Optimal Multi-Period Pricing with Service Guarantees

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Abstract

We consider the multi-period pricing problem of a service firm facing time-varying capacity levels. Customers are assumed to be fully strategic with respect to their purchasing decisions, and heterogeneous with respect to their valuations, and arrival-departure periods. The firm's objective is to set a sequence of prices that maximizes its revenue while guaranteeing service to all paying customers. Although the corresponding optimization problem is non-convex, we provide a polynomial-time algorithm that computes the optimal sequence of prices. We show that due to the presence of strategic customers, available service capacity at a time period may bind the price offered at another time period. Consequently, when customers are more patient for service, the firm offers higher prices. This leads to the underutilization of capacity, lower revenues, and reduced customer welfare. Variants of the pricing algorithm we propose can be used in more general settings, such as a robust optimization formulation of the pricing problem.

1 Introduction

Dynamic pricing is one of the key tools available to a service firm trying to match time-varying supply with time-varying demand. It is, however, a delicate tool to use in the presence of customers who strategically time their purchases. As customers change timing of their purchases, not only the firm might lose revenue, but also its service capacity might be strained in periods where a low price is offered.

We consider the multi-period pricing problem of a service firm that wishes to provide service guarantees to its customers. That is, the firm wants to set its prices in order to maximize the revenue it obtains, while ensuring that any customer willing to pay the price

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at a given time period will be able to obtain service. Service guarantees are an important contract feature that are often used when the customers themselves are businesses that rely on the service they purchase for their own operations.

An example of a market of growing importance where firms set prices over multiple periods in order to maximize profits while providing service guarantees is the business of offering online services, such as cloud computing, where a firm sells computation-on-demand to its customers. The firm's service capacity varies over time depending on the availability of its servers and network. The demand for service also changes over time and customers differ in their willingness-to-wait. Some of the customers are impatient and demand real-time service: they might use, for example, the cloud to run their website. Other customers use the cloud to solve large-scale optimization problems such as the ones that arise in financial analysis or weather forecasting, so they are willing to tolerate delays in obtaining service in exchange for a lower price.

In the market for online business services, customers typically demand reliable service and do not tolerate rationing. That is, customers expect to be able to buy service whenever they need it and it is the firm's responsibility to set prices which ensure that all service requests can be accommodated with the limited service capacity. Note the contrast to settings such as traditional retailing, where customers are exposed to rationing risk. In a traditional retail setting, strategic customers consider the risk of stock-out, and this incentivizes them to purchase the good earlier. This rationing risk mitigates the effect of strategic customer behavior on the firm's ability to set its own prices. In our setting, the firm's need to offer service guarantees places the entire burden of matching supply and demand over time on the firm.

1.1 Our Framework

We consider a monopolist that offers service to customers over a finite horizon. The firm faces a (possibly time-varying) capacity constraint at each time period. The firm's objective is to implement a posted pricing scheme in order to maximize its revenue. At time zero, the monopolist declares (and commits to) a sequence of prices for its service, one for each time period. Given those pre-announced prices, customers decide whether and when to purchase service. The firm needs to solve the constrained optimization problem of determining the prices that maximize revenue while still fulfilling all customer purchase requests.

Each customer is assumed to be infinitesimal and demand a (also infinitesimal) single unit of service. The valuation of a given customer for a unit of service is drawn from a known distribution. She is also associated with an arrival and a departure time. The arrival time corresponds to the time she enters the system and the departure time represents her deadline for obtaining service. All customers are fully strategic about whether and when they purchase service from the firm. That is, each customer either refuses to buy service (if her valuation is below any of the prices offered while she is present) or buys service at the period when it is offered at the lowest price among all the periods in which she is present – if two periods have the same low price, she prefers the earlier one.

We first consider a setting where the monopolist knows the total number of customers that arrive at each given time period, as well as their departure periods. This is a justified assumption when the number of customers is large and fairly predictable, such as in the market for cloud computing. The monopolist uses this information to determine the sequence of prices for the entire time horizon. Note that this modeling choice allows us to study the impact of strategic customers, and time-varying demand and capacity on the optimal sequence of prices but it deliberately removes the element of uncertainty from the model. Interestingly, even the solution of this baseline model is far from trivial. For instance, the set of feasible solutions is not closed; hence, the optimal solution may not exist. Moreover, this feasible set is non-convex. This means no off-the-shelf software can be used to solve this problem efficiently. These challenges reveal the difficult problem faced by a firm who serves strategic customers and offers service guarantees.

We then consider a robust optimization framework (cf. Ben-Tal and Nemirovski (2002), Bertsimas and Thiele (2006)), where there is uncertainty about the firm's capacity and the size of the customer population at any given period, but it is known that these parameters belong to given sets. In this setting, the firm tries to maximize revenues, while ensuring that the capacity constraints are not violated, for any realization of demand and capacities. Finally, we extend the model to a stochastic setting where the seller knows the distribution of the uncertain parameters. We investigate, the pricing rule the firm should use for expected revenue maximization. Both in the robust optimization setting and the stochastic setting, the optimal pricing rule turns out to be closely related to the one obtained for our baseline model.

1.2 Contributions

We offer a few different sets of results. We first characterize the structure of the optimal prices used by the firm. Then, we use this characterization to construct a polynomial-time algorithm to determine those prices. We also extend the model and algorithm to situations where there is uncertainty about the problem parameters, and the firm is interested in maximizing the worst-case revenue or the expected revenue. Finally, we use our algorithm to find optimal prices for randomly generated problem instances and use the numerical results we obtain to derive insights about the impact of customer patience on revenue and customer welfare.

We start with the setting where capacity levels and the mass of customers arriving and leaving at each period are known to the firm. We first observe that due to the presence of strategic customers, the set of feasible price vectors is non-convex. We consider the problem of maximizing revenue subject to feasibility constraints on prices. The set of feasible prices need not be closed, and the optimal solution to this problem may not even exist. We circumvent this issue by reformulating the original optimization problem. In particular, we consider a formulation, where the firm maximizes its revenue by jointly choosing prices and a ranking of prices, and show that this problem is guaranteed to have an optimal solution. We then characterize several structural properties of the optimal sequence of prices. We establish that, under a standard assumption on the distribution of the customer valuations, the firm should only consider prices that are equal to or above the monopoly price (which is the price that maximizes the revenue in the absence of capacity constraints). The rationale is that the firm, as a monopolist, would prefer to set all prices at the monopoly price, but is unable to do so because of the capacity constraints. In order to satisfy the capacity constraints, the firm offers prices above the monopoly price thus decreasing demand.

We also show that at any optimal solution, all prices that are not equal to the monopoly price are determined by the capacity constraints. However, the price at a given time period may be constrained by the capacity constraint at another period, since customers can consider different periods in search for a better price. This implies that if the price in a given period is constrained by the capacity in a different period, then the prices offered in these two periods are identical.

We construct a set of prices, of polynomial size in time horizon, exploiting the above observation. This set contains all the prices that might be used in an optimal sequence of prices. Using this set of prices, we convert the problem of finding optimal prices to a dynamic program, which can be solved in time polynomial in the length of the horizon.

We then relax the assumption that we have full information regarding the capacity levels and the arrivals and departures of customers. We study the model from a robust optimization perspective and show that the results from our baseline model carry over, and a polynomial time algorithm for finding optimal prices can be provided, when the mass of customers and the capacity levels are only known to belong to given sets. We also quantify the revenue loss due to the uncertainty in the problem parameters, and show that if the uncertainty sets are small in size, then our approach still yields a near optimal solution to the underlying revenue maximization problem.

We also extend the model to incorporate soft capacity constraints (i.e., penalties for exceeding available capacity) and stochastic arrivals and departures. In the extended model, our earlier characterization of the set of optimal prices no longer holds, but following a similar approach, we provide a fully polynomial-time approximation scheme to find the optimal sequence of prices.

Finally, we conduct numerical studies using our algorithm, and obtain further insights on the effect of strategic customers and service guarantees on both the firm and its customers. We consider a setting where all the capacity levels and mass of customers arriving and departing at each period are generated randomly. We show that despite the high volatility in the available capacity, the number of price levels that optimal pricing policy employs is small. For instance, in a 24-period model, the optimal price sequence includes 4 different price levels on average. This shows that even in complex multi-period settings, the customers' strategic behavior severely constrains the firms choice of price sequence. We also observe that if patient customers can wait longer for service, both the revenue of the firm and the aggregate customer welfare may decrease. This occurs because the firm is forced to use higher prices to maintain its service guarantees, and consequently the service capacity is underutilized. Thus we conclude that, in a phenomenon similar to Braess's paradox (Başar and Olsder (1999)), when customers have additional freedom in choosing the time period they purchase service, the overall performance of the system may decrease.

1.3 Related Work

In this section, we present a brief overview of the literature on pricing mechanisms in the presence of customers who strategically time their purchases and discuss how the results in the literature relate to ours. There is also an extensive literature on dynamic pricing with myopic customers (see, for example, Lazear (1986), Wang (1993), Gallego and Ryzin (1994), Feng and Gallego (1995), Bitran and Mondschein (1997), Federgruen and Heching (1999)). We do not provide a summary of this line of literature here, but refer the reader to excellent surveys by Talluri and Ryzin (2004), Bitran and Caldentey (2003), Chan et al. (2004), Shen and Su (2007), and Aviv et al. (2009).

The study of monopoly pricing in the presence of strategic customers was pioneered by Coase (1972). Coase conjectured that in a setting in which a monopolist sells a durable good to patient customers, if the monopolist cannot commit to a sequence of posted prices, then the prices would converge to the production cost. Later, Stokey (1979, 1981), Gul et al. (1986) and Besanko and Winston (1990) showed that a decreasing sequence of prices is optimal for selling durable goods when customers face the trade-off of consuming right away versus the possibility of purchasing at the lower prices in the future. They observe that customers with high valuations buy in earlier periods and pay higher prices compared to the low valuation customers.

In the context of revenue management, Aviv and Pazgal (2008) study a model where a monopolist sells k items over a finite time horizon, to customers arriving according to a Poisson process. At a specific time, the seller can reduce the price and sell the items with a discount. The goal of the seller is to optimally choose the discount price. The authors consider two classes of strategies: contingent posted-pricing where the discount may depend on the remaining inventory (before the discounting period begins) and pre-announced posted pricing (in which the seller commits to the amount of discount at the beginning). They observe that commitment (pre-announced discount) can benefit the seller when customers are strategic. Also, ignoring the strategic customer behavior can lead to significant loss of revenue. Mersereau and Zhang (2010) propose a technique that counteract this phenomenon by taking a robust approach to strategic customer behavior.

Elmaghraby et al. (2008) and Dasu and Tong (2010) extend the analysis of Aviv and Pazgal (2008) to a setting where the seller can reduce the prices multiple times. After each price reduction, the buyers bid for the quantity of units given the current price. The seller will randomly allocate the units if there is more supply than demand. The authors show that although the seller can use multiple price reductions, a markdown mechanism with two price steps is optimal when the valuations of the buyers are known in advance. Dasu and Tong (2010) provide numerical examples showing that neither contingent nor posted pricing is dominant, when the vlauations are uncertain. Nevertheless, the difference in the revenues obtained under these schemes is small; see also Arnold and Lippman (2001) and Cachon and Feldman (2010).

The aforementioned works on strategic customer behavior consider only markdown pricing. Su (2007) shows that if the customers are heterogenous regarding their time sensitivity, then the optimal sequence of posted prices might be increasing. In his model, each customer has either high or low valuations and is either patient (strategic) or impatient (myopic). The inventory is fixed at the beginning of the time horizon and customers arrive according to a deterministic flow over time. The author shows that when high-value customers are proportionately patient, then increasing prices are optimal. In addition, the author shows that the revenue of the seller may increase due to the strategic behavior of the customers compared to the revenue of a setting when all customers are impatient (myopic).

Levin et al. (2010) study a dynamic pricing model for a monopolist selling an initial inventory over a finite horizon. They model the "degree of strategic behavior" of each customer by associated different discount factors (e.g., when the discount factor is 0, the customer becomes myopic). The authors look at the subgame perfect equilibrium of the stochastic game defined by the strategic response of the customers to the prices posted dynamically by the seller. The authors show the existence and uniqueness of the equilibria and provide monotonicity properties and analytical solutions under certain assumptions (such as bounded rationality).

Two interesting papers that deal with product availability and pricing with strategic customers are Su and Zhang (2009) and van Ryzin and Liu (2008). Su and Zhang (2009) finds that sellers have an incentive to over-insure consumers against the risk of stockouts, showing that providing service guarantees are possibly in the firm's interest. van Ryzin and Liu (2008) show that, in the absence of commitment power, the firm might want to strategically reduce its available capacity in order to create rationing risk for its customers.

An altogether different approach to this problem is the one taken by the dynamic mechanism design literature. There, the firm offers a direct mechanism that allocates its service as a function of customers reports of their private valuations, entry and departure periods. See Bergemann and Said (2011) for a survey. Solving for optimal mechanisms is generally a challenging problem that often leads to complex mechanisms that are difficult to implement. The paper closest to this one within this literature is Pai and Vohra (2009), where strategic customers arrive and depart over time. The allocation problem studied in that paper is quite dissimilar to the one presented here and the assumptions they introduce to find optimal mechanisms do not apply to our problem.

The model we consider here differs from most papers in the literature in at least four key aspects: in our model, the firm guarantees service to all paying customers and, therefore, the customers do not face rationing risk. Second, the firm is able commit to a sequence of prices upfront and, thus, is not subject to the challenges first pointed out by Coase (1972). Third, instead of having a fixed inventory at time 0, in our model, the firm has a time-varying service capacity, which is non-storable; namely, if the firm doesn't use the capacity in a given period then it is wasted. Hence, strategic behavior of the customers has the potential to increase the utilization of the firm's capacity. Finally, in the previous work, the customers are either present from the beginning of the time horizon, or arrive over time but remain till the end (or after they make a purchase). In our model, buyers arrive over time and they leave the system at different times.

1.4 Organization

We formalize the model in Section 2. In Section 3, we show that the optimal solution of the aforementioned model may not exist and present a reformulation of the original optimization problem to address this. The structural properties of the optimal prices are discussed in Section 4 followed by a polynomial time algorithm for computing to optimal sequence of prices in Section 5. In Section 6, we study our model from a robust optimization perspective. Further generalization of our model, including stochastic arrival and capacity processes, are presented in Section 7. We discuss insights obtained from numerical analysis of the model in Section 8.

2 Model

In this section, we formulate the revenue maximization problem of a monopolist providing guaranteed service. The firm sets a vector of prices over a finite horizon $t = 1, .., T$. The prices, denoted by $\mathbf{p} = (p_1, ..., p_T)$, are announced upfront, one price for each period t. Customers arrive and depart over time and are infinitesimal. We denote the population of customers that arrive at period i and depart at period j by $a_{i,j}$. With slight abuse of notation, we also represent the mass of the population that arrives at period i and departs at period j by $a_{i,j}$.

Each customer wants one unit of service from which she obtains a (non-negative) value, and customers are strategic with respect to timing of their purchases. Given the vector of prices $\mathbf{p}$, a customer from population $a_{i,j}$, with value v for the service, purchases the service at a time period with the lowest price between times i and j, if her value is larger than the lowest price, i.e., if $v \geq \min_{\ell: i \leq \ell \leq j}\{p_\ell\}$. If there is more than one period with the lowest price in $\{i, \cdots, j\}$, the customer chooses the earliest period (with the minimum price) to obtain the service.[1]

Given a price vector $\mathbf{p}$, we can assign to each population $a_{i,j}$, a service period, denoted by $\pi_{i,j}(\mathbf{p})$. This period has the lowest price among periods in $\{i, \cdots, j\}$ and is the earliest one (in $\{i, \cdots, j\}$) with this price. Each member of population $a_{i,j}$ considers purchasing service at time $\pi_{i,j}$ and will purchase service if her value exceeds the price at that period. We call the mass of customers that, given prices $\mathbf{p}$, consider obtaining service at period t as the *potential demand* at time t, and denote it by $\bar{\rho}_t(\mathbf{p})$. Formally, the potential demand is given by

$$\bar{\rho}_t(\mathbf{p}) = \sum_{i,j: 1 \leq i \leq t \leq j \leq T} a_{i,j} \mathbf{1}\{t = \pi_{i,j}(\mathbf{p})\}, \tag{1}$$

where $\mathbf{1}$ is an indicator function.

Each customer assigns a non-negative value for obtaining service. The fraction of customers with value below v is given by $F(v)$. For simplicity of presentation, we assume that F is a continuous function and $v \in [0, 1]$ for all customers. We also assume that customer valuations are independent of their arrival and departure periods, an assumption that we

[1] Our results hold for any other deterministic tie-breaking rule.

relax in Section 7. Hence, given price vector $\mathbf{p}$, the *demand* at time t, denoted by $\bar{D}_t(\mathbf{p})$, is equal to

$$\bar{D}_t(\mathbf{p}) = (1 - F(p_t))\bar{\rho}_t(\mathbf{p}).$$

The firm's objective is to maximize its revenue, which is given by $\sum_{t=1}^{T} p_t \bar{D}_t(\mathbf{p})$. However, the firm is constrained by a service capacity level of c_t, for each $t \in \{1, ..., T\}$. The firm provides service guarantees to its customers, so it must set prices that ensure that the demand $\bar{D}_t(\mathbf{p})$ does not violate the capacity c_t at any period t. Thus, the firm's decision problem is given by:

$$\begin{aligned} \sup_{\mathbf{p} \geq 0} \quad & \sum_{t=1}^{T} p_t \bar{D}_t(\mathbf{p}) \\ s.t. \quad & \bar{D}_t(\mathbf{p}) \leq c_t, \qquad \text{for all } t \in \{1, ..., T\}, \end{aligned} \tag{OPT-1}$$

where $\mathbf{p} \geq 0$ is a short-hand notation for $p_t \geq 0$ for all $t \in \{1, ..., T\}$. The above problem searches for the supremum of the objective function instead of the maximum, since the maximum of OPT-1 does not always exist. We demonstrate non-existence of an optimal solution in Section 3, where we also present our technique for handling this issue.

If there were no capacity constraints, the firm could use a single price p at all periods to maximize its revenue[2], and this would result in a revenue equal to $p(1 - F(p)) \sum_{i \leq j} a_{i,j}$. Since $\sum_{i \leq j} a_{i,j}$ is a constant, we call $p(1 - F(p))$ the *uncapacitated revenue function*. We make the following regularity assumption to simplify our analysis.

Assumption 1 *The* uncapacitated revenue function $p(1 - F(p))$ *is unimodal. That is, there exists some* monopoly price p_M *such that* $p(1 - F(p))$ *is increasing for all* $p < p_M$ *and decreasing for all* $p > p_M$.

Note that this assumption implies that p_M maximizes $p(1 - F(p))$, and it is satisfied for a wide range of distributions, including the uniform, normal, log-normal, and exponential distributions.

We now show, by the means of an example, that the set of feasible prices of OPT-1 is non-convex.

Example 1 *Let the time horizon be* $T = 3$ *and assume that a single unit-mass of customers with uniform valuations in* $[0, 1]$*, arrive at period* 1 *and depart at period* 3. *Assume that* $c_2 = 0$*, and* $c_1, c_3 = 1$. *Then the price vectors* $(0, 0.1, 1)$ *and* $(1, 0.1, 0)$ *are both feasible. However, the average of these two price vectors,* $(0.5, 0.1, 0.5)$*, is infeasible since all customers with valuation above* 0.1 *seek service at period 2, violating the service capacity* $c_2 = 0$. *Therefore, the set of feasible prices of OPT-1 is non-convex.*

The above example illustrates that OPT-1 is a non-convex optimization problem, and we cannot hope to solve it using off-the-shelf optimization tools. We show in Section 5 that despite being non-convex, there exists a polynomial-time algorithm that solves this optimization problem. The construction of this algorithm relies on the structural properties of this pricing problem that are explored in Sections 3 and 4.

[2]Since customer valuations are independent of arrival and departure periods, it can be seen from OPT-1 that if there are no capacity constraints, setting $p_t = \arg\max_p p(1 - F(p))$ for all t maximizes revenue.

3 Optimizing over Prices and Rankings

In this section, we show that there does not always exist a feasible solution achieving the supremum in the firm's optimization problem. To address this issue, we construct a closely related optimization problem where the firm tries to maximize not only over prices, but also over rankings of the prices. We show that this optimization problem always admits an optimal solution which can be used to obtain feasible solutions arbitrarily close to the supremum of the original problem.

We start with an example that shows that the supremum of OPT-1 may not be achieved by a feasible price vector. The main idea is that since the customers always seek the lowest price available, the potential demand function $\bar{\rho}_t$ is a discontinuous function of $\mathbf{p}$; thus, the feasible set of OPT-1 is open.

Example 2 *Consider a two-period model with customer valuations drawn uniformly from* $[0, 1]$*, capacity levels* $c_1 = \frac{1}{2}$ *and* $c_2 = \infty$*, and customer populations* $a_{1,1} = a_{1,2} = 1$ *(and* $a_{2,2} = 0$*). Observe that solutions of the form* $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$ *are feasible for any* $\epsilon > 0$*: the members of population* $a_{1,1}$ *with value above* $\frac{1}{2}$ *obtain service at time* 1 *and the members of population* $a_{1,2}$ *with value above* $\frac{1}{2} - \epsilon$ *are served at time* 2*. Hence,* $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$ *yields the revenue of* $\frac{1}{2} \times \frac{1}{2} + (\frac{1}{2} - \epsilon) \times (\frac{1}{2} + \epsilon) = \frac{1}{2} - \epsilon^2$*. The revenue is decreasing in* ϵ *and as* ϵ *tends to* 0 *the revenue approaches* $\frac{1}{2}$*. The uncapacitated problem provides an upper bound on the revenue obtained, which is* $\frac{1}{2}$*. Therefore, the supremum of OPT-1 is equal to* $\frac{1}{2}$*. However,* $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$ *is not a feasible solution, because under this price vector, both populations will choose the first period for service, and this violates the capacity constraints. Therefore, the feasible set of price vectors is open and the supremum of OPT-1 is not achieved by a feasible price vector.*

The non-existence of an optimal solution can be addressed by finding (feasible) solutions that are arbitrarily close to the (infeasible) supremum. In the remainder of this section, we introduce the notion of rankings and an alternative optimization formulation which allow us to obtain such solutions for OPT-1 (or the optimal solution itself in instances where the optimum is feasible).

We refer to permutations of $\{1, \cdots, T\}$ as *rankings*. We use the notation R_t to denote the rank of time period t under ranking R. We say that a ranking R is *consistent* with a price vector $\mathbf{p}$ if periods with lower rank have lower prices. More precisely, R is consistent with $\mathbf{p}$ if for all t and t', $R_t < R_{t'}$ implies that $p_t \leq p_{t'}$.

We define the *customer-preferred ranking*, denoted by $R^C(\mathbf{p})$, as a ranking consistent with $\mathbf{p}$, such that when there are multiple periods with the same price, the earlier periods are ranked lower. Namely, if $p_t = p'_t$ and $t < t'$ then $R_t < R_{t'}$. It can be seen from the definition of service period $\pi_{i,j}(\mathbf{p})$ (introduced in Section 2) that in OPT-1 for a given price vector $\mathbf{p}$, each population $a_{i,j}$ chooses the time period between i and j, with the lowest customer-preferred ranking to (potentially) receive service. Hence potential demand $\bar{\rho}_t$ can be expressed as a function of this ranking.

More formally, for any period t and ranking of prices R we define the *R-induced potential demand*, denoted by $\rho_t(R)$, as:

$$\rho_t(R) = \sum_{i \leq j} a_{i,j} \mathbf{1} \left\{ R_t = \min_{k: i \leq k \leq j} \{R_k\} \right\}. \tag{2}$$

Similarly, the *R-induced demand*, denoted by $D_t(p_t, R)$, is defined as

$$D_t(p_t, R) = (1 - F(p_t))\rho_t(R). \tag{3}$$

It follows from Eq. (1), Eq. (2) and the definition of customer-preferred ranking that for any price vector $\mathbf{p}$ and customer-preferred ranking $R^C(\mathbf{p})$, we have $\rho_t(R^C(\mathbf{p})) = \bar{\rho}_t(\mathbf{p})$ and $D_t(p_t, R^C(\mathbf{p})) = \bar{D}_t(\mathbf{p})$. That is, it is possible to express demand ($\bar{D}_t$) in terms of the R-induced demand function (D_t) and customer-preferred ranking (R^C).

Suppose that in OPT-1 the firm could select not only the vector of prices $\mathbf{p}$, but also any ranking R consistent with $\mathbf{p}$ (potentially different than the customer-preferred ranking), and customers decided when to obtain service according to this ranking, i.e., each customer chooses the period with the lowest ranking between her arrival and departure time. Then, the demand at any period is given by $D_t(p_t, R)$, and the corresponding revenue maximization problem can be formulated as:

$$\begin{aligned} \max_{\mathbf{p} \geq 0, R \in \mathcal{P}(T)} \quad & \sum_{t=1}^{T} p_t D_t(p_t, R) & \\ s.t. \quad & D_t(p_t, R) \leq c_t & \text{for all } t \in \{1, ..., T\} \\ & R_t < R_{t'} \Rightarrow p_t \leq p_{t'} & \text{for all } t, t' \in \{1, ..., T\}, \end{aligned} \tag{OPT-2}$$

where $\mathcal{P}(T)$ is the set of all possible rankings of $\{1, ..., T\}$. Despite the fact that this problem is different from OPT-1, the solutions of these problems are closely related, as we explain next.

Since $D_t(p_t, R^C(\mathbf{p})) = \bar{D}_t(\mathbf{p})$, it follows that any feasible solution $\mathbf{p}$ of OPT-1 corresponds to a feasible solution of OPT-2 given by $(\mathbf{p}, R^C(\mathbf{p}))$, and these solutions lead to same objective values. Additionally, it can be seen from Eq. (1) and Eq. (2) that for a given price vector $\mathbf{p}$, and any ranking R consistent with $\mathbf{p}$ the (potential) demand levels in OPT-1 and OPT-2 are equal except for periods where the price is equal to the price offered at another period. Intuitively, unlike OPT-1, in OPT-2 the firm can choose how the customers collectively break ties between time periods with equal price, by choosing the ranking R properly, and this may lead to a difference in demand levels only at such time periods. These observations can be used to show that OPT-2 always has an optimal solution, and this solution can be used to construct a solution of OPT-1 that is arbitrarily close to the supremum.

Lemma 1 *The following claims hold:*

1. *The problem OPT-2 admits an optimal solution* $(\boldsymbol{p}^\star, R^\star)$.

2. *Let* $(\mathbf{p}^\star, R^\star)$ *be an optimal solution of OPT-2. For any* $\epsilon > 0$*, the price vector* $\mathbf{p}^\star + \epsilon R^\star$ *is a feasible solution of OPT-1 and the revenue it obtains converges to the supremum of OPT-1 as* ϵ *tends to* 0*.*

3. *If* $\boldsymbol{p}$ *is an optimal solution of OPT-1, then* $(\mathbf{p}, R^C(\mathbf{p}))$ *is an optimal solution of OPT-2.*

The proof is given in the appendix. The idea behind this lemma is that the projection of the set of feasible solutions of OPT-2 onto the set of prices is the closure of the feasible set of OPT-1. Therefore, the optimal prices generated by OPT-2 can be perturbed in a way that maintains the ranking of prices, leading to a solution of OPT-1 that is arbitrarily close to the supremum. In the rest of the paper, we focus on the solution of OPT-2, keeping in mind that an optimal solution (or a solution arbitrarily close to optimal, if optimal solution does not exist) of OPT-1 with (almost) same prices can be constructed using this solution.

4 Structure of the Optimal Prices

In this section, we explore the structure of optimal prices in OPT-2. We first show that at all periods the monopolist has incentive to keep prices higher than the monopoly price p_M. Then, we use this observation to study the optimality conditions in OPT-2. Exploiting these conditions, we construct a set of prices which contains all the possible candidate optimal prices. We show that the cardinality of this set is polynomial in the time horizon T, a result we later use in Section 5 to obtain a polynomial time algorithm to solve OPT-2. Proofs of the results presented in this section can be found in the appendix.

To gain some intuition, we first consider the optimal solution in a single period setting. By Assumption 1, choosing any price $p < p_M$ is suboptimal, and the firm has incentive to increase its price to p_M. If setting the price equal to p_M violates the capacity constraints, then the firm increases its price to the minimum price that respects the capacity constrain. Since customers' values are bounded by 1, such a price exists. Thus, it follows that an optimal price in $[p_M, 1]$ can be found. The following proposition shows that this intuition extends to multi-period settings.

Proposition 1 *There exists an optimal solution* $(\mathbf{p}, R)$ *of OPT-2 such that* $p_M \leq p_t \leq 1$ *for all* $t \in \{1, \ldots, T\}$*.*

To prove this result, we assume that a solution where $p_t < p_M$ for some t, is given, and we raise prices that are below the monopoly price p_M in a way that maintains the ranking of the prices. This ensures that as the prices increase to p_M, the revenue increases, while the demand decreases. Thus, it is possible to obtain a feasible solution that (weakly) improves revenues and satisfies $p_t \geq p_M$.

Note that conditioned on prices being above the monopoly price p_M, by the assumption of unimodality of the uncapacitated revenue function, the incentives of the firm and the customers are aligned: both the firm and the customers prefer lower prices over higher ones. The firm never raises prices to obtain more revenue, only to satisfy capacity constraints.

We next provide a further characterization of the prices that are used at an optimal solution of OPT-2. This characterization significantly narrows down the set of prices that needs to be considered to find an optimal solution.

Proposition 2 *There exists an optimal solution* $(\mathbf{p}, R)$ *of the optimization problem OPT-2 such that for each period* t *one of the following is true:*

1. $p_t = p_M$,

2. $p_t = 1$,

3. $p_t = p_{\hat{t}}$ *for some* $\hat{t}$, *such that* $c_{\hat{t}} = D_{\hat{t}}(p_{\hat{t}}, R)$ *and* $p_{\hat{t}} \in [p_M, 1]$.

The proof of this proposition follows by showing that unless the conditions of the proposition hold, the monopolist can modify the prices in a way that increases its profits, while maintaining the feasibility of capacity constraints. The third condition of the proposition suggests that the price at time t is either such that the capacity constraint at time t is tight, or this price is equal to the price offered at another time period, and the capacity constraint at this other time period is tight. Hence, due to the presence of strategic customers, capacity constraints at one period may bind the prices at another period, but this requires the prices to be identical at these two periods.

This proposition implies that for an optimal solution $(\mathbf{p}, R)$ of OPT-2, each entry of the price vector $\mathbf{p}$ either belongs to $\{p_M, 1\}$ or is above p_M and satisfies the equation

$$c_{\hat{t}} = D_{\hat{t}}(p, R) = \rho_{\hat{t}}(R)(1 - F(p)), \tag{4}$$

for some time period $\hat{t}$. However, to characterize the set of all prices that may appear at an optimal solution, we still need to consider all possible rankings. Although there are $T!$ possible rankings R, there are a significantly smaller number of prices that satisfy equations of the form Eq. (4). In order to formalize this idea, we introduce the notion of attraction range, which is a representation of all the populations that choose the same period for service.

Definition 1 (Attraction Range) *For a given consistent price-ranking pair* $(\mathbf{p}, R)$ *the* attraction range *of a time period* k *is defined as the largest interval* $\{\underline{t}, ..., \bar{t}\} \subseteq \{1, ..., T\}$ *containing* k *such that* $R_k = \min_{\ell \in \{\underline{t}, ..., \bar{t}\}} R_\ell$.

Assume that the attraction range of time period k for a consistent price-ranking pair $(\mathbf{p}, R)$ is $\{\underline{t}, ..., \bar{t}\}$. Since customers choose the time period with the lowest ranking available to them when purchasing service, customers who arrive at the system between periods $\underline{t}$ and k, and who can wait until time period k, but not until after time period $\bar{t}$ are exactly the ones who will seek service at period k. Thus, the attraction range concept can be used to identify customers who are "attracted" to a particular time period for receiving service (see Example 3).

Example 3 (Attraction Range) *Consider a problem instance with* 6 *time periods. Assume that a consistent price-ranking pair* $(\mathbf{p}, R)$ *for this problem is given, and the prices at different time periods are as in Figure 1. Since prices at all time periods are different, there is a unique ranking* R *consistent with these prices. The attraction range of time period* 4 *in this example is* $\{2, ..., 5\}$. *Thus, customers who arrive between time periods* 2 *and* 4 *(inclusive) and who cannot wait until after time period* 5 *are the ones who seek service at period* 4.

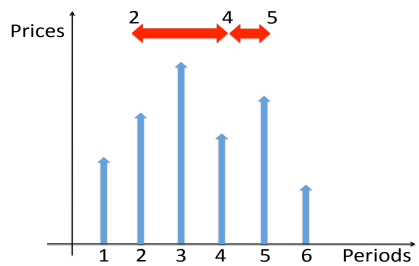


Figure 1: Attraction range of period 4 is $\{2, ..., 5\}$ in this 6 period problem instance. The height of each arrow is assumed to be proportional to the price at the corresponding period.

This example suggests that attraction ranges can be used to determine R-induced potential demand $\rho_t(R)$. Assume that $(\mathbf{p}, R)$ is a consistent price-ranking pair, and consider the attraction range of some time period $k \in \{1, \dots, T\}$, denoted by $\{\underline{t}(k, R), ..., \bar{t}(k, R)\}$. As discussed earlier, customers who arrive at the system between $\underline{t}(k, R)$ and k (inclusive), and who can wait until time k but not until after time $\bar{t}(k, R)$ are the only ones who can request service at time k. Thus, we obtain that

$$\rho_k(R) = \sum_{i=\underline{t}(k,R)}^{k} \sum_{j=k}^{\bar{t}(k,R)} a_{ij}. \tag{5}$$

From this equation it follows that $\rho_k(R)$ can immediately be obtained by specifying the attraction range of time period k. By considering all the possible attraction ranges $\{\underline{t}, \dots, \bar{t}\}$ corresponding to time period k we conclude that for any ranking R

$$\rho_k(R) \in \left\{ \sum_{i=\underline{t}}^{k} \sum_{j=k}^{\bar{t}} a_{ij} \,\middle|\, \underline{t} \le k \le \bar{t} \right\}. \tag{6}$$

Using this observation, it follows that any p satisfying Eq. (4) for some R and $\rho_k(R)$ belongs to the set

$$L_k \triangleq \left\{ \max\left\{ p_M, F^{-1}\left(1 - \left(\frac{c_k}{\sum_{i=\underline{t}}^{k} \sum_{j=k}^{\bar{t}} a_{ij}}\right)\right)\right\} \,\middle|\, c_k \le \sum_{i=\underline{t}}^{k} \sum_{j=k}^{\bar{t}} a_{ij}, \text{ and } \underline{t} \le k \le \bar{t} \right\}. \tag{7}$$

Here the condition $c_k \leq \sum_{i=\underline{t}}^{k} \sum_{j=k}^{\overline{t}} a_{ij}$ is present since F^{-1} is defined over the domain $[0,1]$. The maximum with p_M is taken to make sure that all the prices in L_k are at least equal to p_M, which follows from Proposition 2. By construction each element of L_k corresponds to an attraction range $\{\underline{t}, \ldots, \overline{t}\}$. Since there are $O(T^2)$ attraction ranges (there are $O(T)$ values $\underline{t}$ and $\overline{t}$ can take), the cardinality of L_k is $O(T^2)$. Thus, we reach the following characterization of optimal prices, which is stated without proof as it immediately follows from Propositions 1 and 2, and the definition of L_k given in Eq. (7).

Proposition 3 *Let L be defined as*

$$L \triangleq \left(\cup_{k=1}^{T} L_k\right) \cup \{p_M\} \cup \{1\}. \tag{8}$$

There exists an optimal solution $(\mathbf{p}, R)$ of OPT-2, such that $p_t \in L$ for all $t \in \{1, ..., T\}$. Moreover, the cardinality of L is $O(T^3)$.

The above proposition implies that without actually solving OPT-2, it is possible to characterize a superset of the prices that will be used at an optimal solution. Moreover, this set has polynomially-many elements, and it is sufficient for the monopolist to consider these prices, when making its pricing decisions. However, finding the vector of optimal prices could still be a computationally intractable problem even if L has small cardinality. In the next section, we show that this is not case, and we develop a polynomial-time algorithm that determines the optimal sequence of prices.

5 A Polynomial Time Algorithm

In this section, we use the characterization of the optimal prices obtained in Section 4 to design a polynomial time algorithm for computing the optimal sequence of prices.

As shown in Proposition 3, an optimal solution of OPT-2 can be obtained by restricting attention to set of prices L given in Eq. (8). Thus, an optimal solution to OPT-2 can be obtained by restricting attention to prices in L, and solving the following optimization problem:

$$\begin{aligned} \max_{\mathbf{p} \in L^T, R \in \mathcal{P}(T)} \; & \sum_{t=1}^{T} p_t D_t(p_t, R) \\ s.t. \quad & D_t(p_t, R) \leq c_t \qquad \text{for all } t \in \{1, ..., T\} \\ & R_t < R_{t'} \Rightarrow p_t \leq p_{t'} \qquad \text{for all } t, t' \in \{1, ..., T\}. \end{aligned} \tag{OPT-3}$$

We next show that it is possible to find an optimal solution of OPT-3 by recursively solving problems that are essentially smaller instances of itself.

Consider an optimal solution of OPT-3, denoted by $(\mathbf{p}^\star, R^\star)$. Suppose time period k has the lowest ranking, i.e., $R^\star_k = 1$. In this case the attraction range of k is $\{1, ..., T\}$, and $\rho_k(R^\star) = \sum_{i=1}^{k} \sum_{j=k}^{T} a_{ij}$. Hence, all customers who are present in the system at time k will seek service at time k. This implies that only populations a_{k_1,k_2}, $1 \le k_1 \le k_2 < k$ can receive service at time periods $\{1,\dots,k-1\}$ (similarly, only populations a_{k_1,k_2}, $k < k_1 \le k_2 \le T$ can receive service at time periods $\{k+1,\dots,T\}$). Therefore, if the monopolist knows $p_k^\star$ and that $R_k^\star = 1$, it can solve for optimal prices at other time periods, by solving two separate subproblems for time periods $\{1,\dots,k-1\}$ and $\{k+1,\dots,T\}$: maximize the revenue obtained from time periods $\{1,\dots,k-1\}$ assuming only populations a_{k_1,k_2} are present (with $1 \le k_1 \le k_2 < k$), and similarly for time periods $\{k+1,\dots,T\}$. Note that in the solution of the subproblems we need to impose the condition that prices are weakly larger than $p_k^\star$, as otherwise $p_l^\star < p_k^\star$ for some l, and we obtain a contradiction to $R_k^\star = 1$.

The above observation suggests that given the time period k with the lowest ranking, the pricing problem can be decomposed into two smaller pricing problems, where the prices that can be offered are lower bounded by the price offered at k. We next exploit this observation and obtain a dynamic programming algorithm for the solution of OPT-3.

Let $\omega(i,j,\underline{p})$ denote the maximum revenue obtained from an instance of OPT-3 assuming (i) $a_{k_1,k_2} = 0$ unless $i < k_1 \le k_2 < j$, (ii) restricting prices to be weakly larger than $\underline{p}$. That is,

$$\begin{aligned}
\omega(i,j,\underline{p}) = \max_{\mathbf{p}\in L^T, R\in\mathcal{P}(T)} & \sum_{t=i+1}^{j-1} p_t D_t^{ij}(p_t,R) & \\
s.t. \quad & D_t^{ij}(p_t,R) \le c_t & \text{for all } t \in \{1,\dots,T\} \\
& R_t < R_{t'} \Rightarrow p_t \le p_{t'} & \text{for all } t,t' \in \{1,\dots,T\} \\
& p_t \ge \underline{p} & \text{for all } t \in \{1,\dots,T\},
\end{aligned} \tag{9}$$

where D_t^{ij} is defined similarly to Eq. (3), and denotes the demand at time t, assuming $a_{k_1,k_2} = 0$ unless $i < k_1 \le k_2 < j$. Observe that the optimal objective value of OPT-3 is equal to $\omega(0, T+1, 0)$.

For $i+1 > j-1$, we assume $\omega(i,j,\underline{p})$ equals to 0. On the other hand, for any i,j such that $i+1 \le j-1$, we have

$$\omega(i,j,\underline{p}) \quad = \quad \max_{k\in\{i+1,\dots,j-1\}} \left\{ \max_{p\in L: p\ge \underline{p}} \left\{ \omega(i,k,p) + \gamma_k^{ij}(p) + \omega(k,j,p) \right\} \right\}, \tag{10}$$

where $\gamma_k^{ij}(p)$ is given by:

$$\gamma_k^{ij}(p) = \begin{cases} \left(\displaystyle\sum_{l=i+1}^{k} \sum_{m=k}^{j-1} a_{lm} \right) (1-F(p))p & \text{if } \left(\sum_{l=i+1}^{k} \sum_{m=k}^{j-1} a_{lm} \right)(1-F(p)) \le c_k \\ -\infty & \text{otherwise.} \end{cases} \tag{11}$$

In order to see why the recursion in Eq. (10) holds, consider a solution of Eq. (9), and assume that in this solution k is the time period in $\{i+1,\dots,j-1\}$ with the lowest ranking, and $p_k \ge \underline{p}$ is the corresponding price. Then all populations which are present in the system at k receive service at this time period. The total mass of these populations is $\sum_{l=i+1}^{k} \sum_{m=k}^{j-1} a_{lm}$, since $a_{k_1,k_2} = 0$ unless $i < k_1 \le k_2 < j$ as can be seen from the definition of $\omega(i,j,\underline{p})$.

Since at the optimal solution of Eq. (9) the capacity constraints are satisfied, the revenue obtained from time period k is given by $\gamma_k^{ij}(p_k)$. Since k has the lowest ranking among $\{i+1, \ldots, j-1\}$, only populations a_{k_1,k_2} such that $i < k_1 \leq k_2 < k$ can receive service before time k, and the prices offered at those time periods should be weakly larger than p_k. It follows from the definition of ω that the maximum revenue that can be obtained from these populations (with prices weakly larger than p_k) is given by $\omega(i, k, p_k)$. Similarly, it follows that the maximum revenue that can be obtained from time periods after k equals to $\omega(k, j, p_k)$. Thus, we conclude that $\omega(i, j, \underline{p}) = \omega(i, k, p_k) + \gamma_k^{ij}(p_k) + \omega(k, j, p_k)$. The recursion in Eq. (10) follows since it searches for time period k with the lowest ranking and the corresponding price p_k that maximizes the objective of Eq. (9). Note that since $\gamma_k^{ij}(p) = -\infty$ when a capacity constraint is violated, the solution obtained by solving this recursion also satisfies the capacity constraints.

Theorem 1 shows that a solution of OPT-2, or equivalently a solution of the alternative formulation in OPT-3, can be obtained by solving for prices using the dynamic programming recursion in Eq. (10) and constructing a ranking vector consistent with these prices. The proof is given in the appendix.

Theorem 1 *The optimal solution of OPT-2 can be computed in time $O(T^6)$.*

The above theorem suggests that firms that provide service guarantee can effectively implement optimal pricing policies even in nonstationary environments. In the following sections, we show that this result extends to other general settings as well.

6 A Robust Optimization Formulation

In this section, we consider a robust optimization formulation of the firm's pricing problem. We show that when there is uncertainty about either the service capacity levels or the size of the customer population, a variant of the algorithm of Section 5 can be used to obtain a solution that maximizes revenue, while maintaining feasibility for all possible values of uncertain parameters. Furthermore, we can bound the firm's worst-case revenue loss as a function of the uncertainty in the problem parameters. All proofs of this section can be found in the appendix.

Suppose the firm does not know its service capacity level c_t at a given period, but only knows that it belongs to an interval $\mathcal{C}_t = [c_t^L, c_t^U]$. Similarly, the firm does not know the mass of customers in population $a_{i,j}$, but instead it knows only that $a_{i,j} \in \mathcal{A}_{i,j} = [a_{i,j}^L, a_{i,j}^U]$. We refer to a collection of population sizes $A = \{a_{i,j}\}_{1 \leq i,j \leq T}$ as a population matrix, and represent the set of all possible capacity levels by $\mathcal{C} = \prod_t \mathcal{C}_t$ and the set of all possible population matrices by $\mathcal{A} = \prod_{i,j} \mathcal{A}_{i,j}$. In order to make dependence of the demand (defined in Eq. (3)) on population size explicit, in this section we denote the demand at period t, when population matrix is given by $A \in \mathcal{A}$, and the firm uses price p_t, and ranking $R \in \mathcal{P}(T)$, by $D_t(p_t, R, A)$.

The problem of selecting prices and ranking that are feasible for all capacity levels in $\mathcal{C}$ and population matrices in $\mathcal{A}$, and that maximize the worst case revenue, can be formulated as follows:

$$
\begin{aligned}
\max_{\mathbf{p}\geq 0, R\in\mathcal{P}(T), M} \quad & M \\
s.t. \quad & M \leq \sum_{t=1}^{T} p_t D_t(p_t, R, A) && \text{for all } A \in \mathcal{A} \\
& D_t(p_t, R, A) \leq c_t && \text{for all } t, c_t \in \mathcal{C}_t \text{ and } A \in \mathcal{A} \\
& R_t < R_{t'} \Rightarrow p_t \leq p_{t'} && \text{for all } t, t' \in \{1, ..., T\},
\end{aligned}
\tag{OPT-4}
$$

The following lemma shows how to reformulate the robust optimization problem above into one that is structurally similar to the original problem OPT-2.

Lemma 2 *Let A^L be the population matrix with elements $a^L_{i,j}$ and A^U be the population matrix with elements $a^U_{i,j}$. Then, OPT-4 is equivalent to:*

$$
\begin{aligned}
\max_{\boldsymbol{p}\geq 0, R\in\mathcal{P}(T)} \quad & \sum_{t=1}^{T} p_t D_t(p_t, R, A^L) \\
s.t. \quad & D_t(p_t, R, A^U) \leq c^L_t && \textit{for all } t \in \{1, ..., T\} \\
& R_t < R_{t'} \Rightarrow p_t \leq p_{t'} && \textit{for all } t, t' \in \{1, ..., T\},
\end{aligned}
\tag{OPT-5}
$$

Note that the OPT-5 is different than OPT-2 in that the population matrices used for computing revenue, A^L, and for determining feasibility, A^U, are different. However, the structural insights about the optimal solution obtained in Section 4 and the polynomial-time algorithm developed in Section 5 still apply for this problem with minor modifications.

Proposition 4 *The optimal solution of OPT-4 can be computed in time $O(T^6)$.*

The robust formulation finds a conservative solution that is feasible for all possible values of uncertain parameters. We next quantify the potential revenue loss due to the uncertainty, when the solution obtained from this formulation is used for pricing. Assume that a solution of OPT-4 is obtained using uncertainty sets $\mathcal{C}$ and $\mathcal{A}$. Let $V^{ROB}(\mathcal{C}, \mathcal{A}, \mathbf{c}, A)$ denote the revenue the firm achieves, using this solution, when realized parameter values are $\mathbf{c} \in \mathcal{C}$ and $A \in \mathcal{A}$. If there were no uncertainty in the parameters, i.e., if $\mathbf{c}$ and A were known from the beginning, a solution of OPT-2 could be used for pricing. We denote the revenues that could be obtained in this setting by $V(\mathbf{c}, A)$. The following proposition bounds the decrease in the revenues when there is uncertainty, and the solution of OPT-4 is used to obtain a robust pricing rule.

Proposition 5 *Suppose $c^U_t \leq (1+\theta)c^L_t$ for all $t \in \{1, ..., T\}$ and $a^U_{i,j} \leq (1+\theta)a^L_{i,j}$ for all $i, j \in \{1, ..., T\}$. Then,*

$$
\sup_{\mathbf{c}\in\mathcal{C}, A\in\mathcal{A}} \left(V(c, A) - V^{ROB}(\mathcal{C}, \mathcal{A}, \mathbf{c}, A)\right) \leq 3\theta \sum_{i,j} a^U_{i,j}.
$$

This proposition implies that when the uncertainty in the problem parameters is small (i.e., when θ is small), the revenue loss is also small, provided that a solution of OPT-4 is used for pricing. The proposition also suggests that if a nominal version of the problem with parameters $(\mathbf{c}^{NOM}, A^{NOM})$ is known, and the realized parameters are between $1-\varepsilon$ and $1+\varepsilon$ times the nominal ones (i.e., $1+\theta = 1+\varepsilon/1-\varepsilon$), then the maximum revenue loss due to uncertainty is equal to $\frac{6\epsilon(1+\varepsilon)}{1-\varepsilon}\sum_{i,j} a_{i,j}^{NOM}$.

7 Extensions and an Approximation Scheme

In this section, we provide extensions of our baseline model, introduced in Section 2, by allowing for more general objective functions and constraints. Using the general framework introduced in this section, we obtain solutions to problem instances with

(i) customer valuations that depend on their arrival and departure periods,

(ii) production costs and soft capacity constraints,

(iii) different objective functions such as (weighted) social welfare maximization,

(iv) stochastic arrival and capacity processes and expected utility maximization.

At this level of generality, the results of Section 4 do not hold, and we cannot characterize a set of prices (with size polynomial in the time horizon) that can appear in an optimal solution. Thus, in this general setting, we cannot use the techniques developed in the previous section to compute the optimal sequence of prices in polynomial time. However, we show that modifying our initial framework, a fully polynomial time approximation scheme (FPTAS) can be provided for the solution of the general model. The proofs of our results are presented in the appendix.

We start by introducing an abstract problem that is the focus of this section:

$$\begin{aligned} \max_{\mathbf{p}\in[0,1]^T, R\in\mathcal{P}(T)} \quad & \sum_{t=1}^{T} g_t(p_t, R) \\ s.t. \quad & h_t(p_t, R) \le 0 \qquad \text{for all } t \in \{1,...,T\} \\ & R_t < R_{t'} \Rightarrow p_t \le p_{t'} \qquad \text{for all } t, t' \in \{1,...,T\}. \end{aligned} \tag{OPT-6}$$

We make the following assumption through out this section:

Assumption 2 *For any ranking R and period t, the functions $g_t(\cdot, R)$ and $h_t(\cdot, R)$ satisfy the following properties:*

1. *Each customer prefers the time period with the lowest rank (among those during which she is present) to (potentially) receive service. Hence, the dependence of functions $g_t : \mathbb{R} \times \mathcal{P}(T) \to \mathbb{R}$ and $h_t : \mathbb{R} \times \mathcal{P}(T) \to \mathbb{R}$ on R is through the attraction range of time period t. That is, there exist functions $\hat{g}$, $\hat{h}$ such that*

$$g_t(p_t, R) = \hat{g}_t(p_t, b_t(R), e_t(R)) \quad \text{and} \quad h_t(p_t, R) = \hat{h}_t(p_t, b_t(R), e_t(R)) \tag{12}$$

where $\{b_t(R), ..., e_t(R)\}$, is the attraction range of time period t, when ranking R is chosen.

2. *$h_t(p_t, R)$ is decreasing in p_t.*

3. *$g_t(p_t, R)$ is Lipschitz continuous in p_t with parameter l_t.*

Observe that OPT-2 is a special case of OPT-6, when $g_t(p_t, R) = p_t D_t(p_t, R)$ and $h_t(p_t, R) = D_t(p_t, R) - c_t$, assuming that demand $D_t(p_t, R)$ is Lipschitz continuous in p_t, for a fixed ranking R.

For a constant $\epsilon \in (0, 1)$, consider the set of prices $\mathbf{P}_\epsilon = \{k\epsilon | k \in \mathbb{Z}_+, k\epsilon \leq 1\}$, and assume that we seek a solution to OPT-6 by restricting attention to the prices that belong to this set, i.e.,

$$\begin{aligned} \max_{\mathbf{p} \in \mathbf{P}_\epsilon^T, R \in \mathcal{P}(T)} \quad & \sum_{t=1}^{T} g_t(p_t, R) \\ s.t. \quad & h_t(p_t, R) \leq 0 \qquad \text{for all } t \in \{1, ..., T\} \\ & R_t < R_{t'} \Rightarrow p_t \leq p_{t'} \qquad \text{for all } t, t' \in \{1, ..., T\}. \end{aligned} \tag{OPT-7}$$

Note that any feasible solution of OPT-7 is feasible in OPT-6. We next show that for small ϵ the optimal objective values of these problems are also close. Hence, an optimal solution of OPT-7 can be used to provide a near-optimal solution of OPT-6.

Lemma 3 *Let the optimal solutions of OPT-6 and OPT-7 have objective values v and v_ϵ respectively. Then, $v_\epsilon \geq v - \epsilon \sum_{t=1}^{T} l_t$.*

We next show that a modified version of the algorithm in Section 5 can be used to solve OPT-7. Our approach is again based on obtaining a solution by recursively solving smaller instances of the problem. For this purpose, we first define $\hat{\omega}(i, j, \underline{p})$ to be the maximum utility that can be obtained assuming only populations a_{k_1,k_2}, $i < k_1 \leq k_2 < j$, are present, and the prices that can be used at periods $\{t | i < t < j\}$ are (weakly) larger than $\underline{p}$. It can be seen that the optimal value of OPT-7 is equal to $\hat{\omega}(0, T+1, 0)$.

We set $\hat{\omega}(i, j, \underline{p}) = 0$, for $i + 1 > j - 1$. Using the same argument given in Section 5 to justify the recursion in Eq. (10), it follows that for $i + 1 \leq j - 1$, the following dynamic programming recursion holds:

$$\hat{\omega}(i, j, \underline{p}) = \max_{k \in \{i+1, ..., j-1\}} \left\{ \max_{p \in \mathbf{P}_\epsilon : p \geq \underline{p}} \left\{ \hat{\omega}(i, k, p) + \hat{\gamma}_k^{ij}(p) + \hat{\omega}(k, j, p) \right\} \right\}, \tag{13}$$

where $\hat{\gamma}_k^{ij}(p)$ denotes the utility obtained at time k with price p, from all populations a_{k_1,k_2} that can receive service at this period and that satisfy $i < k_1 \leq k_2 < j$. That is $\hat{\gamma}_k^{ij}(p) = \hat{g}_k(p, i, j)$, if $\hat{h}_k(p, i, j) \leq 0$ and $\hat{\gamma}_k^{ij}(p) = -\infty$ otherwise.

The intuition behind Eq. (13) is similar to the intuition of Eq. (10): in order to find $\hat{\omega}(i, j, \underline{p})$, we search for the time period with the lowest rank (maximization over k in

Eq. (13)), and we search for the best possible price for this time period (maximization over p). Since all populations which are present at the time period with the lowest ranking (say k) receive service at this time period, the payoff obtained from this time period can be given by $\hat{\gamma}_k^{ij}(p)$. We then solve for prices of subproblems for time periods $\{i+1,\ldots,k-1\}$ and $\{k+1,\ldots,j-1\}$. Since the time period with the lowest ranking also has the lowest price, we impose the prices for these subproblems to be weakly larger than p. Thus, the payoffs of the subproblems are given by $\hat{\omega}(i,k,p)$ and $\hat{\omega}(k,j,p)$. Hence, we obtain the recursion in Eq. (13) for computing optimal prices in OPT-7.

In Lemma 4, we use this dynamic program to construct optimal prices and ranking for the solution of OPT-7, and characterize the computational complexity of the solution. Note that since we are dealing with general functions g_t and h_t, our result depends on the computational complexity of evaluating these functions.

Lemma 4 *Assume that for any given t, p, R, computation of $g_t(p,R)$ and $h_t(p,R)$ takes $O(s(T))$ time. An optimal solution of OPT-7 can be found in $O\left(\frac{T^3 s(T)}{\epsilon^2}\right)$ time.*

Lemmas 3 and 4 imply that an approximate solution to OPT-6 can be found in polynomial time provided that $g_t(p,R)$ and $h_t(p,R)$ can be evaluated in polynomial time.

Theorem 2 *Assume that for any given t, p, R, computation of $g_t(p,R)$ and $h_t(p,R)$ takes $O(s(T))$ time. An ϵ-optimal solution of OPT-6 can be found in $O\left(\frac{T^3 s(T)}{\epsilon^2}\right)$ time.*

The proof immediately follows from Lemmas 3 and 4 and is omitted. In many of the relevant cases (such as revenue maximization subject to capacity constraints as introduced in Sections 2 and 3), for given prices and rankings, evaluating constraints and the objective function (h_t and g_t) can be completed in $O(1)$ time. In such settings Theorem 2 implies that an approximate solution can be obtained in $O(T^3/\epsilon^2)$ time.

We conclude this section by discussing some important special cases of this general optimization framework. We show that an approximate solution to these problems can be obtained in polynomial time following the approach introduced in this section.

Population-dependent valuations: Here, we relax the assumption made before that the customers' valuations are independent of their arrival and departure periods. Let $F_{i,j}(v)$ represent the fraction of the $a_{i,j}$ population that values service at most v. We still assume that all valuations are in $[0,1]$ and that $F_{i,j}$ is continuous, but we no longer suppose that Assumption 1 holds, i.e., the corresponding uncapacitated revenue function need not be single peaked. Customer demand at period t as a function of price p_t and the ranking of prices R is now given by

$$D_t(p_t,R) = \sum_{i\le j} a_{i,j}(1-F_{i,j}(p_t))\mathbf{1}\{R_t \le R_k \text{ for all } i \le k \le j\}.$$

By choosing $g_t(p_t,R) = p_t D_t(p_t,R)$ and $h_t(p_t,R) = D_t(p_t,R) - c_t$, the corresponding revenue maximization problem is an instance of OPT-6. Note that for a fixed R, denoting

$f_{i,j}(p) = dF_{i,j}(p)/dp$, and assuming $f_{i,j}$ is bounded by $l_{i,j}$ we conclude

$$\left|\frac{\partial D_t(p_t, R)}{\partial p_t}\right| = \sum_{i \le j} a_{i,j} f_{i,j}(p_t) \mathbf{1}\{R_t \le R_k \text{ for all } i \le k \le j\} \le \sum_{i \le j} a_{i,j} l_{i,j}.$$

Thus, it follows that when $f_{i,j}$ is bounded for all i, j, $D_t(\cdot, R)$ is Lipschitz continuous. This implies that $g_t(\cdot, R)$ is also Lipschitz continuous, for all t and R. Moreover, h_t is decreasing in p_t (since demand is decreasing in p_t). Furthermore, for any t, p, R evaluating $D_t(p_t, R)$, and in turn $g_t(p, R)$ and $h_t(p, R)$ takes $O(T^2)$ time. Thus, Theorem 2 applies and we conclude that the approximate revenue maximization problem can be solved in $O\left(\frac{T^5}{\epsilon^2}\right)$.

Production costs and soft capacity constraints: We also incorporate productions costs into the model. We assume that it costs the firm $\mu_t(d)$ to provide service to mass d of customers at period t. We make the following regularity assumption on the production costs:

Assumption 3 *Assume that for any period t, the production cost μ_t is a non-negative, non-decreasing, λ-Lipschitz continuous function.*

Besides the cost of producing the service to be delivered, the function μ_t can also capture a soft capacity constraint: if $\bar{c}_t$ represents a capacity level above which any unit produced costs $\bar{\mu}$, then we can capture this by setting $\mu_t(d) = \max\{0, \bar{\mu}(d - \bar{c}_t)\}$. Even with soft-capacity constraints, we still assume that the firm provides service guarantees. Whenever the firm is incapable of providing the purchased service itself, it contracts service delivery out to a third-party with a unit cost of $\bar{\mu}$.

By letting $g_t(p_t, R) = p_t D_t(p_t, R) - \mu_t(D_t(p_t, R))$ and $h_t(p_t, R) = D_t(p_t, R) - c_t$, we obtain an instance of OPT-6. From Assumption 3, it follows that $g_t(p_t, R)$ is Lipschitz continuous in p_t. Since demand is decreasing with price, we observe that $h_t(p_t, R)$ decreases with price. Thus, Theorem 2 applies and since $D_t(p, R)$ (and hence $g_t(p_t, R)$, $h_t(p_t, R)$) can be evaluated in polynomial time for any given p and R, it follows that an approximate solution of the problems with production costs and soft capacity constraints can be obtained in polynomial time.

Weighted welfare maximization: We can also modify the firm's objective function so that it cares not only about its own revenue, but also about the welfare obtained by its customers. Consider a unit mass of customers belonging to population $a_{i,j}$. If the monopolist offers the good to this population at price p, the welfare of the customers (CW_{ij}) and the revenue of the firm (REV_{ij}) are given by:

$$\begin{aligned} CW_{ij}(p) &= \int_p^1 (x - p) f_{i,j}(x) dx, \\ REV_{ij}(p) &= \int_p^1 p f_{i,j}(x) dx. \end{aligned} \tag{14}$$

For any given parameter $\alpha \in [0, 1]$, and population $a_{i,j}$, we define the *weighted welfare function* as

$$\begin{aligned} w_{i,j,\alpha}(p) &\triangleq \alpha REV_{ij}(p) + (1-\alpha) CW_{ij}(p). \\ &= \int_p^1 (\alpha p + (1-\alpha)(x-p)) f_{i,j}(x) dx \\ &= (1-\alpha) \int_p^1 x f_{i,j}(x) dx + (2\alpha - 1) p \int_p^1 f_{i,j}(x) dx. \end{aligned} \tag{15}$$

Denote the total weighted welfare at time t by

$$W_t(p_t, R) = \sum_{i \le j} w_{i,j,\alpha}(p) \mathbf{1}\{R_t \le R_k \text{ for all } i \le k \le j\}.$$

Assume that the monopolist prices service so as to maximize the weighted social welfare function, i.e., it maximizes $\sum_t W_t(p_t, R)$ subject to capacity constraints $D_t(p_t, R) \le c_t$.

By choosing $g_t(p_t, R) = W_t(p_t, R)$, and $h_t(p_t, R) = D_t(p_t, R) - c_t$, it follows that weighted welfare maximization problem is an instance of OPT-6. If $f_{i,j}$ is bounded for all i, j, it follows that CW_{ij} and REV_{ij} are Lipschitz continuous. Consequently, $W_t(p_t, R)$ is Lipschitz continuous in p_t for all R. Since $REV_{ij}(p)$ and $CW_{ij}(p)$ take values independent of the length of the horizon, it follows that the welfare function $W_t(p, R)$ can be evaluated in polynomial time. Thus, Theorem 2 suggests that the dynamic programming recursion in Eq. (13) can be used to solve the weighted welfare maximization problem in polynomial time.

Stochastic arrival and capacity processes: Assume that population sizes $\{a_{i,j}\}_{i,j}$ and capacities $\{c_t\}_t$ are random variables with known distributions. Let $E[a_{i,j}] = \hat{a}_{i,j}$ and $E[c_t] = \hat{c}_t$. In this setting, if monopolist wants to guarantee that the total service request does not exceed the capacity for any realization of the parameters, it can use the robust optimization framework in Section 6. On the other hand, if the firm has the capability to contract service delivery out whenever the capacity is exceeded (hence it has soft capacity constraints), it can solve the following expected revenue maximization problem:

$$\begin{aligned} \max_{\mathbf{p} \in \mathbf{P}_\epsilon, R \in \mathcal{P}(T)} \quad & \sum_{t=1}^{T} E[p_t D_t(p_t, R) - \mu_t(D_t(p_t, R))] \\ s.t. \quad & R_t < R_{t'} \Rightarrow p_t \le p_{t'} \qquad \text{for all } t, t' \in \{1, ..., T\}, \end{aligned} \tag{16}$$

By choosing $h_t(p_t, R) = 0$ and $g_t(p_t, R) = E[p_t D_t(p_t, R) - \mu_t(D_t(p_t, R))]$ we obtain an instance of OPT-6. Note that $\frac{\partial g_t(p_t, R)}{\partial p_t} = E\left[\frac{\partial(p_t D_t(p_t, R) - \mu_t(D_t(p_t, R)))}{\partial p_t}\right]$ is bounded when μ_t is Lipschitz continuous and $f_{i,j}$ is bounded for all i, j. Thus, under these assumptions, provided that the expectation in $E[p_t D_t(p_t, R) - \mu_t(D_t(p_t, R))]$ can be evaluated in polynomial time, Eq. (16) can be solved using the dynamic programming recursion in Eq. (13) in polynomial time.

8 Numerical Insights

In this section, we consider generic instances of the firm's pricing problem and obtain qualitative insights about the optimal pricing scheme introduced in this paper. We show that due to presence of strategic customers, the firm uses only a few different prices in its optimal price sequence. Surprisingly our simulations also indicate that increased patience levels for customers translate to higher prices, underutilization of capacity and reduced firm revenue and customer welfare.

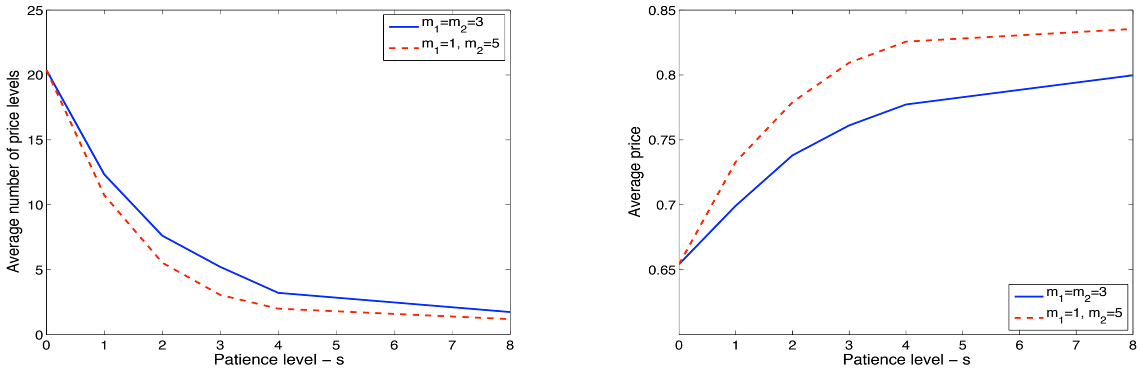
Our numerical results are based on simulations with 48 time periods, where we plot the middle 24 periods to avoid potential boundary effects.[3] We let customer valuations to be uniformly distributed between 0 and 1, and capacities to be generated independently and uniformly in between 0.5 and 1.5 for each time period. We assume that there are two types of populations arriving at each time period: (i) impatient (or myopic) customers, i.e., customers who are only interested in purchasing service at the period they arrived, (ii) strategic (or s-patient) customers, who are willing to wait up to s periods to purchase service. This is captured by setting all $a_{i,j}$ equal to 0 unless $j = i$ (myopic customers) or $j = i + s$ (strategic ones). For each i, $a_{i,i}$ is generated at random from a uniform distribution between 0 and m_1, while $a_{i,i+s}$ is generated from a uniform distribution between 0 and m_2. Our analysis focuses on the revenues, capacity usage and customer welfare (as defined in Section 7), as the parameters s (willingness to wait for strategic customers) and $\frac{m_2}{m_1+m_2}$ (fraction of customers who are strategic) change. Since parameters are generated randomly, we present our results by averaging them over 50 problem instances.

Patient Customers Lead to Fewer Price Levels: Proposition 2 stated that the optimal price in a given time period must either belong to $\{p_M, 1\}$, or be equal to the price of a time period where the capacity is tight. This implies that the total number of price levels used might be smaller than the total number of periods. Our simulation analysis shows that this is indeed the case.

Figure 2(a) shows that the average number of price levels over the 24-period horizon drops somewhat when a higher fraction of the population is willing to wait for service (see the difference the between the two curves) and drops dramatically when the customers who are willing to wait become more patient. For example, while roughly 14 prices are needed when customers are willing to wait only up to 1 period, this number drops to 8 if they are willing to wait for 2 periods and 5 if they are willing to wait for 3 periods (see the simulation with $m_1 = m_2 = 3$). We note that when an optimal solution for the original pricing problem OPT-1 does not exist, Lemma 1 suggests using perturbed prices $\{p_t + \epsilon R_t\}_t$ for an arbitrarily small ϵ to obtain solutions arbitrarily close to the optimal. In such cases, our results indicate that the firm uses "essentially" few prices.

Patient Customers Also Lead to Higher Prices: As customers become more patient, the firm becomes more constrained in the prices it can offer. Consequently, to maintain

[3] We note that no significant changes are observed in our results when the entire time horizon is used for the analysis.



(a) Number of price levels. (b) Average price.

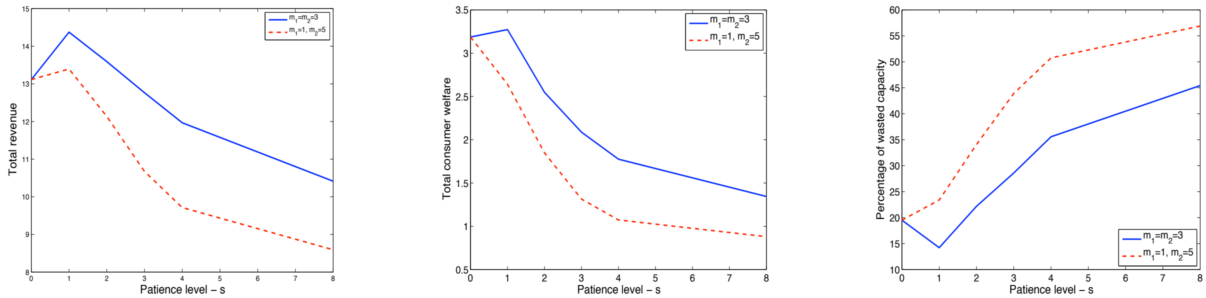
Figure 2: Number of price levels, and average price over 24 time periods of interest.

feasibility, and sustain its service guarantees, it may need to increase the prices at some periods. Recall that the prices were already at or above monopoly price to begin with, so both the firm and the customers lose as the prices go up. Even a small increase in customer patience causes a fairly large increase in average prices (see Figure 2(b)) and, as expected, the effect is more pronounced when a larger fraction of the population is strategic.

Fewer and Higher Prices Lead to Capacity Waste: At first glance, the presence of customers that are more patient would seem to lead to better use of resources. After all, high demand and low supply in one period, followed by low demand and high supply in the next period could be properly matched if customers are willing to wait. Indeed this phenomenon does show up in our numerical analysis to some extent, when customers switch from being completely impatient to willing to wait for one period (see the case $m_1 = m_2 = 3$, in Figure 3(c)). However, we mainly observe the opposite effect. As customers become more patient (for $s \geq 1$), the firm is forced to use fewer and higher prices. These prices lead to inefficient use of the firm's resources.

Capacity Waste Leads to Low Revenue and Reduced Welfare: The presence of patient customers forces the firm to offer few high prices and this causes its service to be underutilized. This phenomenon lowers the firm's revenue and reduces customer welfare. As customers become more patient, both revenue loss and customer welfare reduction become quite significant (for the case $m_1 = 1$, $m_2 = 5$, it can be seen that revenue and welfare for $s = 8$ are respectively 35% and 75%, of those for $s = 0$), as can be seen in Figure 3.

Thus, increased patience levels for customers (or flexibility of purchase time), leads to worse outcomes for the customers and the firm. This is analogous to Braess' paradox that arises in transportation problems (where opening a new road may lead to higher overall congestion in the network). However, the mechanism at work here is different than the one at Braess' paradox, since in our setting the lower welfare is a consequence of the firm's price adjustment (raising prices to maintain feasibility of solution).



(a) Revenue of the monopolist. (b) Welfare of customers. (c) Wasted capacity.

Figure 3: Revenue, customer welfare, and (wasted) capacity over 24 time periods of interest.

9 Conclusions

In this paper, we study a service firm's multi-period pricing problem in the presence of customers that are fully strategic with respect to their purchasing decisions, and heterogeneous with respect to their valuations, and arrival-departure times. A distinct feature of our model is the service guarantees provided by the firm, which ensure that any customer willing to pay the announced service price will be able to receive service. These guarantees force the firm to take the complex problem of how to ensure it has sufficient service capacity at every period onto itself, while tremendously simplifying the customers' decision-making process by removing all rationing risk.

Finding the optimal sequence of service prices turns out to be a non-convex optimization problem. Nevertheless by exploiting the optimality conditions and using dynamic programming techniques, we provide a tractable computational method for the solution of this problem. The prices used by the firm are higher and fewer in number than one would naively expect. This is because, presence of strategic customers, who are willing to wait for a cheaper future price, prevents the monopolist from setting low prices. A slight increase in the patience level of customers can lead to better usage of resources and, therefore, better outcomes for both the firm and the customers. However, as the population becomes more patient, the firm is quickly forced to use higher prices to guarantee availability of service. This leads to less efficient usage of capacity, lower revenues and lower utility for the customers.

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A Proof of Section 3

A.1 Proof of Lemma 1

First note that we can add the constraint $0 \leq \mathbf{p} \leq 1$ to the problem without loss of optimality, since customer valuations are bounded by 1. Consequently, it follows that for a given fixed ranking R, the set of consistent and feasible prices defines a closed and bounded set. Since the objective function is continuous in prices (for a fixed ranking), we conclude that optimal prices exist for any given ranking R. By maximizing over the finitely many possible rankings, we conclude that an optimal solution of OPT-2 exists.

For the second claim, observe that if $\mathbf{p}$ is a feasible solution of OPT-1, then $(\mathbf{p}, R^C(\mathbf{p}))$ is a feasible solution of OPT-2 with the same objective value. Thus, the maximum of OPT-2 is an upper bound on the supremum of OPT-1. Given an optimal solution $(\mathbf{p}^\star, R^\star)$ of OPT-2, and any $\epsilon > 0$, $\mathbf{p}^\star + \epsilon R^\star$ is a feasible vector of prices that is consistent with the ranking $R^\star$, and hence $(\mathbf{p}^\star + \epsilon R^\star, R^\star)$ is a feasible solution of OPT-2. This is because, if $R^\star_t < R^\star_{t'}$, then $p^\star_t \leq p^\star_{t'}$, and consequently $p^\star_t + \epsilon R^\star_t < p^\star_{t'} + \epsilon R^\star_{t'}$. Moreover, this inequality also implies that in $\mathbf{p}^\star + \epsilon R^\star$ no price is repeated, and hence the only consistent ranking with this price vector is $R^\star$. This implies that $R^\star$ is the customer-preferred ranking corresponding to $\mathbf{p}^\star + \epsilon R^\star$, and thus this price vector is feasible in OPT-1 with the same objective value. Since the objective of OPT-2 is continuous in prices for a fixed ranking $R^\star$, the value of $(\mathbf{p}^\star + \epsilon R^\star, R^\star)$ approaches to that of $(\mathbf{p}^\star, R^\star)$, as ϵ goes to 0. Thus for $\epsilon > 0$, $\mathbf{p}^\star + \epsilon R^\star$ is a feasible solution of OPT-1, value of which converges to maximum of OPT-2 as ϵ goes to 0. Since maximum of OPT-2 is an upper bound on the supremum of OPT-1, it follows that these values are equal, and $\mathbf{p}^\star + \epsilon R^\star$ converges to the supremum of OPT-1, as claimed.

If $\mathbf{p}$ is an optimal solution of OPT-1, then its value equals to the supremum value. However, as explained earlier this value equals to the maximum of OPT-2, and $(\mathbf{p}, R^C(\mathbf{p}))$ is a feasible solution of this problem with the same value. Thus, the claim follows.

B Proofs of Section 4

B.1 Proof of Proposition 1

Since the valuations are bounded by 1, it is not beneficial to set a price above 1. Now, suppose $(\mathbf{p}, R)$ is a feasible and consistent price ranking. Let $\mathbf{p}'$ be the price vector such that $p'_t = \max\{p_M, p_t\}$. We claim that $(\mathbf{p}', R)$ is both consistent and feasible. For consistency, note that if $R_t < R_{t'}$ then $p_t \le p_{t'}$. Hence, $\max\{p_M, p_t\} \le \max\{p_M, p_{t'}\}$. Therefore, $(\mathbf{p}', R)$ is consistent. Moreover, because we have (weakly) increased the prices, it is a feasible solution. Finally, observe that the revenue obtained from $(\mathbf{p}', R)$ is at least equal to the revenue of $(\mathbf{p}', R)$. The reason is $\rho_t(R)$ does not change, but the uncapacitated revenue function, $p(1 - F(p))$, increases. Namely,

$$\sum_t p_t(1 - F(p_t))\rho_t(R) \le \sum_t p'_t(1 - F(p'_t))\rho_t(R).$$

since by definition p_M maximizes $p(1 - F(p))$. Therefore, starting from a feasible solution, we can construct another one with weakly better objective value, where all prices are weakly above p_M, thus the claim follows.

B.2 Proof of Proposition 2:

Note that if for a ranking R, we have $\rho_t(R) = 0$, then, without loss of generality, we can let $p_t = 1$. Now, by Proposition 1, we can assume that $(\mathbf{p}, R)$ is an optimal solution of OPT-2 such that $p_M \le p_t \le 1$ and $p_t = 1$ if $\rho_t(R)(1-F(p_t)) = 0$ for all $t \in \{1, \ldots, T\}$. We will prove that $(\mathbf{p}, R)$ is such that p_t necessarily satisfies one of the conditions 1-3 of the proposition for all $t \in \{1, \ldots, T\}$.

By contradiction, assume that for $(\mathbf{p}, R)$ none of the conditions 1-3 hold at time t_0. Since conditions 1-2 do not hold $1 > p_{t_0} > p_M$. Note that $p_{t_0} \neq 1$ implies that $\rho_{t_0}(R)(1-F(p_{t_0})) \neq 0$, and hence $\rho_{t_0}(R) > 0$. Since condition 3 does not hold, for $\tilde{t} \in S \triangleq \{t \in \{1, \ldots, T\} | p_t = p_{t_0}\}$ we have that $c_{\tilde{t}}$ is not tight, i.e.,

$$\rho_{\tilde{t}}(R)(1 - F(p_{\tilde{t}})) < c_{\tilde{t}} \quad \text{for all } \tilde{t} \in S. \tag{17}$$

Let δ be a constant such that

$$\delta = \begin{cases} p_{t_0} - p_M & \text{if } p_{t_0} \leq p_k \text{ for all } k \in \{1, \dots, T\}, \\ p_{t_0} - \max_{\{t_1 | p_{t_1} < p_{t_0}\}} p_{t_1} & \text{otherwise.} \end{cases}$$

Consider the price vector $\hat{\mathbf{p}}$, for which $p_k = \hat{p}_k$ for $k \notin S$, and $\hat{p}_k = p_k - \epsilon$ otherwise, for some $0 < \epsilon < \delta$. It follows from the definition of δ that if $p_i \leq p_j$ for some $i, j \in \{1, \dots, T\}$ then $\hat{p}_i \leq \hat{p}_j$. Hence, the price vector $\hat{\mathbf{p}}$ is also consistent with ranking R. Moreover, since $(1 - F(p))$ is a continuous function, by Eq. (17) we conclude that ϵ can be chosen small enough to guarantee that for time periods $t \in S$, $\rho_t(R)(1 - F(\hat{p}_t)) < c_t$. Since $(\mathbf{p}, R)$ is feasible and $p_t = \hat{p}_t$ for $t \notin S$, it also follows that for $t \notin S$, we have $\rho_t(R)(1 - F(\hat{p}_t)) = \rho_t(R)(1 - F(p_t)) \leq c_t$. Consequently, $(\hat{\mathbf{p}}, R)$ is feasible in OPT-2.

The definition of δ also suggests that $p_M < \hat{p}_t < p_t = p_{t_0}$ for $t \in S$. It follows by the definition of p_M and the unimodality of the uncapacitated revenue function that $p_t(1 - F(p_t)) < \hat{p}_t(1 - F(\hat{p}_t))$ for $t \in S$. Thus, using the fact that $\rho_{t_0}(R) \neq 0$ we conclude that the revenue obtained from time periods $t \in S$, increases under $\hat{\mathbf{p}}$, i.e.,

$$\sum_{t \in S} \hat{p}_t(1 - F(\hat{p}_t))\rho_t(R) > \sum_{t \in S} p_t(1 - F(p_t))\rho_t(R). \tag{18}$$

Since, $p_t = \hat{p}_t$ for $t \notin S$, it also follows that $\sum_{t \notin S} \hat{p}_t(1 - F(\hat{p}_t))\rho_t(R) = \sum_{t \notin S} p_t(1 - F(p_t))\rho_t(R)$. Hence, we conclude that the overall revenue improves when $(\hat{\mathbf{p}}, R)$ is used. Therefore, we reach a contradiction and $(\mathbf{p}, R)$ has to satisfy one of the conditions 1-3 of the proposition.

C Proof of Section 5

C.1 Proof of Theorem 1

We first describe how the optimal prices and ranking in OPT-2 are obtained, and then we consider the computational complexity of the solution.

As explained in the text, given the set L, the optimal solution of OPT-2 can be obtained by solving OPT-3. The solution of the latter problem is identical to that of Eq. (9), with $i = 0$, $j = T + 1$, $\underline{p} = 0$, and hence the optimal value is equal to $\omega(0, T + 1, 0)$. Given $\omega(i, j, \underline{p})$, for $0 \leq i \leq j \leq T + 1$, one can construct the optimal sequence of prices in this problem using the recursion in Eq. (10): We say that k is the solution for $\omega(i, j, \underline{p})$ if the r.h.s. of Eq. (10) takes its maximum at k and k is the earliest time period that achieves the maximum. Let (k^*, p_{k^*}) be the optimal solution of $\omega(0, T + 1, 0)$ in Eq. (10). Then the price of time period k^* in the optimal solution of Eq. (9) is p_{k^*}, and prices for time periods earlier and later than k^* can be obtained by solving for the prices in the subproblems $\omega(0, k^*, p_{k^*})$ and $\omega(k^*, T + 1, p_{k^*})$.

We assume that at each step the left most subproblem is solved first. We say that the time period k^* which solves the ith subproblem has priority i (hence the time period which solves $\omega(0, T+1, 0)$ has priority 1). Using these priorities together with prices, we next construct the ranking vector (consistent with the already obtained prices) that appear in the solution of OPT-3 (or equivalently to Eq. (10) with $i = 0$, $j = T+1$, $\underline{p} = 0$). Consider time periods k_1 and k_2. If $p_{k_1} \neq p_{k_2}$, it is clear how to rank them: the lower price will have a smaller rank. Now suppose $p_{k_1} = p_{k_2}$, then the time period with lower priority receives lower ranking. Note that under this ranking, the ranking vector is consistent with prices. Moreover, when there are multiple time periods with the same price, the time period that has lower ranking is the one that is used by the algorithm to solve an earlier subproblem. This implies that the ranking is consistent with the time period each population receives service in the solution of the recursion Eq. (10).

We next characterize the computational complexity of providing a solution to OPT-2. Note that by Proposition 3, there exists an optimal solution for OPT-2 with prices that belong to set L (Eq. (8)). It can be seen from Eq. (7) that to compute the prices in this set we need quantities of the form $z_{ijk} = \sum_{t_1=i}^{k} \sum_{t_2=k}^{j} a_{t_1,t_2}$ for all $i \leq j \leq k$. Note that there are $O(T^3)$ values z_{ijk} can take, and each value takes at most $O(T^2)$ to compute. Thus, all values of z_{ijk}, and the set L can be computed in $O(T^5)$ time.[4] Thus, in $O(T^5)$ time we can reduce OPT-2 to OPT-3. We characterize the computational complexity of the latter problem.

Observe that the algorithm relies on characterizing $\omega(i, j, p)$ for all time periods $i \leq j$ and $p \in L$. Since cardinality of L is $O(T^3)$ (each z_{ijk} corresponds to an element as can be seen from Eq. (8)), there are $O(T^5)$ values of $\omega(i, j, p)$ that needs to be characterized. These can be computed, using the condition $\omega(i, j, \underline{p}) = 0$ if $i + 1 \geq j - 1$, and the recursion in Eq. (10). At each step of the recursion there are $O(T)$ different values k can take. On the other hand, for a given value of k, the corresponding optimal p_k can be computed in $O(1)$: Since for all $p \in L$ we have $p \geq p_M$, it follows that $\gamma_k^{ij}(p)$ is decreasing in p, provided that $p \in L$. Moreover, $\omega(i, j, p)$ is also decreasing in p for all i, j, since larger p corresponds to tighter constraints in Eq. (9). Thus, the p_k that solves Eq. (10) is the smallest $p \geq p_M$ that makes the capacity constraint feasible. Therefore, it follows that $p_k = \max\left\{p_M, F^{-1}\left(1 - c_k / \sum_{l=i+1}^{k} \sum_{m=k}^{j-1} a_{lm}\right)\right\}$, where the latter is the price that makes the capacity at time k tight. Since by construction both these prices belong to L, and elements of L were computed earlier, it follows that given k, p_k can be constructed in $O(1)$. Thus, we conclude that each step of the recursion in Eq. (10) can be computed in $O(T)$. Thus, the overall complexity of computing all $\omega(i, j, p)$ is $O(T^6)$.

Finally, given all values of $\omega(i, j, p)$, the construction of the prices, that solve Eq. (9) takes $O(T^2)$ following the procedure described in the beginning of the proof: to solve for each p_k, an instance of the recursion Eq. (10) needs to be solved. This takes $O(T)$ time, and there are $O(T)$ prices to be solved for. Similarly constructing priorities and rankings consistent with these prices takes another $O(T)$. Thus, the overall complexity of the algorithm is $O(T^5 + T^6 + T^2 + T) = O(T^6)$.

[4]The computation time can be decreased using the relation between different values of z_{ijk}. This is omitted, as it does not affect our final complexity result.

D Proofs of Section 6

D.1 Proof of Lemma 2:

The function $D_t(p_t, R, A)$ is weakly increasing in all the elements of the matrix A. Therefore, for all $A \in \mathcal{A}$, the tightest constraint among all of constraints of the form $M \leq \sum_{t=1}^{T} p_t D_t(p_t, R, A)$ is the one given by A^L. At optimal solutions of OPT-4, M should be replaced by the maximum value it can attain, which is $\sum_{t=1}^{T} p_t D_t(p_t, R, A^L)$. Similarly, the tightest constraint among of the constraints of the form $D_t(p_t, R, A) \leq c_t$ is the one given by A^U and c_t^L. Thus, the claim follows replacing constraints of this form by $D_t(p_t, R, A^U) \leq c_t^L$.

D.2 Proof of Proposition 4:

The constraint set of OPT-5 is identical to that of an instance of OPT-2 with parameters $(\mathbf{c}^L, A^U)$. Additionally, the objective functions of both problems are nonincreasing for all $p \geq p_M$. Since, Proposition 3 relied on the monotonicity of revenue in prices, and the properties of constraint sets, it follows that for OPT-5, a set L with $O(T^3)$ prices that contains all candidate optimal prices can be constructed (using parameters $(\mathbf{c}^L, A^U)$). Thus, we can still use the recursion in Eq. (10) to find the optimal sequence of prices. However, $\gamma_k^{ij}(p)$ needs to be modified slightly since in OPT-5 the feasibility constraints involve A^U, whereas, the revenue function involves A^L. Therefore, the recursion in Eq. (10) solves OPT-5 (again in $O(T^6)$), using the following modified definition of $\gamma_k^{ij}(p)$:

$$\gamma_k^{ij}(p) = \begin{cases} \left(\displaystyle\sum_{l=i+1}^{k} \sum_{m=k}^{j-1} a_{lm}^L \right) (1 - F(p))p & \text{if } \left(\sum_{l=i+1}^{k} \sum_{m=k}^{j-1} a_{lm}^U \right) (1 - F(p)) \leq c_k^L \\ -\infty & \text{otherwise.} \end{cases}$$

D.3 Proof of Proposition 5

Let $\mathbf{c}^*$, A^* denote the capacity and arrivals in a given problem instance. We will show that $V(\mathbf{c}^*, A^*) - V^{ROB}(\mathcal{C}, \mathcal{A}, \mathbf{c}^*, A^*) \leq \theta(2H+1)P(A^U)$. Since, $\mathbf{c}^*$, and A^* are arbitrary, the claim then follows from taking supremum over all $\mathbf{c}^*$ and A^*.

Let $V_R(\mathbf{c}, A, A^*)$ denote the revenue obtained by (i) offering a price vector consistent with ranking R, (ii) ensuring that prices are feasible for arrival matrix A and capacity vector $\mathbf{c}$, (iii) having arrival realization A^*, i.e.,

$$\begin{aligned} V_R(\mathbf{c}, A, A^*) = \max_{\mathbf{p} \geq 0} \quad & \sum_{t=1}^{T} p_t D_t(p_t, R, A^*) \\ s.t. \quad & D_t(p_t, R, A) \leq c_t \qquad \text{for all } t \in \{1, ..., T\} \\ & p_t \leq p_{t'} \text{ if } R_t < R_{t'} \qquad \text{for all } t, t' \in \{1, ..., T\}. \end{aligned}$$

Note that imposing the constraint $p_t \leq p_{t'}$ if $R_{t'} = R_t + 1$ (for all t, t') is equivalent to imposing the constraint $p_t \leq p_{t'}$ if $R_t < R_{t'}$ in the above optimization problem, due to the transitivity of the inequalities. Thus, we conclude

$$
\begin{aligned}
V_R(\mathbf{c}, A, A^*) = \max_{\mathbf{p}\geq 0} \quad & \sum_{t=1}^{T} p_t D_t(p_t, R, A^*) & \\
s.t. \quad & D_t(p_t, R, A) \leq c_t & \text{for all } t \in \{1, ..., T\} \\
& p_t \leq p_{t'} \text{ if } R_{t'} = R_t + 1 & \text{for all } t, t' \in \{1, ..., T\}.
\end{aligned} \tag{19}
$$

Let $\lambda_t \geq 0$ denote the Lagrange multiplier corresponding to the capacity constraint associated with time t, and $\mu_{t,t'} \geq 0$ be the Lagrange multiplier associated with the ranking constraint $p_t \leq p_{t'}$, assuming $R_{t'} = R_t + 1$. The KKT conditions (see, for example, Acemoglu (2008)) imply that for all t, the optimal prices satisfy:

$$D_t(p_t, R, A^*) + p_t \frac{\partial D_t(p_t, R, A^*)}{\partial p_t} - \lambda_t \frac{\partial D_t(p_t, R, A)}{\partial p_t} + \mu_{t'',t} - \mu_{t,t'} = 0, \tag{20}$$

where t, t', and t'' are such that $R_{t'} = R_t + 1$ and $R_t = R_{t''} + 1$. By the complementary slackness conditions, if two prices p_t and $p_{t'}$ are different, then $\mu_{t,t'} = 0$. Thus, summing the KKT conditions for all periods that have the same price p (and noting that ranking of such periods are necessarily consecutive), μ_t terms cancel, and we obtain:

$$\sum_{t:\ p_t = p} \left[D_t(p, R, A^*) + p \frac{\partial D_t(p, R, A^*)}{\partial p} - \lambda_t \frac{\partial D_t(p, R, A)}{\partial p} \right] = 0.$$

By definition $D_t(p, R, A) = \rho_t(R, A)(1 - F(p))$, where $\rho_t(R, A)$ is the R-induced potential demand for population matrix A. Hence, using the notation $F'(p) = f(p)$, we obtain

$$\sum_{t:\ p_t = p} \left[\rho_t(R, A^*)(1 - F(p)) - p\ \rho_t(R, A^*) f(p) + \lambda_t \rho_t(R, A) f(p) \right] = 0.$$

Rearranging terms, this equation leads to

$$\sum_{t:\ p_t = p} \rho_t(R, A) \lambda_t = \sum_{t:\ p_t = p} \rho_t(R, A^*) \left[p - \frac{1 - F(p)}{f(p)} \right] \leq \sum_{t:\ p_t = p} \rho_t(R, A^*),$$

where the inequality follows from the fact that optimal prices are bounded by 1, and $\frac{1-F(p)}{f(p)} \geq 0$. Thus, summing the above equality over all periods t (or all different price levels p that appear in an optimal solution) we obtain

$$\sum_{t=1}^{T} \rho_t(R, A) \lambda_t \leq \sum_{t=1}^{T} \rho_t(R, A^*) = P(A^*) \leq P(A^U), \tag{21}$$

where $P(A) = \sum_{i,j} a_{i,j}$. By the complementary slackness conditions, $c_t = D_t(p_t, R, A) = \rho_t(R, A)(1 - F(p_t)) \leq \rho_t(R, A)$ whenever the Lagrange multiplier $\lambda_t \neq 0$. Hence, the above inequality also implies

$$\sum_{t=1}^{T} c_t \lambda_t \leq P(A^U). \tag{22}$$

We next consider how $V_R(\mathbf{c}, A, A^*)$ changes as $\mathbf{c}$ increases and A decreases. The Envelope Theorem (see Acemoglu (2008)) suggests that the derivatives $\frac{\partial V_R(\mathbf{c},A,A^*)}{\partial c_t}$ and $\frac{\partial V_R(\mathbf{c},A,A^*)}{\partial a_{i,j}}$ are equal to

$$\frac{\partial V_R(\mathbf{c}, A, A^*)}{\partial c_t} = \lambda_t \quad \text{and} \quad \frac{\partial V_R(\mathbf{c}, A, A^*)}{\partial a_{i,j}} = -\lambda_{t'(i,j,R)}(1 - F(p_{t'(i,j,R)})), \tag{23}$$

where $t'(i, j, R)$ represents the period t' that has minimum ranking in R within $\{i, ..., j\}$, i.e., the time period population $a_{i,j}$ receives service.

Observe that by definition V_R is increasing in $\mathbf{c}$ and decreasing in A. Since $\frac{c_t^U}{c_t^L}$ and $\frac{a_{i,j}^U}{a_{i,j}^L} \leq 1 + \theta$, it follows that

$$\begin{aligned} 0 \leq V_R(\mathbf{c}^*, A^*, A^*) - V_R(\mathbf{c}_L, A^U, A^*) &\leq V_R(\mathbf{c}_U, A^L, A^*) - V_R(\mathbf{c}_L, A^U, A^*) \\ &\leq V_R((1+\theta)\mathbf{c}_L, A^L, A^*) - V_R(\mathbf{c}_L, A^L(1+\theta), A^*) \end{aligned} \tag{24}$$

Using the Fundamental Theorem of Calculus (and the notation $g_R(x) = V_R((1+x)\mathbf{c}^L, A^L, A^*)$) it follows that

$$\begin{aligned} 0 \leq V_R((1+\theta)\mathbf{c}_L, A^L, A^*) - V_R(\mathbf{c}_L, A^L, A^*) &= \int_{x=0}^{\theta} \frac{dg_R(x)}{dx} dx = \int_{x=0}^{\theta} \sum_{t=1}^{T} \frac{\partial V_R}{\partial c_t}((1+x)\mathbf{c}_L, A^L, A^*) c_t^L dx \\ &\leq \int_{x=0}^{\theta} \sum_{t=1}^{T} \frac{\partial V_R}{\partial c_t}((1+x)\mathbf{c}_L, A^L, A^*)(1+x) c_t^L dx. \end{aligned} \tag{25}$$

Observing from Eq. (23) that $\frac{\partial V_R}{\partial c_t}((1+x)\mathbf{c}_L, A^L, A^*)$ equals to the Lagrange multiplier λ_t for the problem instance with capacity vector $(1 + x)\mathbf{c}_L$, and using Eq. (22) and Eq. (25), we obtain

$$V_R((1+\theta)\mathbf{c}_L, A^L, A^*) - V_R(\mathbf{c}_L, A^L, A^*) \leq \int_{x=0}^{\theta} P(A^U) dx = \theta P(A^U). \tag{26}$$

Following a similar approach, we also obtain

$$\begin{aligned} 0 \leq V_R(\mathbf{c}_L, A^L, A^*) - V_R(\mathbf{c}_L, (1+\theta)A^L, A^*) &= -\int_{x=0}^{\theta} \sum_{i,j} \frac{\partial V_R}{\partial a_{i,j}}(\mathbf{c}_L, (1+x)A^L, A^*) a_{i,j} dx \\ &\leq -\int_{x=0}^{\theta} \sum_{i,j} \frac{\partial V_R}{\partial a_{i,j}}(\mathbf{c}_L, (1+x)A^L, A^*)(1+x) a_{i,j} dx \end{aligned} \tag{27}$$

Using Eq. (23), it follows that $-\frac{\partial V_R}{\partial a_{i,j}}(\mathbf{c}_L, (1+x)A^L, A^*) = \lambda_{t'(i,j,R)}(1 - F(p_{t'(i,j,R)})) \leq \lambda_{t'(i,j,R)}$, where λ_t denotes the Lagrange multiplier in a problem instance with parameters $\mathbf{c}_L, (1 + x)A^L, A^*$. Thus, using Eq. (27) and noting from the definition of $t'(i, j, R)$ that $\rho_t(R, A) = \sum_{i,j:t'(i,j,R)=t} a_{i,j}$, we obtain,

$$V_R(\mathbf{c}_L, A^L, A^*) - V_R(\mathbf{c}_L, (1+\theta)A^L, A^*) \leq \int_{x=0}^{\theta} \sum_{t=1}^{T} \lambda_t \rho_t(R, A^L(1+x)) dx. \tag{28}$$

Thus it follows from Eq. (21) that

$$V_R(\mathbf{c}_L, A^L, A^*) - V_R(\mathbf{c}_L, (1+\theta)A^L, A^*) \leq \int_{x=0}^{\theta} P(A^U)dx = \theta P(A^U). \tag{29}$$

Adding Eq. (26) and Eq. (29), and using it in the right hand side of Eq. (24) it follows that

$$V_R(\mathbf{c}^*, A^*, A^*) - V_R(\mathbf{c}_L, A^U, A^*) \leq 2\theta P(A^U). \tag{30}$$

Note that by linearity of the objective of Eq. (19) in its third argument, and the fact that $a^L_{i,j} \leq a^*_{i,j} \leq a^U_{i,j} \leq (1+\theta)a^L_{i,j}$, it follows that $V_R(\mathbf{c}_L, A^U, A^*) \leq V_R(\mathbf{c}_L, A^U, A^L)(1+\theta)$. On the other hand, since maximum price customers can pay for service is 1, it follows from the definition of $P(A)$ that $V_R(\mathbf{c}_L, A^U, A^L) \leq P(A^L) \leq P(A^U)$. Thus, we conclude $V_R(\mathbf{c}_L, A^U, A^*) - V_R(\mathbf{c}_L, A^U, A^L) \leq \theta P(A^U)$. Combining this with Eq. (30) we obtain

$$V_R(\mathbf{c}^*, A^*, A^*) \leq V_R(\mathbf{c}_L, A^U, A^L) + 3\theta P(A^U). \tag{31}$$

Maximizing both sides of this inequality over R and noting that $\max_R V_R(\mathbf{c}^*, A^*, A^*) = V(\mathbf{c}^*, A^*)$, we conclude $V(\mathbf{c}^*, A^*) \leq \max_R V_R(\mathbf{c}_L, A^U, A^L) + 3\theta P(A^U)$. Note that by definition $\max_R V_R(\mathbf{c}_L, A^U, A^L)$ equals the solution of OPT-5 and $V^{ROB}(\mathcal{C}, \mathcal{A}, \mathbf{c}^*, A^*)$ is larger than this solution (OPT-5 gives the worst case profits for optimal prices that are feasible for all capacities in $\mathcal{C}$, and arrivals in $\mathcal{A}$, whereas $V^{ROB}(\mathcal{C}, \mathcal{A}, \mathbf{c}^*, A^*)$ is the realized profit). Thus, we conclude $V(\mathbf{c}^*, A^*) \leq V^{ROB}(\mathcal{C}, \mathcal{A}, \mathbf{c}^*, A^*) + 3\theta P(A^U)$, and the claim follows.

E Proofs of Section 7

E.1 Proof of Lemma 3

Let $\mathbf{p}^\star$ and $R^\star$ denote an optimal solution of OPT-6. Observe that for all t, the set $\mathbf{P}_\epsilon \cap [p^\star_t, p^\star_t + \epsilon)$ contains a single element. Denote this element by $\hat{p}_t$.

We first show that $\hat{\mathbf{p}}$ is consistent with ranking $R^\star$. Note that if $R^\star_t < R^\star_{t'}$ then $p^\star_{t'} \geq p^\star_t$. Moreover, since we have $p^\star_{t'} + \epsilon \geq p^\star_t + \epsilon$, and $\hat{p}_k$ is characterized by intersection of $[p^\star_k, p^\star_k + \epsilon)$ with $\mathbf{P}_\epsilon$ for all k, it follows that $\hat{p}_{t'} \geq \hat{p}_t$, and hence the consistency claim.

By Assumption 2, $h_t(p, R^\star)$ is decreasing in p, for any R. Therefore, $(\{\hat{p}_t\}, R^\star)$ is a feasible solution of OPT-6. By Assumption 2 again, and the fact that $\hat{p}_t \in [p^\star_t, p^\star_t + \epsilon)$ for all t, it follows that

$$v = \sum_t g_t(p^\star_t, R^\star) \leq \sum_t \Big(g_t(\hat{p}_t, R^\star) + l_t \epsilon \Big). \tag{32}$$

On the other hand, by construction $\hat{p}_t \in \mathbf{P}_\epsilon$ for all t, thus $(\{\hat{p}_t\}, R^\star)$ is a feasible solution of OPT-7. Hence $v_\epsilon \geq \sum_t g_t(\hat{p}_t, R^\star)$, and together with Eq. (32), this implies that $v_\epsilon \geq v - \epsilon \sum_t l_t$.

E.2 Proof of Lemma 4

Construction of optimal prices and ranking, using the dynamic programming recursion in Eq. (13) is identical to the construction given in Theorem 1, and is omitted. In the rest of the proof we characterize the computational complexity of this construction.

In order to solve the recursion in Eq. (13) we compute all values of $\omega(i, j, p)$ by solving $O(T^2|\mathbf{P}_\epsilon|)$ subproblems. At each step of the recursion we solve for the optimal k and p. Finding these requires at most $O(T|P_\epsilon|)$ trials. Given a value of p and k, we need to evaluate $\hat{\gamma}_k^{ij}(p)$. This requires checking if constraints are satisfied in the subproblem (hence computing $h_k(p, R)$), and evaluating the corresponding objective value ($g_k(p, R)$) in the relevant subproblem. Thus, computation of $\hat{\gamma}_k^{ij}(p)$ can be completed in $O(s(T))$ time, and the overall complexity is $O\left(\frac{T^3 s(T)}{\epsilon^2}\right)$.