Fair Allocation through Competitive Equilibrium from Generic Incomes

ABSTRACT

Two food banks catering to populations of different sizes with different needs must divide among themselves a donation of food items. What constitutes a "fair" allocation of the items among them?

Competitive equilibrium from equal incomes (CEEI) is a classic solution to the problem of fair and efficient allocation of goods among agents [Foley 1967, Varian 1974]. Every agent (foodbank) receives an equal endowment of artificial currency with which to "purchase" bundles of goods (food items). Prices for the goods are set high enough such that the agents can simultaneously get their favorite within-budget bundle, and low enough such that all goods are allocated (no waste). A CEEI satisfies mathematical notions of fairness like fair-share, and also has built-in transparency – prices can be published so the agents can verify they're being treated equally. However, a CEEI is not guaranteed to exist when the items are indivisible.

We study competitive equilibrium from generic incomes (CEGI), which is based on the idea of slightly perturbed endowments, and enjoys similar fairness, efficiency and transparency properties as CEEI. We show that when the two agents have almost equal endowments and additive preferences for the items, a CEGI always exists. We then consider agents who are a priori non-equal (like different-sized foodbanks); we formulate a new notion of fair allocation among non-equals satisfied by CEGI, and show existence in cases of interest (like when the agents have identical preferences). Experiments on simulated and Spliddit data (a popular fair division website) indicate more general existence. Our results open opportunities for future research on fairness through generic endowments, and on fair treatment of non-equals.

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1 INTRODUCTION.

We study fair and efficient allocation of indivisible goods in settings without money, where agents have possibly very different a priori entitlements to the goods. Such settings arise in the context of real-life allocation decisions, such as allocating donations to food banks [48], allocating courses to students or shifts to workers [17], or sharing scientific/computational resources within a university or company. Important policy objectives often shape different entitlements to the resources [28]. We wish to develop notions of fairness that apply to such settings; our approach is to study fairness through the prism of competitive market equilibrium – a connection made long ago by [31, 56].

1.1 The economic concept of equilibrium in Fisher markets with indivisible goods.

Let us suspend for a few paragraphs the discussion of fairness and introduce the market equilibrium concept from microeconomics. Imagine endowing the agents with budgets of some fake currency. In a competitive equilibrium of the resulting market, goods are assigned prices, each player takes his preferred set of goods among those that are within his budget, and the market clears. By the first welfare theorem, the resulting allocation is Pareto efficient.

In this paper we focus on the simple Fisher market model [10], where a (hypothetical) seller brings m goods to the market and the n agents bring certain positive amounts of artificial currency in the form of budgets $b_1, \ldots, b_n$. The seller has no use for the goods and the players have no use for their budget, but each agent has a preference order among all possible bundles of goods. In this model, if some goods are divisible, a competitive equilibrium is known to exist when agents' preferences satisfy mild conditions (e.g., [29]). We focus on the case where *all* goods are indivisible. It is not hard to see that an equilibrium need not exist – just consider a single indivisible item sold among two agents, both with the same budget of 1. Indeed, if the item's price is at most 1 then both agents desire it, while if the price is strictly above 1 then neither can afford it and the market does not clear.[1]

In this simple example, non-existence is a knife-edge outcome: if the budgets are $1 + \epsilon$ and 1 instead of precisely equal, then an equilibrium exists by setting the item price in between the two budgets. To avoid such knife-edge non-existence results, Budish [17] initiates the study of *almost-equal* budgets.[2] In this paper we further advance this idea by considering *generic* budgets – arbitrary budgets (possibly far from equal) to which small perturbations have been added (see Section 2.2.2 for a definition).

While generic budgets may not be a silver bullet solution to equilibrium non-existence, anecdotal evidence gathered from computer simulations and real-life data suggests that existence in our model is quite common with generic budgets. A computerized search we ran over 2-player markets with generic budgets produced no non-existence examples (details appear in Appendix C). This means that a competitive equilibrium can be a useful way to allocate goods in Fisher markets with indivisibilities and generic budgets, which brings us to the question of its fairness properties.

[1]Recall that players have no value for money, so an agent is not satisfied if the price is 1 but he does not get the item.

[2]Budish [17] couples almost-equal budgets with an *approximate* equilibrium notion that requires giving up market clearance; in contrast, our results will not require market clearance relaxation.

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1.2 Fairness with indivisible goods and different entitlements.

Which allocations of goods among players should be considered "fair"? A vast literature is devoted to this question (for expositions see, e.g., [12, 13, Chapters 11–13]). Many standard notions of fairness in the literature reflect an underlying assumption that all players have an equally-strong claim to the goods. However, some players may be *a priori* much more entitled to the goods than others (see, e.g. [50, Chapters 3 and 11]). Real-life examples include – in addition to the ones mentioned above – partners who own different shares of the partnership's holdings [23], different departments sharing a company's computational resources [34], or family members splitting an heirloom [47]. Our model captures these situations, since the entitlement of player i can be encoded by his budget b_i (we may assume without loss of generality that $\sum_i b_i = 1$).

An appropriate notion of fairness should of course take into account the indivisible goods and different entitlements, and capture the idea of "proportional satisfaction of claims" [16, p. 95]. In the special case where all entitlements are equal ($b_1 = \ldots = b_n = 1/n$), *envy-freeness* (no player prefers another's allocation to his own) is an important fairness criterion [31], but it is not clear how to generalize this notion to heterogeneous entitlements. *Fair share* (every player i prefers his allocation to a b_i-fraction of all items) is also an important fairness criterion [53], but it does not easily extend to indivisible items.

In this paper we adapt a well-known approach to fair allocation with equal entitlements – finding a *competitive equilibrium from equal incomes* (*CEEI*) [7, 56] – to the case of unequal entitlements. We treat the players as buyers and their entitlements as budgets, and seek a competitive equilibrium in the resulting Fisher market. Such an equilibrium is not guaranteed to exist, but this is to be expected – indivisibility can indeed undermine the ability to fairly allocate. Our idea is to first perturb the entitlements (budgets), since competitive equilibrium from *generic* incomes (CEGI) may exist more widely and offer approximate fairness guarantees that are the best possible given indivisibilities. *Thus, a better understanding of competitive equilibrium with generic budgets can help reach fair or approximately fair division of indivisible items among players with heterogeneous entitlements.*

1.3 Our results.

We begin by showing in Section 3 that when a CE exists, its allocation has a natural fairness property related to the well-known cut-and-choose protocol. Recall that in the cut-and-choose protocol, say with 2 players, one player (the "cutter") divides the items into 2 sets, and the other player chooses her favorite set; the resulting allocation is intuitively fair. More precisely, each player gets a bundle she prefers at least as much as the one she can guarantee for herself as the cutter, called her *maximin share (MMS)*. We build upon [17] to show that CE allocations have a generalized maximin share guarantee which is appropriate for different budgets. Namely, an *ℓ-out-of-d maximin share* of a player is any bundle at least as good as the one she can guarantee by the following (hypothetical) protocol: the player partitions the items into d parts (some may be empty), and then takes the worst ℓ of these parts (since the other players were entitled to $d - \ell$ of the parts and presumably their choice was worst-case for the cutter).

Theorem 1.1 (Fairness). *In any CE allocation, for every player $i \in N$ and rational number $\ell/d \leq b_i$, player i gets her ℓ-out-of-d maximin share.*

In Section 3 we discuss in what sense the guarantee of Theorem 1.1 improves upon that of [17], besides being applicable to a priori non-equal players.

We then turn to the question of existence, focusing on the 2-player case with additive preferences. Besides being the simplest case in which it is not known whether generic budgets guarantee CE existence, the 2-player case is probably the most important in practice (e.g., Fair Outcomes Inc. is a website dedicated to 2 players). As we shall see, it already provides rich technical challenges that require novel techniques.

Our initial result for this setting is establishing the second theorem of welfare (the first welfare theorem is known to hold):

Theorem 1.2 (Second welfare theorem). *Consider 2 players with additive preferences. For every Pareto efficient allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$, there exist budgets b_1, b_2 and prices p such that $(\mathcal{S}, p)$ is a CE.*

We then identify large families of instances for which a CE exists:

Theorem 1.3 (Existence results). *Consider 2 players with additive preferences v_1, v_2. A CE with generic budgets always exists in markets satisfying any one of the following conditions:*

- *Existence of a* proportional *allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ in which every player i receives at least a b_i-fraction of his total value.*
- *Budgets are almost equal, as in Budish's setting (but without revoking market clearance).*
- *The 2 players have identical preferences.*

Moreover, the CEs guaranteed to exist assign identical prices to identical copies of the same item.

The first condition of Theorem 1.3 shows that CEs are no rarer than allocations which satisfy the fairness property of proportionality. As for the second condition, we know from [17] that a CE with almost equal budgets satisfies "envy-freeness up to one item", so if the goal is to reach such envy-freeness then it is always achievable for 2 additive players. The third condition establishes CE existence for the case that would seem to be hardest in terms of fairness, in which players are in direct competition.

Our results for 2 additive players are via a characterization we develop of CEs for 2 players, which has a particularly nice form for additive preferences. We use this to show that linear combinations of the additive preferences form equilibrium prices under a certain condition. To see when this condition is fulfilled, we use a graphical representation of all allocations and their values for the players, as depicted in the figures accompanying the text.

Finally, we also show that in all cases to which Theorem 1.3 applies, the CE has the following fairness property: each player i receives his *truncated share*, which is as close to proportionality as the indivisibility of the items allows when no proportional allocation exists.[3]

We leave the following as our main open problem:

Open Problem 1. *Does there always exist a CE for 2 players with additive preferences and generic budgets?*

1.4 Additional related work.

Discrete Fisher markets. Our model is a special case of Arrow-Debreu exchange economies with indivisible items. Several other variants have been studied: [1, 32, 41, 43, 54] consider markets with indivisibilities in which an infinitely divisible good plays the role of money, and so money carries inherent value for the players. Shapley and Scarf [51], Svensson [55] and subsequent works focus on the house allocation problem with *unit-demand* players. Several works assume a continuum of players [e.g., 39], and/or study relaxed CE notions [e.g., 25, 27, 49]. Closest are the models of *combinatorial assignment* [17], which allows non-monotonic preferences, and of *linear markets* [25], which crucially relies on non-generic budgets.

CEEI. Budish [17] circumvents the non-existence of CEEI due to indivisibilities by weakening the equilibrium concept and allowing market clearance to hold only approximately. He focuses exclusively on budgets that are almost equal. In the same model, Othman et al. [44] show PPAD-completeness of computing an approximate CEEI, and NP-completeness of deciding the existence of an approximate CEEI with better approximation factors than those shown to exist by Budish. The preferences used in the hardness proofs are non-monotone, leading Othman et al. to suggest the research direction of restricting the preferences (as we do here) as a way around their negative results. Brânzei et al. [15] study (exact) CEEI existence for two valuation classes (perfect substitutes and complements) with non-generic budgets. For *divisible* items, there has been renewed interest in CEEI under additive preferences due to their succinctness and practicality. Bogomolnaia and Moulin [7] offer a characterization based on natural axioms, and Bogomolnaia et al. [8] analyze CEEI allocations of "bads" rather than goods.

Fairness with indivisibilities. Most notions of fair allocation in the classic literature apply to divisible items. Bouveret et al. [9] survey research on fairness with *indivisible* items, emphasizing computational challenges (see also [6, 24]). The works of [37, 38] study envy minimization (rather than elimination) with indivisibilities. Guruswami et al. [36] study envy-freeness in the context of profit-maximizing pricing. Several recent works [3, 14, 19, 21, 22, 33] consider the approximation of Nash social welfare and related fairness guarantees. As for practical implementations of fair division with indivisibilities, these are discussed by Budish and Cantillon [18] (for responsive preferences), Othman et al. [45] (implementing findings of Budish [17]), Goldman and Procaccia [35] (presenting the Spliddit website for additive preferences), and Brams and Taylor [12] (presenting the adjusted winner algorithm in which one item may need to be divided). Mechanism design aspects appear in [2]. For further discussion see Section 2.3.

Concurrent and subsequent work. Recently there has been increasing interest in fair division of indivisible goods; the following works appeared concurrently or subsequently to early versions of our paper. Several additional results related to CEs with generic budgets appear in our companion paper, which studies other classes of preferences and more than 2 players (citation omitted to accommodate the double-blind policy). Farhadi et al. [30] independently develop a new fairness notion for allocation among agents with cardinal preferences and different entitlements. Their notion is distinct from ours and is not directly related to the solution concept of a CE. For example, their fairness notion is not always guaranteed when allocating 3 items among players with generic budgets (as implied in their Theorem 2.1). This is at odds with our CE existence results and resulting fairness guarantees according to our notions of fairness (Proposition 3.3 and companion paper), demonstrating the difference between the approaches. Several recent papers have advanced our understanding of envy-freeness, including [46] which studies the envy-free relaxation EF-1* suggested by [19], and the works of [5] and [20] which study envy-freeness with incomplete knowledge.

2 MODEL.

In this section we formulate our market model and competitive equilibrium notion (with generic budgets), and present fairness preliminaries.

2.1 Market setting.

We study discrete Fisher markets which consist of m indivisible items M and $n = 2$ agents $N = \{1, 2\}$, whom we refer to by the game-theoretic term *players*. We refer to subsets of items as bundles and often denote them by S or T. Each player $i \in N$ has a *cardinal* preference, represented by a valuation function $v_i : 2^M \to \Re_+$ which assigns to every bundle S a nonnegative value $v_i(S)$. Player i prefers bundle S to bundle T iff $v_i(S) > v_i(T)$, and cardinality allows us to compare by how much – by considering the ratio $v_i(S)/v_i(T)$. The absolute values themselves however don't matter, hence a cardinal valuation can be normalized without loss of generality (wlog).

Our results hold primarily for the class of *additive* preferences. A cardinal preference v_i is additive if $v_i(S) = \sum_{j \in S} v_i(\{j\})$ for every bundle S. We assume wlog that v_i is normalized, i.e., $v_i(M) = 1$. We also assume that preferences are *monotone* (satisfy free disposal), i.e., $v_i(S) < v_i(T)$ whenever $S \subset T$. Moreover we assume *strict* preferences (no indifferences), i.e., either $v_i(S) < v_i(T)$ or $v_i(S) > v_i(T)$ whenever $S \neq T$. We allow one exception to strictness – items may be identical in which case preferences over them are allowed to be weakly-monotone rather than strictly so.[4]

In addition to preferences, players in our model have *budgets*. Let $b = (b_1, b_2, \ldots, b_n)$ be a budget profile, where $b_i > 0$ is player i's budget. Unless stated otherwise, we assume wlog that $\sum_{i=1}^{n} b_i = 1$, and $b_1 \geq b_2 \geq \ldots \geq b_n$ (every market can be converted to satisfy these properties by renaming and normalization).

[3] We define truncated share as any bundle the player prefers at least as much as his most preferred Pareto efficient allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ for which $v_i(\mathcal{S}_i) \leq b_i \cdot v_i(\{1, \ldots, m\})$.

[4] All the CEs in our existence results assign the same prices to identical items. This is similar to the treatment of identical items in [17], where prices are assigned to classes rather than to individual seats.

We emphasize that budgets and "money" in our model are only means for allocating items among players with different entitlements. Money has no intrinsic value for the players, whose preferences are over subsets of items and who disregard any leftover budget. For this reason, it is important that the preferences are scale-free, in contrast to a model with money where a value $v_{i,j}$ can be interpreted as the amount player i is willing to pay for item j. Similarly, scale-dependent measures of social efficiency and fairness like welfare and egalitarianism (maximizing the value of the worst-off player) are inappropriate in our model.

2.2 Competitive equilibrium (CE).

The goal of the market is to allocate the items among the players. An *allocation* $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$ is a partition of *all* items among the players, i.e., $\mathcal{S}_i \cap \mathcal{S}_k = \emptyset$ for $i \neq k$ and $\bigcup_i \mathcal{S}_i = M$. By definition, an allocation is *feasible* (no item is allocated more than once) and *market clearing* (every item is allocated).

A competitive equilibrium is an allocation together with item prices that "stabilize" it. Let $p = (p_1, p_2, \dots, p_m)$ denote a vector of non-negative item prices. The price $p(S)$ of a bundle S is then $\sum_{j \in S} p_j$. We say that S is *within budget* b_i if $p(S) \leq b_i$, and that it is *demanded* by player i at price vector p if it is the most preferred bundle within her budget at these prices. Formally, $p(\mathcal{S}_i) \leq b_i$, and $p(T) > b_i$ for every T such that $v(T) > v(\mathcal{S}_i)$. We can now define our equilibrium notion:

Definition 2.1 (CE). A *competitive equilibrium (CE)* is a pair $(\mathcal{S}, p)$ of allocation $\mathcal{S}$ and item prices p, such that $\mathcal{S}_i$ is demanded by player i at prices p for every $i \in N$.

Given a market with preferences $\{v_i\}_{i \in N}$ and budget profile b, an allocation $\mathcal{S}$ is *supported* in a CE if there exist item prices p such that $(\mathcal{S}, p)$ is a CE. Where only preferences are given, we overload this notion and say that $\mathcal{S}$ is supported in a CE if there exist prices p and budgets b such that $(\mathcal{S}, p)$ is a CE.

Given budgets b and an allocation $\mathcal{S}$, we say that prices p are *budget-exhausting* if $p(\mathcal{S}_i) = b_i$ for every player i. Note that if p is budget-exhausting then every player is allocated, i.e., $\mathcal{S}_i \neq \emptyset$ for every i (since $b_i > 0$). We observe that budget-exhaustion is wlog when every player is allocated, and use this throughout the paper:

Claim 1 (Budget-exhausting prices are wlog). *For every CE $(\mathcal{S}, p)$ such that $S_i \neq \emptyset$ for every $i \in N$, there exists a CE $(\mathcal{S}, p')$ in which p' is budget-exhausting.*

Proof. As every player is allocated at least one item, we can raise the price of that item in his allocated bundle until her budget is exhausted. The new prices form a CE with the original allocation since every player still gets her demanded set. □

2.2.1 *Pareto optimality (PO).*

What does it means for an allocation (CE or otherwise) to be "efficient"?

Definition 2.2 (PO). Consider a market with preferences $\{v_i\}_{i \in N}$. An allocation $\mathcal{S}$ is *Pareto optimal (PO)* (a.k.a. Pareto efficient) if no allocation $\mathcal{S}'$ *dominates* $\mathcal{S}$, i.e., if for every $\mathcal{S}' \neq \mathcal{S}$ there exists a player i for whom $v_i(\mathcal{S}_i) > v_i(\mathcal{S}'_i)$.

We use the notation $\widetilde{\text{PO}} = \widetilde{\text{PO}}(v_1, v_2)$ to denote the set of all different PO allocations for preferences v_1, v_2.

There are two kinds of fundamental welfare theorems in economics that apply to various market equilibrium notions. The first is about Pareto optimality and holds for CEs in our setting; we discuss the second in Section 4.

Theorem 2.3 (First welfare theorem). *Let $(\mathcal{S}, p)$ be a CE. Then $\mathcal{S}$ is PO.*

Proof (for completeness). Assume for contradiction an alternative allocation $\mathcal{S}'$, such that for every player i for whom $\mathcal{S}_i \neq \mathcal{S}'_i$ it holds that $v_i(\mathcal{S}_i) < v_i(\mathcal{S}'_i)$. Consider the total payment $\sum_i p(\mathcal{S}'_i)$ for the alternative allocation given the CE prices p. By market clearance, $\sum_i p(\mathcal{S}'_i) = \sum_i p(\mathcal{S}_i)$. Therefore there must exist a player i for whom $\mathcal{S}_i \neq \mathcal{S}'_i$ but $p(\mathcal{S}'_i) \leq p(\mathcal{S}_i)$. This means that $\mathcal{S}_i$ cannot be demanded by player i, leading to a contradiction. □

2.2.2 *Generic budgets (vs. different budgets).*

We are interested in showing *generic* existence of a CE for classes of markets. We use the standard notion of genericity, i.e., "all except for a zero-measure", or equivalently, "with tiny random perturbations". By generic existence we thus mean that for every market in the class, for every vector of budgets except for a zero-measure subset, a CE exists. An equivalent way to say this is: for every market in the class, for every vector of budgets, by adding tiny random perturbations to the budgets we get a new instance in which a CE exists with probability 1.

A useful way of specifying a zero-measure subset of budgets is as those which satisfy some condition, for instance, $b_1 = 2b_2$. The conditions we use differ among different CE existence results, and for concreteness we shall list them explicitly within each result; we emphasize however that the conditions themselves are entirely irrelevant to our contribution, as long as the measure of budgets satisfying them is zero.

Generic budgets are not to be confused with *different* (or *arbitrary*) budgets, by which we mean budgets that are not necessarily close to one another. While fair allocation among (almost) equal players is well-studied, much less is known for players who have a priori different entitlements, as modeled by different budgets.

2.3 Fairness preliminaries.

We include here the fairness preliminaries most related to our results; a more detailed account – including *ordinal* preferences in addition to cardinal ones, envy-freeness in addition to fair share, and Nash social welfare – appears in Appendix 2.3 and is summarized in Tables 1-2. These tables also show where our new fairness notions fit in with existing ones.

The "two most important tests of equity" according to Moulin [42, p.166] are (i) guaranteeing each player his *fair share (FS)*; and (ii) *envy-freeness (EF)*. Our main concern is FS, "probably the least controversial fairness requirement in the literature" [8] – in the setting of indivisible items and players with different budgets for which not much is known. Intuitively, FS for *divisible* items and *equal-budget* players guarantees that each player believes she receives at least $1/n$ of the divisible "cake" (while EF guarantees she believes no one else receives a better slice than hers). More formally, FS requires for each player to receive a bundle that she prefers at least as much as the bundle consisting of a $1/n$-fraction of every divisible item on the market.

When items are *indivisible*, to define FS we must use the cardinal nature of the preferences in our model. The parallel of FS is the notion of proportionality, which extends naturally to players with different budgets: Given a budget profile b, an allocation $\mathcal{S}$ gives player i his *budget-proportional share* if player i receives at least a b_i-fraction of his value for all items, that is $v_i(\mathcal{S}_i) \geq b_i \cdot v_i(M)$. An allocation is *budget-proportional* (a.k.a. weighted-proportional) if every player receives his proportional share. When all budgets are equal, such an allocation is simply called *proportional*.

Unfortunately, budget-proportionality is a very restrictive fairness requirement when dealing with indivisible items (see Section 5). Budish [17] studies the following weaker notion:

Definition 2.4 (FS with indivisibilities). An allocation $\mathcal{S}$ guarantees *1-out-of-n maximin share (MMS)* if every player receives a bundle she prefers at least as much as the bundle she can guarantee for herself by the following procedure: partitioning the items into n parts, and allowing the $n-1$ other players to chose their parts first (assuming their choice is the worst possible for her).

Every CE for n players with *equal* budgets gives every player his 1-out-of-n maximin share and achieves EF. Budish [17] shows that every CE for n players with *almost* equal budgets guarantees 1-out-of-$(n+1)$ maximin share (as if there were an extra player). In Proposition 3.3 we generalize this result to *different* budgets, by defining the fairness notion of ℓ-out-of-d maximin share.

3 FAIRNESS PROPERTIES OF EQUILIBRIUM.

We are interested in fairness properties that are guaranteed by the existence of a CE when players have arbitrary budgets (not necessarily equal or almost equal). In a sense, we are building upon the classic connection between CE with *equal* budgets and fairness, expanding it to *different* budgets, and using it to derive a natural fairness notion appropriate for a priori non-equal players.

In Section 3.1 we define our notion (Def. 3.2): a parameterized version of the 1-out-of-n maximin share guarantee (Def. 2.4), generalizing it to accommodate arbitrary (possibly very different) budgets. In Section 3.2 we show that every CE guarantees fairness according to our notion (Proposition 3.3), and in Section 3.3 we briefly discuss implications. (We remark that in subsequent sections we discuss a different fairness notion − truncated share − which is guaranteed by all of our CE existence results but not by every CE in general.)

3.1 Definition of ℓ-out-of-d maximin share.

Consider a player i. Her ℓ-out-of-d maximin bundle is the bundle she can guarantee for herself by the following (hypothetical) protocol: the player partitions the items into d parts, lets the other players choose $d-\ell$ of these parts (their choice is assumed to be worst-case), then receives the remaining ℓ parts.

Example 3.1. The 2-out-of-3 maximin bundle of an additive player who values items (A, B, C) at $(1, 2, 3)$ is $\{A, B\}$.

Player i's ℓ-out-of-d maximin share reflects how preferable her ℓ-out-of-d maximin bundle is for her. We now give a formal definition:

Definition 3.2. An allocation $\mathcal{S}$ guarantees player i her *ℓ-out-of-d maximin share* if

$$v_i(\mathcal{S}_i) \geq \max_{\text{partition } (T_1,\dots,T_d)} \left\{ \min_{L\subseteq[d], |L|=\ell} \left\{ v_i\left(\bigcup_{t\in L} T_t \right) \right\} \right\}.$$

3.2 ℓ-out-of-d maximin share in equilibrium.

The following proposition holds generally for any market setting (for any number of players, preference class, etc.), and shows that a CE guarantees for every player her fair share in the sense of Definition 3.2. The parameters ℓ and d in the ℓ-out-of-d maximin share guarantee correspond to the budget b_i of the player, thus mirroring her a priori entitlement to the items.

Proposition 3.3. *Let b be an arbitrary budget profile. Every CE guarantees player i her ℓ-out-of-d maximin share for every rational number $\ell/d \leq b_i$.*

Proof. Let (S, p) be a CE and let P denote the sum of prices $\sum_{j\in M} p_j$. Since S is an allocation of all items, every item is "purchased" by a player and so $P = \sum_j p_j \leq \sum_i b_i = 1$. Let $(T_1, \dots, T_d)$ be any partition of the items into d parts, and observe that $1 \geq P = \sum_j p_j = p(\bigcup_{t=1}^d T_t) = \sum_{t=1}^d p(T_t)$ (using linearity of the prices). By the pigeonhole principle, there exists a subset of ℓ parts whose total price is at most $\frac{\ell}{d}P \leq \frac{\ell}{d}$. Let us call this "the cheap subset". By the proposition's assumption, $\frac{\ell}{d} \leq b_i$. Therefore, agent i can afford the cheap subset, and by definition of a CE the bundle actually allocated to agent i must be at least as preferred by him as the cheap subset. □

3.3 Discussion.

For the case of almost equal budgets, Proposition 3.3 subsumes the celebrated result of Budish [17] that every CE with almost equal budgets gives every player her 1-out-of-$(n+1)$ maximin share. To see this notice that if budgets are almost equal then $b_1 \geq b_2 \geq \cdots \geq b_n \geq \frac{n}{n+1} b_1 \geq \frac{1}{n+1}$ for every player i (the last inequality follows since b_1 must be $\geq 1/n$ for the budgets to sum up to 1). Thus the result of Budish can be deduced from Proposition 3.3.

Proposition 3.3 also strengthens the result of Budish in the following sense. Consider $n = 3$ a priori equal players and $m = 7$ items. The 1-out-of-$(n+1)$ maximin share in this case guarantees the worst part out of a partition of the 7 items into 4 parts; the ℓ-out-of-d maximin share applies with $\ell = 2, d = 7$ (since $\ell/d = 2/7 \ll 1/3$), guaranteeing the worst 2 parts out of a partition of the 7 items into 7 parts. If a player views the items as roughly equal, the latter guarantee is better.

The flexibility allowed by the parameters ℓ and d is even more important when dealing with different (i.e. not almost equal) budgets. Consider $m = 3$ items and a player with budget $5/13$. Dividing the 3 items into 13 parts and taking the worst 5 parts does not guarantee the player anything beyond an empty bundle, whereas taking the worst part among 3 (using that $1/3 \leq 5/13$) guarantees her at least one item.

4 SUFFICIENT CONDITION FOR EXISTENCE AND 2ND WELFARE THEOREM.

In this section we state and prove Lemma 4.3 – a sufficient condition for CE existence which is the workhorse of our existence results in subsequent sections. We demonstrate its usefulness by showing a fundamental theorem of welfare (Theorem 4.4) that was not previously known to hold in our setting.

4.1 Equilibrium Characterization.

We begin by presenting necessary and sufficient conditions for a budget-exhausting pricing and PO allocation to form a CE, when each of the $n = 2$ players gets a non-empty set (in this case budget-exhaustion is wlog by Claim 1). The characterization holds beyond additive preferences for any pair of cardinal preferences. To state it we use the following standard notation: for a preference v_i and disjoint sets S, T, the *marginal value* of S given T is denoted by $v_i(S \mid T) = v_i(S \cup T) - v_i(T)$.

Proposition 4.1 (Characterization). *Given 2 players with monotone cardinal preferences v_1, v_2, consider a PO allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ in which $\mathcal{S}_i \neq \emptyset$ for $i \in \{1, 2\}$, and budget-exhausting item prices p. Then $(\mathcal{S}, p)$ forms a CE if and only if for $i, k \in \{1, 2\}$, $i \neq k$, and for every two bundles $S \subseteq \mathcal{S}_i$, $T \subseteq \mathcal{S}_k$,*

$$\begin{aligned} &v_i(S \mid \mathcal{S}_i \setminus S) > v_i(T \mid \mathcal{S}_i \setminus S) \quad \text{and} \\ &v_k(S \mid \mathcal{S}_k \setminus T) > v_k(T \mid \mathcal{S}_k \setminus T) \\ &\implies p(S) > p(T). \end{aligned} \tag{1}$$

Proof. For the first direction, assume by way of contradiction that $(\mathcal{S}, p)$ is a CE but Condition (1) is violated. Wlog assume that this is the case for $S \subseteq \mathcal{S}_1$ and $T \subseteq \mathcal{S}_2$, i.e., it holds that $v_1(S \mid \mathcal{S}_1 \setminus S) > v_1(T \mid \mathcal{S}_1 \setminus S)$ and $v_2(S \mid \mathcal{S}_2 \setminus T) > v_2(T \mid \mathcal{S}_2 \setminus T)$ while $p(S) \leq p(T)$. Then player 2 prefers to swap T for S and has enough budget to do so, in contradiction to the fact that he gets his demanded set in the CE.

For the other direction, consider a pair $(\mathcal{S}, p)$ such that Condition (1) holds for every S, T as in the proposition statement. Assume by way of contradiction that (wlog) player 2 is not allocated his demanded set. Then since player 2's budget is exhausted, this means there must be bundles $S \subseteq \mathcal{S}_1$ and $T \subseteq \mathcal{S}_2$ such that $v_2(S \mid \mathcal{S}_2 \setminus T) > v_2(T \mid \mathcal{S}_2 \setminus T)$ and $p(S) \leq p(T)$. Therefore, $v_1(S \mid \mathcal{S}_1 \setminus S) \leq v_1(T \mid \mathcal{S}_1 \setminus S)$, and so by swapping S, T in the allocation we arrive at a new allocation strictly preferred player 2, and no worse for player 1, in contradiction to the Pareto optimality of $\mathcal{S}$. □

4.2 Sufficient condition for existence.

When preferences are additive, prices can be derived from weighted linear combinations of the preferences:

Definition 4.2. Consider 2 additive preferences v_1, v_2, and parameters $\alpha, \beta \in \mathfrak{R}_+$ such that $\max\{\alpha, \beta\} > 0$. The *combination pricing* p with parameters α, β is an item pricing that assigns every item j the price $p_j = \alpha v_1(\{j\}) + \beta v_2(\{j\})$.

Observe that by additivity of v_1, v_2 in Definition 4.2, the combination pricing p with parameters α, β assigns every bundle S the price $p(S) = \alpha v_1(S) + \beta v_2(S)$. Note that identical items have identical prices (items j, j' are identical precisely if $v_i(\{j\}) = v_i(\{j'\})$ for every player i).

The following lemma presents a sufficient condition for CE existence:

Lemma 4.3 (Budget-exhausting combination pricing is sufficient). *Consider 2 players with additive preferences and budgets $b_1 \geq b_2 > 0$ (possibly equal). If for a PO allocation $\mathcal{S}$ there exists a budget-exhausting combination pricing p, then $(\mathcal{S}, p)$ is a CE.*

Proof. The existence of a budget-exhausting pricing indicates that both players are allocated nonempty bundles in $\mathcal{S}$. Thus by Proposition 4.1, to prove the lemma it is sufficient to show that Condition (1) holds. For additive preferences this condition can be written as: for every $i, k \in \{1, 2\}$, $i \neq k$, and for every two bundles $S \subseteq \mathcal{S}_i$, $T \subseteq \mathcal{S}_k$,

$$v_i(S) > v_i(T) \text{ and } v_k(S) > v_k(T) \implies p(S) > p(T).$$

Plugging in the combination pricing, for every S, T such that $v_1(S) > v_1(T)$ and $v_2(S) > v_2(T)$, it holds that $p(S) = \alpha v_1(S) + \beta v_2(S) > \alpha v_1(T) + \beta v_2(T) = p(T)$. So Condition (1) holds for any $i \neq k \in \{1, 2\}$, and $(\mathcal{S}, p)$ is a CE. □

4.3 Second welfare theorem.

A second fundamental theorem of welfare economics is of the form [40, Part III, p. 308]:

> "*[A]ny Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*"

In particular, this means that any socially-efficient allocation that a social planner deems desirable for its equitability can be realized in equilibrium. In our context, such a theorem would say that for every set of players and their preferences, for every PO allocation $\mathcal{S}$ of items among them, we can find budgets for the players and prices for the items which support $\mathcal{S}$ as a CE. From the companion paper we know that such a theorem does *not* hold for 2 players with general preferences. We use Lemma 4.3 to establish the second welfare theorem for two players with additive preferences:

Theorem 4.4 (Second welfare theorem). *Consider 2 players with additive preferences. For every PO allocation $\mathcal{S}$, there exist budgets b_1, b_2 and prices p for which $(\mathcal{S}, p)$ is a CE.*

Moreover, if both players are allocated non-empty bundles in $\mathcal{S}$, then $(\mathcal{S}, p)$ is a CE for any *combination pricing p and corresponding budgets $b_1 = p(\mathcal{S}_1)$, $b_2 = p(\mathcal{S}_2)$.*

Proof. If $\mathcal{S}$ allocates all items to a single player, wlog player 1, then we can get a CE by pricing every item at some arbitrary price $\rho > 0$, setting $b_1 = m\rho$, and setting $b_2 < \rho$ (the budgets can of course be normalized). Otherwise, fix any combination pricing p; in particular, a combination pricing with parameters $\alpha = \beta = 1$. Set $b_i = p(\mathcal{S}_i)$ for every player i. By Lemma 4.3, $(\mathcal{S}, p)$ is a CE, completing the proof. □

5 BUDGET-PROPORTIONALITY.

Section 5.1 shows that if we are fortunate enough to face a setting in which a budget-proportional allocation exists despite item indivisibility, then a CE is also guaranteed to exist (Theorem 5.2). This existence result does *not* rely on generic budgets. Section 5.2 deals with the case of no budget-proportional allocation, identifying two alternative candidates for CE allocations which are as close to budget-proportional as possible. These are the basis for the existence results in Sections 6-8.

5.1 Equilibrium existence when a budget-proportional allocation exists.

Since preferences are normalized, player i gets his budget-proportional share precisely when $v_i(\mathcal{S}_i) \geq b_i$. We say that player i gets *at most* his budget-proportional share if $v_i(\mathcal{S}_i) \leq b_i$, and call an allocation *anti-proportional* if every player gets at most his budget-proportional share and at least one player gets strictly below it (Claim 3 in Appendix B demonstrates the existence of markets in which the allocation of every CE is anti-proportional.)

We show that every budget-proportional PO allocation is supported in a CE. By the same argument, anti-proportional PO allocations are also supported in a CE. The proof is by exploiting (anti-)proportionality to construct a budget-exhausting combination pricing.

Proposition 5.1. *For 2 players with additive preferences and* any *budgets, every budget-proportional PO allocation is supported in a CE. Additionally, every anti-proportional PO allocation is supported in a CE.*

Proof. Let v_1, v_2 be the preferences and b_1, b_2 the budgets (all normalized), and let $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ be a budget-proportional PO allocation.

Since $v_1(\mathcal{S}_1) \geq b_1$, $v_2(\mathcal{S}_2) \geq b_2$ and by normalization, we have that $v_1(\mathcal{S}_2) = 1 - v_1(\mathcal{S}_1) \leq 1 - b_1 = b_2$ and $v_2(\mathcal{S}_1) = 1 - v_2(\mathcal{S}_2) \leq 1 - b_2 = b_1$.

We now construct a budget-exhausting combination pricing with parameters α, β. If it is the case that $v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) = 1$ (and thus $b_1 = v_1(\mathcal{S}_1)$ and $b_2 = v_2(\mathcal{S}_2)$), we set $\alpha = 1$ and $\beta = 0$. Otherwise, $v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1 \neq 0$, and we can set

$$\alpha = \frac{v_2(\mathcal{S}_2) - b_2}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1}, \beta = 1 - \alpha = \frac{v_1(\mathcal{S}_1) - b_1}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1}. \quad (2)$$

Observe that

$$\begin{aligned} \alpha v_1(\mathcal{S}_1) + \beta v_2(\mathcal{S}_1) &= \\ \frac{v_2(\mathcal{S}_2) - b_2}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1} v_1(\mathcal{S}_1) &+ \\ \frac{v_1(\mathcal{S}_1) - b_1}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1}(1 - v_2(\mathcal{S}_2)) &= b_1, \end{aligned}$$

and that similarly $\alpha v_1(\mathcal{S}_2) + \beta v_2(\mathcal{S}_2) = b_2$. Since the allocation is budget-proportional, it holds that $\alpha, \beta \geq 0$. In each of the cases we thus have a combination pricing p with parameters α, β such that $p(\mathcal{S}_1) = b_1$ and $p(\mathcal{S}_2) = b_2$, and thus by Lemma 4.3, $(\mathcal{S}, p)$ is a CE.

It remains to consider anti-proportional PO allocations. For every such allocation in which one player gets at most his budget-proportional share and the other gets less, the same pair of parameters α, β defined in Eq. (2) will give a budget-exhausting combination pricing with non-negative parameters, and thus a CE. □

Observe that every budget-proportional allocation is dominated by a budget-proportional PO allocation. The following theorem thus follows directly from Proposition 5.1.

Theorem 5.2. *If there exists a budget-proportional allocation then a CE exists.*

5.2 As close as possible to budget-proportionality.

Let us assume from now on that a budget-proportional allocation does not exist (and nor does an anti-proportional PO allocation). When it is not possible to simultaneously give each player her budget-proportional share, the next best thing in terms of fairness is her "truncated" share: the best she can obtain in any PO allocation in which she is allocated *at most* her budget-proportional share. In this section we formalize this notion, and show two PO allocations that give both players their truncated shares. Both of these are natural candidates for CEs, and we indeed validate this intuition in Sections 6-8. Thus, all of our positive results establish the existence of CEs for allocations that are "as fair as possible". In Example 5.6 we show that not every CE has this property.

Figure 1 depicts the setting and fixes our notation.

Definition 5.3 (**Truncated share**). Let

$$b_i^- = \max_{\mathcal{S} \in \widetilde{\text{PO}} | v_i(\mathcal{S}_i) \leq b_i} \{v_i(\mathcal{S}_i)\}$$

be the maximum share player i can obtain in any PO allocation in which she gets at most her budget-proportional share. Denote by $\hat{\mathcal{S}}^i = \hat{\mathcal{S}}^i(b_i)$ the maximizing PO allocation, i.e., $b_i^- = v_i(\hat{\mathcal{S}}_i^i)$.[5] An allocation $\mathcal{S}$ gives player i her truncated budget-proportional share, or *truncated share* for short, if $v_i(\mathcal{S}_i) \geq b_i^-$.

An analogous definition is the following:

Definition 5.4 (**Augmented share**). Let

$$b_i^+ = \min_{\mathcal{S} \in \widetilde{\text{PO}} | v_i(\mathcal{S}_i) \geq b_i} \{v_i(\mathcal{S}_i)\}$$

be the minimum share player i can obtain in any PO allocation in which she gets at least her budget-proportional share. Denote by $\check{\mathcal{S}}^i = \check{\mathcal{S}}^i(b_i)$ the minimizing PO allocation, i.e., $b_i^+ = v_i(\check{\mathcal{S}}_i^i)$. An allocation $\mathcal{S}$ gives player i her *augmented share* (and thus in particular her truncated share) if $v_i(\mathcal{S}_i) \geq b_i^+$.

The following lemma establishes a simple but important fact about the four allocations $\hat{\mathcal{S}}^i, \hat{\mathcal{S}}^k, \check{\mathcal{S}}^i, \check{\mathcal{S}}^k$, which give players i, k their truncated or augmented shares, respectively. Namely, it turns out that these four allocations are in fact two, since $\hat{\mathcal{S}}^i = \check{\mathcal{S}}^k$ and $\hat{\mathcal{S}}^k = \check{\mathcal{S}}^i$. By definition, each of these two allocations gives each player at least her truncated share.

Lemma 5.5 (Two "as fair as possible" allocations). *Consider 2 players $i \neq k \in \{1, 2\}$ with additive preferences and arbitrary budgets. Assume there are no budget-proportional allocations nor PO anti-proportional allocations. Then the PO allocation $\check{\mathcal{S}}^i$ coincides with the PO allocation $\hat{\mathcal{S}}^k$. That is, $\check{\mathcal{S}}^i$ obtains share b_i^+ for player i and share b_k^- for player k.*

[5]The allocation $\hat{\mathcal{S}}^i(b_i)$ is well-defined: it is possible to give nothing to player i, so the maximum is taken over a non-empty set of allocations.

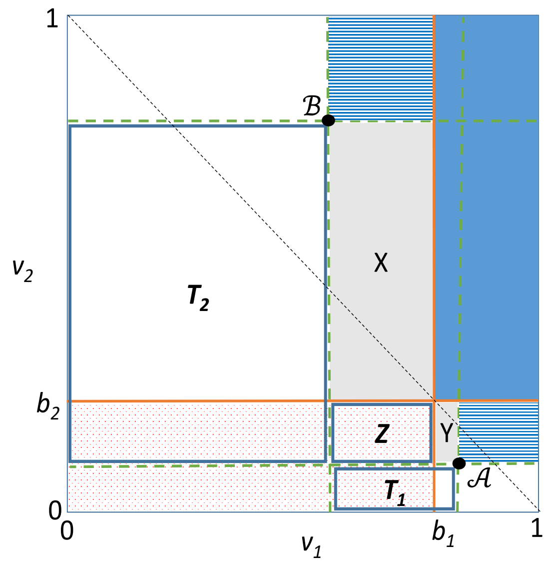

Figure 1: This figure illustrates the setting and notation when neither a budget-proportional nor an anti-proportional PO allocation exists. It shows the value of an allocation for player 1 on the v_1-axis and the value of an allocation for player 2 on the v_2-axis. Every allocation $\mathcal{S}$ can be represented by the point $(v_1(\mathcal{S}_1), v_2(\mathcal{S}_2))$ on the $(v_1 \times v_2)$-plane. The two points $\mathcal{A}, \mathcal{B} \in \widetilde{\text{PO}}$ represent two PO allocations. The players' budgets b_1, b_2 are shown on the same axes.
The closure of the solid blue area (at or above b_2 and at or to the right of b_1) includes all allocations that are budget-proportional, and is empty by assumption. The closure of the red dotted area represents anti-proportional allocations and has no PO allocations by assumption. By Pareto optimality, the blue striped areas to the right and above $\mathcal{A}$ and $\mathcal{B}$ are both empty (the only allocation in their closures are $\mathcal{A}$ and $\mathcal{B}$). The figure also depicts rectangles T_1, T_2, X, Y, and Z, which will play a role in our arguments in Sections 6-8.

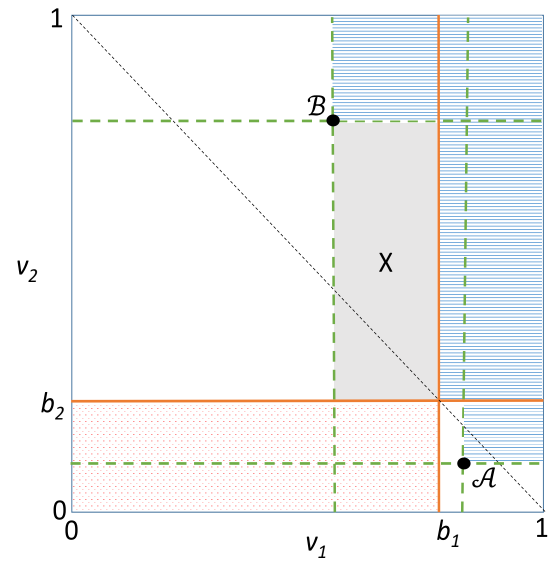

Figure 2: Illustration of the proof of Lemma 5.5.

Proof. Let $k = 1, i = 2$ (the complementary case $k = 2, i = 1$ is similar). Denote $\mathcal{A} = \hat{\mathcal{S}}^1$ and $\mathcal{B} = \hat{\mathcal{S}}^2$. The notation we use for the proof is depicted in Figure 2, where indeed in allocation $\mathcal{A}$ player 1 can be seen to receive value above b_1, and in allocation $\mathcal{B}$ player 2 can be seen to receive value above b_2. We now use Figure 2 to argue that $\mathcal{B} = \hat{\mathcal{S}}^1$ (showing that $\mathcal{A} = \hat{\mathcal{S}}^2$ is similar).

By assumption, the closure of the blue striped area is empty of allocations except for $\mathcal{A}$ and $\mathcal{B}$ (recall Figure 1). By definition, $\mathcal{B}$ is the lowest PO allocation at or above b_2. Any PO allocation at or to the left of b_1 that is to the right of $\mathcal{B}$ must be in the interior of the gray rectangle X (using that there are no PO anti-proportional allocations, i.e., no PO allocations in the closure of the dotted red area). Yet such a point in the interior of X is not only to the right of $\mathcal{B}$, it is also below it. This means that it is closer than $\mathcal{B}$ to b_2 from above, yielding a contradiction.

The closed rectangle X must therefore be empty of PO allocations except for $\mathcal{B}$ (and thus must also be empty of non-PO allocations). But the point corresponding to allocation $\hat{\mathcal{S}}^1$ must fall within the closure of rectangle X by definition (it is the rightmost PO point at or to the left of b_1, and it cannot lie to the left of $\mathcal{B}$). Thus $\hat{\mathcal{S}}^1 = \mathcal{B}$, completing the proof. □

The following example demonstrates that not every CE gives every player her truncated share.

Example 5.6 (Less fair than possible). Consider 2 additive players who both value items (A, B, C, D) at $(7.9, 1, 5, 2)$ (unnormalized). The budgets are $b_1 = \frac{1}{2}+\epsilon$, $b_2 = \frac{1}{2}-\epsilon$ for some sufficiently small ϵ. Every allocation is PO. The allocation $(\{B, C, D\}, \{A\})$ is an equilibrium allocation in which player 2 gets a share of $\frac{7.9}{15.9} < b_2$. The allocation $(\{A, B\}, \{C, D\})$ is also an equilibrium allocation, despite the fact that player 2's share drops to $\frac{7}{15.9}$.[6]

6 MAIN TECHNICAL TOOL.

In this section we build our main technical tool for establishing generic existence of CEs for players with almost equal budgets (Section 7), and for players with the same preferences and different budgets (Section 8). This tool is formalized in Lemma 6.3, which establishes two conditions that together are sufficient for a CE to exist. Namely, for a fixed player i, the conditions are:

(1) Genericity of the budgets, defined as *not* belonging to a zero-measure subset R_i;
(2) Emptiness of "rectangle" T_i from any allocations (see Fig. 1).

The genericity condition is what drives our existence results, and is therefore to be expected. The condition that T_i is empty, however, is a necessary artifact of our proof techniques.[7] Dropping this

[6]The supporting prices for the two allocations are, respectively, $p = (\frac{1}{2}-\epsilon, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}+\epsilon)$ and $p = (\frac{1+\epsilon}{2}, \frac{\epsilon}{2}, \frac{1}{4}, \frac{1}{4}-\epsilon)$.
[7]Our simulations identified an example in which for each player i, T_i is not empty, and there is no CE with item prices based on scaling $v_i(\{j\})$ (other CEs were found). The example includes 7 items. Player 1's values

condition requires novel ideas, and we leave this as an open question for future research.

6.1 Definitions.

We now formally define R_i, T_i. For the definition of T_i, recall the allocations $\hat{\mathcal{S}}^i, \hat{\mathcal{S}}^k$ (Definition 5.3).

Definition 6.1 (Rectangle of allocations T_i). Let $T_i = T_i(b_i, v_1, v_2)$ be the set of allocations $\mathcal{S}$ satisfying $v_i(\hat{\mathcal{S}}^i_i) < v_i(\mathcal{S}_i) < v_i(\hat{\mathcal{S}}^k_i)$ and $0 < v_k(\mathcal{S}_k) < v_k(\hat{\mathcal{S}}^k_k)$.

For the definition of R_i, let $d = |\widetilde{\text{PO}}|$ be the total number of PO allocations. Order all allocations in $\widetilde{\text{PO}}$ by player i's preference, such that his r-th least preferred PO allocation is at index $r \le d$. Denote this allocation by $\mathcal{S}(r)$, so that $v_i(\mathcal{S}(r+1)_i) > v_i(\mathcal{S}(r)_i)$ for every index $r \le d-1$.

Definition 6.2 (Zero-measure subset of budgets R_i). Every budget pair $(b_i, 1-b_i)$ for players (i,k), respectively, belongs to $R_i = R_i(v_1, v_2)$ iff there exists an index r such that $\frac{b_i}{v_i(\mathcal{S}(r+1)_i)} = \frac{1-b_i}{1-v_i(\mathcal{S}(r)_i)}$.

Note that R_i is a zero-measure subset of the budget pairs.

6.2 Statement and proof.

Lemma 6.3 (Main technical tool). *Consider 2 players with additive preferences v_1, v_2 and budgets $b_1 > b_2$. Assume there are no budget-proportional allocations nor PO anti-proportional allocations. If for some player i, $(b_1, b_2) \notin R_i(v_1, v_2)$ and the set $T_i = T_i(b_i, v_1, v_2)$ is empty, then a CE exists. Moreover, in this CE every player gets his truncated share.*

For space considerations the proof is deferred to Appendix A.

7 EXISTENCE FOR ALMOST EQUAL BUDGETS.

In this section we present our main result for almost equal budgets: the generic existence of a CE for 2 additive players who are a priori equal. The genericity of the budgets serves as a tie-breaking mechanism among the players, and is sufficient to ensure CE existence for any number of items. The proof utilizes Lemma 6.3.

Theorem 7.1. *Consider 2 players with additive preferences and budgets $b_1 > b_2$. For sufficiently small $\epsilon > 0$, if $b_1 - b_2 \le \epsilon$ then there exists a CE that gives every player his truncated share.*

For space considerations the proof is deferred to Appendix A.

8 EXISTENCE FOR DIFFERENT BUDGETS.

In this section we present our main result for different budgets: the generic existence of a CE for 2 additive players *with the same preferences* who can be a priori *non*-equal (and in fact quite different) in their entitlement to the items.

Theorem 8.1. *Consider 2 players with additive preferences and budgets $b_1 > b_2$, such that the pair (b_1, b_2) does not belong to the zero-measure subset R_i (Def. 6.2) for some player i. If the players have the same preferences then there exists a CE that gives every player his truncated share.*

When both players share the same additive preferences, we have a "constant-sum game" – whatever one player gains the other loses. As a consequence, every allocation among such players is PO, and in addition there is no anti-proportional allocation, in which both players would get at most their truncated share and one of them strictly so. Theorem 8.1 thus follows directly from the next lemma, whose proof utilizes Lemma 6.3:

Lemma 8.2. *Consider 2 players with additive preferences and budgets $b_1 > b_2$, such that the pair (b_1, b_2) does not belong to the zero-measure subset R_i (Def. 6.2) for some player i. If every allocation among the players is PO then there exists a CE. Moreover, if there is no anti-proportional PO allocation then this CE gives each player his truncated share.*

Proof. If there exists a budget-proportional or anti-proportional PO allocation, then there exists a CE by Proposition 5.1. In the former case, this CE clearly gives each player his truncated share. Otherwise, the conditions of Lemma 6.3 hold: any allocation in T_1 would be dominated by $\hat{\mathcal{S}}^2$, and any allocation in T_2 would be dominated by $\hat{\mathcal{S}}^1$, but since there are no Pareto dominated allocations these two rectangles must be empty. There thus exists a CE in which every player gets at least his truncated share, completing the proof. □

9 SUMMARY.

Fairness notions applicable to the allocation of indivisible items among players with different entitlements are an under-explored area of theory. In this paper we develop new such notions through a classic connection to competitive equilibrium. For the question of equilibrium existence, our main conceptual contribution is to show that for interesting classes of 2-player additive markets, it is sufficient to exclude degenerate market instances to get existence. This is done by adding small noise to the players' budgets to make them generic (in the spirit of smoothed analysis for getting positive computational tractability results [52]). Unlike Budish's approach [17], there is no need to relax the equilibrium notion, and our model allows for possibly very different budgets. In a companion paper we establish additional existence results and we expect more to be added to this list in the future. Many exciting open directions remain, including:

- What is the appropriate notion of envy-free fairness (as opposed to fair share) for players with different entitlements and indivisible items? For a preliminary discussion see Appendix B.
- For which classes of markets are fair allocations according to our notions guaranteed to exist?
- How far can we push CE generic existence? E.g., when preferences are the same does it hold for more than 2 additive players?

For the last two questions, the widespread existence for simulated and real data (Appendix C) seems suggestive.

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are $(0.1420, 0.0808, 0.1921, 0.1717, 0.1651, 0.1200, 0.1283)$, player 2's values are $(0.0827, 0.1056, 0.1743, 0.1515, 0.1862, 0.1123, 0.1874)$. The budgets are 0.8093 and 0.1907.

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A MISSING PROOFS.

Proof of Lemma 6.3. Let i be the player for which the conditions of the lemma hold. By Lemma 5.5, both PO allocations $\check{\mathcal{S}}^1$ and $\check{\mathcal{S}}^2$ give both players their truncated share. To prove the claim it is thus sufficient to show that at least one of these allocations is supported in a CE. We next show that indeed for some $\gamma \in (0,1)$, at least one of these two allocations is supported by item prices of the form $p_j = \gamma v_i(\{j\})$ for every item j.

We first characterize the set of allocations that are within the budget of each player when prices are set to $p_j = \gamma v_i(\{j\})$ for every item j, and $\gamma \in (0,1)$ (i.e., prices are a linearly scaled down version of player i's valuation). Player i can afford any allocation $\mathcal{S}$ such that $\gamma v_i(\mathcal{S}_i) \le b_i$. Player k can afford any allocation $\mathcal{S}$ such that $\gamma v_i(\mathcal{S}_k) \le b_k$, or equivalently $\gamma(1 - v_i(\mathcal{S}_i)) \le 1 - b_i$ (using that both valuations and budgets are normalized, that is, $b_1 + b_2 = v_1(M) = v_2(M)$). We illustrate this for $\gamma = 1$ in Figure 3.

Now define

$$\gamma_i = \max\left\{\frac{b_i}{v_i(\check{\mathcal{S}}^i_i)}, \frac{1-b_i}{1-v_i(\check{\mathcal{S}}^k_i)}\right\} = \max\left\{\frac{b_i}{b_i^+}, \frac{b_k}{v_i(\check{\mathcal{S}}^k_k)}\right\} < 1,$$

and note that γ_i is well-defined and less than 1. The assumption that the pair of budgets does not belong to $R_i(v_1, v_2)$ implies that the maximum is obtained by only one of the terms. The proof follows by analyzing two cases, as illustrated in Figures 4 and 5, respectively:

- **Case 1.** $\gamma_i = b_i/b_i^+$. We show that $\check{\mathcal{S}}^i$ is supported by item prices $p_j = \gamma_i v_i(\{j\})$. For player i, every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $v_i(\mathcal{S}_i) \le b_i/\gamma_i$, and this holds with equality for $\check{\mathcal{S}}^i_i$. For player k, every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $(b_i/b_i^+)v_i(\mathcal{S}_k) \le 1 - b_i$. Since we are in the case that $b_i/b_i^+ > (1-b_i)/(1-v_i(\check{\mathcal{S}}^k_i))$, we derive:

$$\frac{b_i}{b_i^+}(1 - v_i(\mathcal{S}_i)) = \frac{b_i}{b_i^+}v_i(\mathcal{S}_k) \le 1 - b_i < \frac{b_i}{b_i^+}(1 - v_i(\check{\mathcal{S}}^k_i)),$$

  or equivalently $v_i(\mathcal{S}_i) > v_i(\check{\mathcal{S}}^k_i)$. We claim that player k's most preferred allocation that satisfies this is $\check{\mathcal{S}}^i$: By Lemma 5.5 it holds that $\check{\mathcal{S}}^i = \hat{\mathcal{S}}^k$, i.e., $\check{\mathcal{S}}^i$ is also the PO allocation in which player k gets at most b_k while maximizing his share. Since there is no allocation $\mathcal{S}$ in which $v_i(\mathcal{S}_i) > v_i(\check{\mathcal{S}}^k_i)$ and $v_k(\mathcal{S}_k) > v_k(\check{\mathcal{S}}^i_k)$, the claim follows.
- **Case 2.** $\gamma_i = b_k/v_i(\check{\mathcal{S}}^k_k)$. We show that $\check{\mathcal{S}}^k$ is supported by item prices $p_j = \gamma_i v_i(\{j\})$. For player k, every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $v_i(\mathcal{S}_k) \le b_k/\gamma_i$, and this holds as equality for $\check{\mathcal{S}}^k_k$. For player i, every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $v_i(\mathcal{S}_i) \le b_i/\gamma_i < \frac{b_i}{b_i/v_i(\check{\mathcal{S}}^i_i)} = v_i(\check{\mathcal{S}}^i_i)$, as $b_i/v_i(\check{\mathcal{S}}^i_i) < \frac{1-b_i}{1-v_i(\check{\mathcal{S}}^k_i)} = \gamma_i$. Since T_i is empty, it cannot be the case that $v_i(\check{\mathcal{S}}^k_i) < v_i(\mathcal{S}_i) < v_i(\check{\mathcal{S}}^i_i)$. Thus the most preferred allocation that player i can afford gives him at most $v_i(\check{\mathcal{S}}^k_i)$. This is indeed what he gets in allocation $\check{\mathcal{S}}^k$, thus $\check{\mathcal{S}}^k_i$ is demanded by player i.

□

Proof of Theorem 7.1. **Case 1.** Assume first that there exists an allocation which gives each player a value of exactly $1/2$. Then

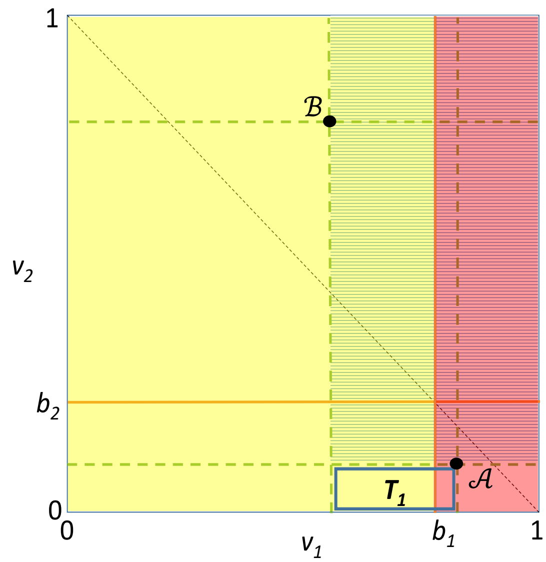

Figure 3: This figure illustrates the first part of the proof of Lemma 6.3, using the notation of Figures 1 and 2. It shows the allocations that each player can afford given his budget when prices are $p = v_1$ (i.e., according to player 1's valuation). Allocations in the yellow rectangle (at or to the left of b_1) have value at most b_1 for player 1, and thus also price at most b_1, so player 1 can afford them. Allocations in the red rectangle (at or to the right of b_1) have value at least b_1 for player 1, and thus player 1 values player 2's allocation at most at $1 - b_1 = b_2$ (by normalization), so the price is at most b_2 and affordable for player 2.
The blue striped area marks allocations with value for player 1 that is above his value for $\mathcal{B}$, and value for player 2 that is above his value for $\mathcal{A}$. This area has no allocation at all, as it is subset of the union of the following areas: the blue areas from Figure 1 without allocations below $\mathcal{B}$ and to the left of $\mathcal{A}$; the interiors of X and Y that are empty by the proof of Lemma 5.5 in Figure 2; and the interior of Z that must be empty, as an allocation there must be dominated by some PO allocation in the areas we just argued are empty, or by an anti-proportional PO allocation (which does not exist by assumption).
Therefore, if rectangle T_1 is empty then at these prices player 1 demands the allocation $\mathcal{B} = \check{\mathcal{S}}^2$ (the rightmost allocation within the yellow area – his budget), while player 2 demands the allocation $\mathcal{A} = \check{\mathcal{S}}^1$ (the highest allocation within the red area – his budget).

there is a PO allocation $\mathcal{S}$ that gives each player at least $1/2$. Such an allocation is budget-proportional for $b_1 = b_2 = 1/2$, and thus by Proposition 5.1, there exists a CE $(\mathcal{S}, p)$. For sufficiently small $\epsilon > 0$, let $b_1 = (\frac{1}{2} + \epsilon)/(1+\epsilon) > \frac{1}{2}$ and $b_2 = 1 - b_1$ (that is, we slightly increase the budget of player 1 while normalizing the sum $b_1 + b_2$ to 1). We claim that $(\mathcal{S}, p)$ is also a CE with the perturbed budgets. Indeed, as prices have not changed, player 2 gets his demand. As

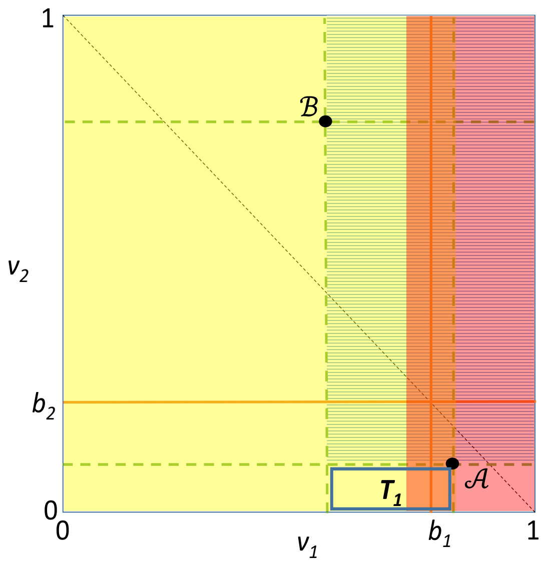

Figure 4: This figure illustrates Case 1 in the proof of Lemma 6.3, using the notation of Figures 1 and 2, for player $i = 1$. Prices are $p_j = \gamma_1 v_1(\{j\})$ for every item j, where $\gamma_1 = b_1/b_+^1$. The yellow and red rectangles are the allocations that players 1 and 2 can afford, respectively. Both players can afford more allocations than when $\gamma = 1$ (cf. Figure 3). The overlap (the closure of the orange rectangle) contains the allocations that both players can afford. The value of γ_1 is such that player 1 can exactly afford allocation $\mathcal{A}$, which is clearly demanded by him at these prices (the rightmost allocation within his budget). We show in the proof that player 2 cannot yet afford allocation $\mathcal{B}$ and any other allocation that gives him the same value, and so allocation $\mathcal{A}$ is in his demand (the highest allocation within his budget, using that the blue striped area above $\mathcal{A}$ that is within his budget, is empty). Thus $(\mathcal{A}, p)$ is a CE.

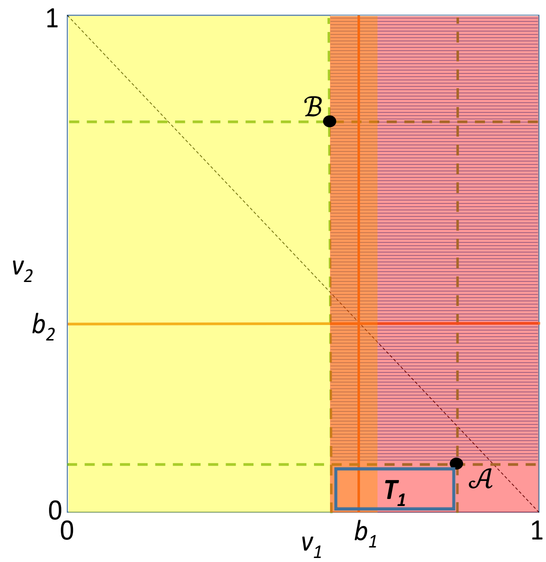

Figure 5: This figure illustrates Case 2 in the proof of Lemma 6.3, using the notation of Figures 1 and 2, for player $i = 1$. Prices are $p_j = \gamma_1 v_1(\{j\})$ for every item j where $\gamma_1 = b_2/v_1(\check{\mathcal{S}}_2^2)$. The yellow and red rectangles are as in Figure 4. The value of γ_1 is such that player 2 can exactly afford allocation $\mathcal{B}$, which is clearly demanded by him at these prices (the highest allocation within his budget, using that the blue striped area is empty and that $\mathcal{B}$ is PO). We show in the proof that player 1 cannot yet afford allocation $\mathcal{A}$, and so allocation $\mathcal{B}$ is in his demand (the rightmost allocation within his budget, using the fact that within his budget, there are no allocations that are also in the blue striped area right of $\mathcal{B}$ or in T_1). Thus $(\mathcal{B}, p)$ is a CE.

for player 1, while his budget is slightly larger, he cannot afford any set that is more expensive than his set $\mathcal{S}_1$, provided ϵ is smaller than the difference in prices of any two bundles with non-identical prices.

Case 2. Consider now the complementary case, in which no allocation gives each player a value of exactly $1/2$. **Case 2(a).** If there is an allocation that gives both players strictly more than $1/2$, consider any PO allocation that dominates it. For sufficiently small ϵ, the PO allocation is budget-proportional for any budgets $b_1 > b_2 \geq b_1 - \epsilon$, and so the result follows from Proposition 5.1. Note that if there is an allocation that gives both players strictly less than $1/2$ then the allocation in which the two players swap their bundles gives both players more than $1/2$.

Case 2(b). From now on we assume that every allocation gives strictly more than $1/2$ to one player, and strictly less than $1/2$ to the other player. For sufficiently small ϵ, such an allocation is neither budget-proportional nor anti-proportional for any budgets $b_1 > b_2 \geq b_1 - \epsilon$.

Recall from Definition 5.3 that $\hat{\mathcal{S}}^1(b_1), \hat{\mathcal{S}}^2(b_2)$ are PO allocations that give players 1, 2 their truncated shares. As the set of PO allocations is finite, we can find $\epsilon > 0$ such that there is no PO allocation $\mathcal{S}$ such that $1/2 - 2\epsilon < v_1(\mathcal{S}_1) < 1/2 + 2\epsilon$. For such an ϵ, consider budgets $b_1 > b_2 \geq b_1 - \epsilon$. Using the notation of Figure 2, let $\mathcal{A} = \hat{\mathcal{S}}^2(b_2)$ and $\mathcal{B} = \hat{\mathcal{S}}^1(b_1)$.

We first claim that $\mathcal{A} = \hat{\mathcal{S}}^2(1/2)$ and $\mathcal{B} = \hat{\mathcal{S}}^1(1/2)$. This holds because $\mathcal{B}$ is the PO allocation that gives the largest value to player 1 that is below $1/2 - 2\epsilon$, but there are no PO allocations that give player 1 value between $1/2 - 2\epsilon$ and $1/2$, and thus it also gives the largest value to player 1 that is below $1/2$. A similar argument holds for $\mathcal{A}$ and the truncated share of player 2.

Because every allocation gives strictly more than $1/2$ to one player and strictly less to the other, $v_2(\mathcal{A}_2) < 1/2 \implies v_1(\mathcal{A}_1) > 1/2$, and $v_1(\mathcal{B}_1) < 1/2 \implies v_2(\mathcal{B}_2) > 1/2$.

We now show that $\mathcal{A}$ and $\mathcal{B}$ are "symmetric" in the sense that one is obtained from the other by swapping bundles among the players (and so $v_1(\mathcal{A}_1) = v_1(\mathcal{B}_2) = 1 - v_1(\mathcal{B}_1)$ and $v_2(\mathcal{A}_2) = 1 - v_2(\mathcal{B}_2)$, as can be seen in Figure 6). The proof of the symmetry claim is depicted in Figure 6, which also shows that T_1, T_2 (as defined in Definition 6.1) are both empty. The budgets b_1, b_2 are almost equal,

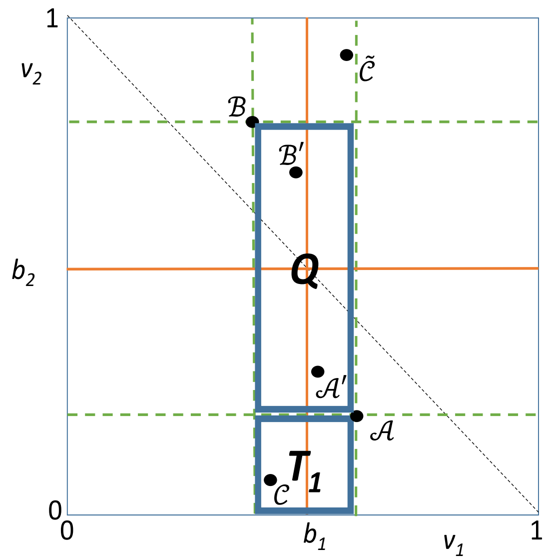

Figure 6: Illustration of the proof of Theorem 7.1.

i.e., both are very close to 1/2. For every allocation $\mathcal{S}$ there is a "symmetric" allocation $\tilde{\mathcal{S}}$ obtained by swapping the allocated bundles, which corresponds to a 180° rotation around the point (1/2, 1/2). Assume for contradiction that $\mathcal{A}$ is not symmetric to $\mathcal{B}$, that is $\mathcal{B} \neq \tilde{\mathcal{A}} = \mathcal{B}'$. Then it is also the case that $\mathcal{A} \neq \tilde{\mathcal{B}} = \mathcal{A}'$. Notice that for sufficiently small ϵ, one of $\mathcal{A}', \mathcal{B}'$ must be located in the interior of the axes-parallel rectangle Q as illustrated in the figure. But since $\mathcal{A} = \hat{\mathcal{S}}^2(b_2)$ gives player 2 his truncated share closest to $b_2 \approx 1/2$ from below, $\mathcal{A}' \neq \mathcal{A}$ cannot be located as in the figure. Similarly, $\mathcal{B}' \neq \mathcal{B}$ cannot be located in the interior of Q as in the figure, a contradiction. We have thus established that $\mathcal{A}, \mathcal{B}$ are symmetric. We now show that the closure of T_1 must be empty, except for $\mathcal{A}$: If that were not the case – say, T_1 contained an allocation $C \neq \mathcal{A}$ – then its symmetric allocation $\tilde{C}$ would Pareto dominate $\mathcal{B}$ (due to the symmetry of $\mathcal{A}, \mathcal{B}$), a contradiction. T_2 only contains the allocation $\mathcal{B}$ by a similar argument using the Pareto optimality of $\mathcal{A}$.

We can set ϵ to be sufficiently small such that $(b_1, b_2) \notin R_i(v_1, v_2)$. The proof is complete by invoking Lemma 6.3. □

B CE AND FAIRNESS.

B.1 Detailed fairness preliminaries.

The discussion in this section is summarized in Tables 1 and 2, which also show where our fairness concepts fit in with some of the existing concepts.

B.1.1 Ordinal preferences. A player $i \in N$ has an *ordinal* preference $\prec_i$ among bundles of items if $S \prec_i T$ for every T preferred to S, and there is no representation of the preference via a numerical valuation function v_i. Fair share and envy-freeness are well-defined for ordinal preferences when items are *divisible*. Intuitively, fair share guarantees that each player believes he receives at least $1/n$ of the "cake" being divided, and envy-freeness guarantees he believes no one else receives a better slice than him. More formally, *fair share* (FS) requires for each player to receive a bundle that he prefers at least as much as the bundle consisting of a $1/n$-fraction of every item on the market. Player i *envies* player k given allocation $\mathcal{S}$ if $S_i \prec_i S_k$, and an allocation is *envy-free* (EF) if no player envies another player.

Table 1: Fair share and envy-free notions for ordinal preferences (our contribution in bold).

	Equal budgets	Arbitrary budgets
Divisible items	Fair share (FS) [53] Envy-freeness (EF) [31]	Budget-FS Budget-EF[a]
Indivisible items	1-out-of-n MMS [17] EF-1 [17], EF-1* [19]	**ℓ-out-of-d MMS (Definition 2.4)** Justified-EF (Definition B.5), open

[a]Budget-FS and budget-EF are natural generalizations of FS and EF, defined in Appendix B.1.

Table 2: Fair share notions for cardinal preferences (our contribution in bold).

	Equal budgets	Arbitrary budgets
Divisible items	Proportionality [53]	Budget-proportionality [11, 50]
Indivisible items	Proportionality [53]	**Truncated share (Def. 5.3)**

We observe that for divisible items, both notions extend naturally to budgeted players: *Budget-FS* requires every player to prefer his bundle at least as much as a b_i-fraction of the bundle of all items. Player i envies player k with a larger budget if he prefers a b_i/b_k-fraction of S_k to S_i, and player k' with a smaller budget if he prefers $S_{k'}$ to a $b_{k'}/b_i$-fraction of S_i; the *budget-EF* property excludes such envy.

When items are indivisible, FS is not well-defined; EF is well-defined but often cannot be satisfied [26]. To circumvent the definition and existence issues stemming from indivisibilities, Budish [17] proposes appropriate variants: 1-out-of-n maximin share (Definition 2.4) and envy-free up to one good (Definition B.1). Caragiannis et al. [19] introduce a strengthening of EF-1 called EF-1* (or EFX), envy-freeness up to *any* good, in which for every players i, k and *any* item $j \in S_k$ it holds that $S_k \setminus \{j\} \prec_i S_i$.

We discuss how to generalize the maximin share guarantee for budgeted players in Section 3, defining the notion of ℓ-out-of-d maximin share (Definition 2.4). We also define a notion called justified-EF for budgets (Definition B.5), and leave the question of how to generalize (non-justified) EF to budgets as an open direction.

Application to CEs. It is well-known that every CE with n equal budgets gives every player his 1-out-of-n maximin share and achieves EF. Budish [17] shows that every CE with almost equal budgets guarantees 1-out-of-$(n + 1)$ maximin share (whereas 1-out-of-n

maximin share cannot always be guaranteed). Proposition 3.3 generalizes this to arbitrary budgets using the notion of ℓ-out-of-d maximin share.

Budish [17] also shows that every CE with almost equal budgets is EF-1 (a short proof appears for completeness in Proposition B.2). We demonstrate that a CE with almost equal budgets is not necessarily EF-1* (Claim 4), but any CE with arbitrary budgets is justified-EF (Claim 5).

B.1.2 Cardinal preferences. For cardinal preferences, the parallel of FS is the notion of proportionality, which extends naturally to players with different budgets: Given a budget profile b, an allocation $\mathcal{S}$ gives player i his *budget-proportional share* if player i receives at least a b_i-fraction of his value for all items, that is $v_i(\mathcal{S}_i) \geq b_i \cdot v_i(M)$. An allocation is *budget-proportional* (a.k.a. weighted-proportional) if every player receives his proportional share. When all budgets are equal, such an allocation is simply called *proportional*. Budget-proportional allocations play a central role in our positive results in Section 5. It is clear that a budget-proportional allocation does not always exist with indivisible items (e.g., with a single one). In Section 5.2 we present a relaxation of budget-proportional share that we call truncated share, which is guaranteed to exist (Definition 5.3).

Budget-EF also extends naturally to players with cardinal preferences and budgets, by saying that player i envies player k if $v(S_i) < b_i \cdot v_i(S_k)/b_k$. It is not hard to see that budget-EF implies budget-proportionality. Another fairness notion for cardinal preferences which naturally extends to budgets (e.g., [14]) is *Nash social welfare* maximization. Given a budget profile b, an allocation $\mathcal{S}$ is Nash social welfare maximizing if it maximizes $\prod_i (v_i(\mathcal{S}_i))^{b_i}$, or equivalently, $\sum_i b_i \log v_i(\mathcal{S}_i)$, among all allocations (notice that the maximizer is invariant to budget or valuation scaling).

Application to CEs. In all settings for which we prove CE existence in Sections 7 and 8, the CEs guarantee every player his truncated share. A CE with equal budgets maximizes the (unweighted) Nash social welfare; in contrast, we show a simple market with different budgets in which a CE exists, but the allocation that maximizes Nash social welfare is not supported by a CE:

Claim 2. *There exists a market of 2 items and 2 players with additive preferences and unequal budgets, such that the unique PO allocation that maximizes (weighted) Nash social welfare is not supported in a CE, but a CE exists.*

Proof. Proof. Consider 2 players Alice and Bob with budgets 101, 100, and 2 items A, B valued by Alice $v_1(A) = 5, v_1(B) = 4$ and by Bob $v_2(A) = 1000, v_2(B) = 1$. Clearly the only CE gives item A to Alice and item B to Bob (since Bob cannot prevent Alice from taking item A even if he pays his full budget for it), but the only allocation maximizing the Nash social welfare (i.e., maximizing $101 \log v_1(\mathcal{S}_1) + 100 \log v_2(\mathcal{S}_2)$) gives item A to Bob and item B to Alice. □

Claim 3. *There exists a market of 2 items and 2 players with unequal budgets, for which a CE exists and every CE allocation is anti-proportional.*

Proof. Proof. Let $b_1 = 5/8, b_2 = 3/8, v_1(A) = 100, v_1(B) = 101, v_2(A) = 1, v_2(B) = 1000$. Since $b_1 > b_2$, in every CE player 1 gets his preferred item, item B. Moreover, he cannot get both items, as $b_2 > b_1/2$ so player 2 can always afford at least one item. So in every CE, player 1 gets item B and player 2 gets item A, and their shares are $100/201 < 5/8$ and $1/1001 < 3/8$, respectively. Equilibrium prices that support this allocation are $p(A) = 3/8, p(B) = 5/8$. □

B.2 Variants of envy-freeness.

In this section we discuss variants of envy-freeness, and define a notion called justified-EF for budgets (Definition B.5). We leave the question of how to generalize (non-justified) EF to budgets as an open direction.

Definition B.1 (EF with indivisibilities [17]). An allocation $\mathcal{S}$ is *envy-free up to one good* (EF-1) if for every two players i, k, for *some* item $j \in S_k$ it holds that $S_k \setminus \{j\} \prec_i S_i$.

Budish [17] establishes an envy-free property of CEs with almost equal budgets, as follows (we include a short proof for completeness):

Proposition B.2 (Budish [17]). *Consider a CE $(\mathcal{S}, p)$ with almost equal budgets $b_1 \geq b_2 \geq \cdots \geq b_n \geq \frac{m-1}{m} b_1$, then the CE allocation $\mathcal{S}$ is* EF-1.

Proof. Proof. Fix any two players i, k. We show that there is some item $j^* \in \mathcal{S}_k$ such that $\mathcal{S}_i \succ_i \mathcal{S}_k \setminus \{j^*\}$. By Claim 1, we may assume without loss of generality that the budget of player k is exhausted. So there exists some item $j^* \in \mathcal{S}_k$ such that its price p_{j^*} is at least $b_k/|\mathcal{S}_k| \geq b_k/m$. The price of $\mathcal{S}_k \setminus \{j^*\}$ is therefore at most $\frac{m-1}{m} b_k \leq b_n \leq b_i$, and so i can afford $\mathcal{S}_k \setminus \{j^*\}$. Since $\mathcal{S}$ is a CE allocation and bundle $\mathcal{S}_k \setminus \{j^*\}$ is within i's budget, $\mathcal{S}_i \succ_i \mathcal{S}_k \setminus \{j^*\}$ as needed. □

A requirement stronger than (implying) EF-1 and weaker than (implied by) EF is the following:

Definition B.3 (Caragiannis et al. [19], Definition 4.4). An allocation $\mathcal{S}$ is EF-1* if for every two players i and k, for *every* item $j \in \mathcal{S}_k$ it holds that $\mathcal{S}_k \setminus \{j\} \prec_i \mathcal{S}_i$.

An EF-1* allocation always exists for 2 players – the cut-and-choose procedure from cake-cutting results in such an allocation. It is an open question whether it always exists in general. We demonstrate (by an example with non-strict preferences) that an EF-1* allocation is not necessarily EF even when an EF allocation exists in the market:

Example B.4. Consider 3 symmetric additive players, 22 "small" items worth 1 each, and 2 "large" items worth 7 each. An EF allocation is two bundles of 1 large item and 5 small items each, and one bundle of 12 small items. An EF-1* allocation that is not EF is one bundle of 2 large items, and two bundles of 11 small items each.

While Proposition B.2 shows that a CE implies EF-1 for almost equal budgets, we next prove that a CE does not imply the stronger property EF-1*.

Claim 4. *The allocation of a CE from almost equal budgets is not necessarily* EF-1**, even for 2 symmetric players with an additive preference over 4 items.*

Proof. Proof. The proof follows from Example 5.6. Recall that the allocation $(\{A, B\}, \{C, D\})$ is a CE allocation, but it is not EF-1*: player 2 envies player 1 even if he gives up item B. □

We conclude this section by defining a notion of envy-freeness that a CE with different budgets guarantees. Borrowing from the matching literature, we define *justified envy* as the envy of a player with a higher budget towards a player with a lower budget. The intuition is that any envy of a lower-budget player towards the allocation of a higher-budget player isn't justified and so "doesn't count", because the higher-budget player "deserves" a better allocation. We thus only care about eliminating justified envy.

Definition B.5. An allocation $\mathcal{S}$ is *justified-EF* given budgets $b_1 \geq \cdots \geq b_n$ if for every two players $i < k$, player i (with the higher budget) does not envy player k (with the lower budget). An allocation $\mathcal{S}$ is justified-EF *for coalitions* if for every player i and set of players K such that $i \notin K$ and $b_i \geq \sum_{k \in K} b_k$, player i does not envy K, i.e., $\bigcup_{k \in K} \mathcal{S}_k \prec_i \mathcal{S}_i$.

Claim 5. *Every CE allocation (with possibly very different budgets) is justified-EF for coalitions.*

Proof. Proof. Assume that $b_i \geq \sum_{k \in K} b_k$. Since the total price $\sum_{k \in K} p(\mathcal{S}_k)$ is at most $\sum_{k \in K} b_k$, player i can afford the bundle $\bigcup_{k \in K} \mathcal{S}_k$. Because $\mathcal{S}$ is a CE allocation, it must hold that $\bigcup_{k \in K} \mathcal{S}_k \prec_i \mathcal{S}_i$. □

C COMPUTERIZED SEARCH FOR EQUILIBRIA.

We attempted to find a market with no CEs for additive as well as general preferences. The instances examined were either randomly generated by sampling from distributions, or taken from real-world Spliddit data. The computational results suggest that CE existence is a wider phenomenon than theoretically verified at this point.

C.1 Two players with randomly sampled preferences.

Setup. Our computerized search ran on instances with between 4 and 8 items and 2 players with randomly generated preferences.[8] We generated both random additive preferences, where the values for the items were drawn from the uniform or Pareto distributions and then normalized to sum up to 1, as well as random general monotone preferences. To generate the monotone preferences we randomly picked an order for all singletons, then randomly placed all pairs among the singletons while maintaining monotonicity, then placed all triplets and so on.

In choosing budgets for the random additive instances, our goal was to avoid instances for which we know from Lemma 6.3 or from our companion paper that a CE exists. We thus iterated over consecutive pairs of allocations on the Pareto optimal frontier, and for each such pair tested several budgets that "crossed" in between those allocations. To illustrate this, recall Figure 2 in which the budgets "cross" between A and B. This choice ruled out the existence of budget-proportional allocations. We used additional such considerations to carefully chose the budgets in order to rule out all "easy cases". For random non-additive instances, we simply used several choices of arbitrary non-equal budgets.

[8]The instances with 4 items were generated as a "sanity check", as we know from our companion paper that a CE exists for these instances.

Running the search. For each of the resulting instances we conducted an exhaustive search for an equilibrium: we iterated over all possible PO allocations, and for each one of them we used CVX with the LP solver MOSEK 7 to look for equilibrium prices. Note that although the problem is possibly computationally hard, our instances were small enough that they could be completely solved by the solver in a matter of seconds. We verified the equilibria found by the LP solver by implementing a demand oracle. The run time for 10,000 instances of 4 items was several minutes, and run time increased noticeably as the number of items increased.

In all instances with additive preferences that we tested, we found and verified an equilibrium. As for general preferences, in all cases with 4 items we found an equilibrium (as expected), and even for 5 items we needed to go over several hundred instances before we found one that does not have an equilibrium. Instances with general preferences that do not have an equilibrium seemed to become more rare as the number of items increased.

C.2 Spliddit data with additive preferences.

Setup. We ran our second computerized search on instances of Spliddit data, specifically, 803 instances created so far through Spliddit's "divide goods" application that were kindly provided to us by the Spliddit team [*cf.* 19, Sec. 4.3]. In every Spliddit instance, every player divides a pool of 1000 points among the instance's indivisible items in order to indicate his values for the items; the resulting preference is additive in these values.

Running the search. We implemented a simple tâtonnement process: Prices start at 0, and all players are asked for their demand at these prices. Then the price of over-demanded items is increased by 1, and the price of undemanded items is decreased by 1. Prices thus remain integral throughout the process, and since our budgets are reasonably-sized integers the process is likely to converge reasonably quickly (we do not allow it to run for more than $20,000$ iterations). The running time was typically well under a minute, usually no more than a second or two. One issue that deserves mention (and possibly further research) is how to update the prices when more than a single item is over- or under-demanded. Our first attempts either updated only a single such item's price in every iteration, or updated all such items' prices – both variants converged to a CE fairly often. We improved upon this by randomly deciding after each price update whether or not to continue updating prices in the current iteration.

Special case of interest. An anecdotal but interesting case is non-demo instances with between 5 and 10 items. There were 14 such instances available in the data, with between 3 and 9 players each. As Spliddit assumes that players have equal entitlements, we started by giving all players equal budgets of 100, in which case an equilibrium was found for less than half of the instances. When we added small perturbations to make the budgets only *almost* equal (resulting in the budget vector $(100, 103, 106, \dots)$), we found a CE in all instances. The same was true for other small perturbations that we tried (resulting in budget vectors like $(100, 101, 104, 109, \dots)$). We also tried several other budget vectors with budgets that are far from equal (such as $(100, 151, 202, \ldots)$ or $(100, 200, 300, \ldots)$), and CEs were always found for these as well.