  $x \geq 0$ , attains its minimum value at  $x_0$ . We choose  $b = a(c/a)^m/m$  so that

$$\begin{aligned} p(x_0) & = (ax_0 - c)x_0^{m-1} + b \\ & = b - \frac{ac^m}{ma^m} \left(1 - \frac{1}{m}\right)^{m-1} \geq 0, \end{aligned}$$

where for the last step, we use  $f(x) = (1 - 1/x)^{x-1} \leq 1$  for  $x \geq 1$  and  $\lim_{x \downarrow 1} f(x) = 1$ . □

**Lemma F.4.** For a three times differentiable matrix valued function  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{m \times k}$ , we have the following chain rules

$$\begin{aligned}
\nabla \sigma^{-1}(x)[v] &= -\sigma^{-1}(x) \nabla \sigma(x)[v] \sigma^{-1}(x), \\
\nabla^2 \sigma^{-1}(x)[v, u] &= \sigma^{-1}(x) [\nabla \sigma(x)[u] \sigma^{-1}(x) \nabla \sigma(x)[v] - \nabla^2 \sigma(x)[v, u] \\
&\quad + \nabla \sigma(x)[v] \sigma^{-1}(x) \nabla \sigma(x)[u]] \sigma^{-1}(x), \\
\nabla^3 \sigma^{-1}(x)[v, u, w] &= \sigma^{-1}(x) \left[ -\nabla \sigma(x)[w] \sigma^{-1}(x) \nabla \sigma(x)[u] \sigma^{-1}(x) \nabla \sigma(x)[v] \right. \\
&\quad + \nabla^2 \sigma(x)[u, w] \sigma^{-1}(x) \nabla \sigma(x)[v] - \nabla \sigma(x)[u] \sigma^{-1}(x) \nabla \sigma(x)[w] \sigma^{-1}(x) \nabla \sigma(x)[v] \\
&\quad + \nabla \sigma(x)[u] \sigma^{-1}(x) \nabla^2 \sigma(x)[v, w] - \nabla \sigma(x)[u] \sigma^{-1}(x) \nabla^2 \sigma(x)[v] \sigma^{-1}(x) \nabla \sigma(x)[w] \\
&\quad + \nabla \sigma(x)[w] \sigma^{-1}(x) \nabla^2 \sigma(x)[v, u] - \nabla^3 \sigma(x)[v, u, w] \\
&\quad + \nabla^2 \sigma(x)[v, u] \sigma^{-1}(x) \nabla \sigma(x)[v] - \nabla \sigma(x)[w] \sigma^{-1}(x) \nabla \sigma(x)[v] \sigma^{-1}(x) \nabla \sigma(x)[u] \\
&\quad + \nabla^2 \sigma(x)[v, w] \sigma^{-1}(x) \nabla \sigma(x)[u] - \nabla \sigma(x)[v] \sigma^{-1}(x) \nabla \sigma(x)[w] \sigma^{-1}(x) \nabla \sigma(x)[u] \\
&\quad \left. + \nabla \sigma(x)[v] \sigma^{-1}(x) \nabla^2 \sigma(x)[u, w] - \nabla \sigma(x)[v] \sigma^{-1}(x) \nabla \sigma(x)[u] \sigma^{-1}(x) \nabla \sigma(x)[w] \right] \sigma^{-1}(x).
\end{aligned}$$

*Proof.* Results follow from matrix differentiation and the chain rule. □