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Stable Maximum Throughput Broadcast in Wireless Fading Channels

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***Abstract*—This research considers network coded broadcast system with multi-rate transmission and dual queue stability constraints. Existing network coded broadcast systems consider single rate transmission without receiver queue constraints. First, we shall illustrate that broadcast without network coding cannot support maximum throughput in wireless fading channels. However, the network coded broadcast poses new constraints for the receivers to manage stable queues while the fading channel characteristics suggest the broadcast to operate at multi-rate to achieve higher throughput. In this research, we propose a joint scheduling and network coding (JSNC) strategy for such network coded broadcast system to achieve maximum throughput under queue stability constraint. In a single cell broadcast networks with exogenous arrivals of packets at the base station, we prove that JSNC can stabilize the system as long as the rate of the exogenous arrival flow is within the capacity region. Sufficient control parameters are provided in JSNC for trading off between sender's buffer and the receivers' buffers. JSNC can be viewed as a generalization of the classical backpressure scheduling rule to coded information flow. Alternatively, JSNC can also be viewed as an extension of network coding theory to queuing system.**

***Index Terms*-Fading, network coding, scheduling, stability.**

I. INTRODUCTION

Network coding theory is established by Ahlswede et al [1], and Li et al [2], mainly to address the multicast information flow problem in wired networks. Due to the broadcast nature of wireless channels, network coding can naturally be extended to wireless scenarios. In this research, we study wireless network coding for downlink broadcast, as illustrated in Fig. 1. Previous research along this direction [3][4] concluded that network coding can improve network throughput when link error exists. They show that the benefit of network coding depends on the link error rate. The higher the link error rate is, the larger benefit we can expect from network coding. Are these results the whole story? In this paper, we present our new insights into wireless network coding and view the classic problem from a completely new perspective.

To the best of our knowledge, all previous researches on network coded broadcast focus on applying different forms of network coding in order to resolve transmission errors. However, we believe that some of the core problems of network coded broadcast has neither been well defined nor been solved due to the following two reasons.

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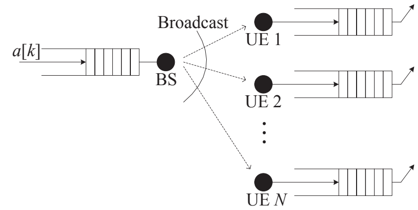

Fig. 1. System diagram.

1) When the base station (BS) can dynamically adjusting its transmission rate according to channel status, as will be clear later, network coding can support higher throughput than schemes without network coding, even the links can be made error free. However, network coded broadcast in multi-rate system has not yet been well studied.
2) When network coding is applied, the receivers need to buffer the received coded packets until they can decode them. Without well designed control algorithm, the undecoded packets may cause the receiver queues to be unstable. Traditional scheduling policies only study the stability of the sender's queue without considering the receivers' decoding queues.

To incorporate these two considerations, we formulate the network coded broadcast to be a new joint scheduling and network coding problem. Specifically, we investigate the throughput optimal scheduling and network coding policy under both the sender's and the receivers' queue stability constraints. The main contributions of this paper are summarized as follows.

First, we derive an optimal joint scheduling and network coding policy that is able to support maximum source rate while keeping the system queues stable. Specifically, we propose a network coding policy called *convolutional random linear network coding* (CRNC) and a scheduling policy called *bidirectional pressure* (BiP) and prove that a combination of CRNC and BiP can support maximum arrival rate. CRNC can be viewed as a new derivative of block random network coding while BiP can be viewed as a novel generalization of the well known *backpressure* principle to network coded information flows. However, the true innovation of the proposed JSNC lies in its integrated consideration of both the encoder queue

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and the decoder queues in its analysis.

Second, we discover the strategic tradeoffs among the queue lengths through extensive simulations using different control parameters and weighting functions, which will be useful in the design of practical network coded broadcast system.

II. Related Work

A. Linear network coding

The original linear network coding [2] generally requires central code construction. Ho et al [5] propose random network coding (RNC), whose network codewords are randomly generated. RNC can be seen as mixing packets along 'spatial' direction. Chou et al [6] propose block random network coding, which combines packet along both 'temporal' and 'spatial' directions. In essential, the 'spatial' direction network coding resolves network topology related bottlenecks and the 'temporal' directional network coding resolves transmission delay and link error related issues.

In the particular scenario of network coded broadcast, Keller et al [7] and Sundararajan et al [4][8] propose online network coding, which can dynamically combine packets in the sender's buffer according to receivers' feedbacks. Such approach can guarantee that the sender always broadcasts innovative information to the receivers if available. They investigate the error mitigation property of network coding, hence GF(2) is adequate. However, when applying network coding to fading channels with the sender side multi-rate support, the size of the Galois field must be expanded. More recent papers [9][10] discuss delay related issues in network coded broadcast. They use combinatorics based approaches with each packet attached a hard deadline.

B. Multiuser scheduling

Stoylar [11] use fluid limit technique to study scheduling policy under queueing model. Using this tool, Shakkottai et al [12] propose EXP rule and Andrews et al [13] popose M-LWDF/M-LWWF rule, which theoretically support maximal source rates under buffer stability constraints and show great performance in practice. Tassiulas et al [14] use Foster-Lyapunov criteria to study stochastic network scheduling problem and establish the well known backpressure scheduling policy. Eryilmaz et al [15] use Lyapunov drift to prove the stabilities of a broad class of scheduling policies in wireless fading channels. Neely [16] develops a powerful systematic technique that uses joint Lyapunov drift and convex optimization to design scheduling strategies under power or delay constraints in stochastic queueing networks. However, these results are not applicable to network coded broadcast because in this scenario, both the encoder queue and the decoder queues should be managed to keep the system stable.

C. Joint scheduling and network coding

Ho et al [17] study intra-session network coding in multicast scenario with queue stability constraint. Eryilmaz et al [18] use backpressure policy to adaptively perform inter-session network coding. Yan et al [19] study utility maximization problem in network coding enabled wireless multicast. Lakshminarayana et al [20] study the same problem in multi-rate scenario. The above approaches study JSNC problem from the information flow level, which are not able to characterize some important low level issues, including network code construction (i.e. deciding which packets should be combined together at each transmission opportunity) and decoder buffer stability. Chaporkar et al [21] report that applying network coding whenever possible in multi-rate multi-hop wireless networks may hamper network throughput. They propose a variant of backpressure policy to achieve optimal throughput in such scenario. However, their uncast model without considering decoder queue stability is significantly different from the multicast model in our work. The cellular topology used in our research allows us to use advanced network code and makes it possible to explore decoder queue stability.

Sagduyu et al [22] study the stability region of one-to-two or two-to-two topology with network coding in lossy channels. In this research, we propose a different approach focusing on the benefit of network coding in wireless fading channels facilitating multi-rate support. In fact, network coded broadcast has intrinsic connection with the network coding problem in multicast switch [8]. However, the underlying system and the analysis method in this research are significantly different from that of [8].

III. Motivation

In this section, we use an example to illustrate that network coding is essential in achieving the capacity of network coded broadcast. To make the illustration clear, we use infinite backlog model, i.e. all source packets have already been buffered in the BS rather than stochastic arrival. The same conclusion can be drawn for queueing model. The term plain broadcast is used for the scheme without network coding.

In this example, th BS broadcasts packets to three users (UEs). The channels can support two different rates, 1 or 4 packet/timeslot (ppt) depending on instantaneous channel quality. Let $r_i[t]$, $1 \le i \le 3$, $t \in \mathbb{N}$ represent UE i's channel rate at timeslot t, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. UE i can receive the broadcast packets at timeslot t if the BS's sending rate $r[t] \le r_i[t]$ and can receive nothing if $r[t] > r_i[t]$. The channel rates are i.i.d. between different timeslots and satisfy $\mathrm{P_r}(r_i[t] = 1) = \frac{1}{3}$, $\mathrm{P_r}(r_i[t] = 4) = \frac{2}{3}$, $1 \le i \le 3$. C_{pb} and C_{nc} represent the maximum throughput with and without network coding. From these assumptions, there are 8 channel states, indexed from 0 to 7, as illustrated in Table I. We use a 8-tuple to represent the static rate allocation policy. For example, $\Phi_0 = (1, 4, 4, 4, 4, 4, 4, 4)$ means that when the channel is in state 0, the broadcasting rate is 1ppt and when the channel is in state 1 to 7, the rate is 4ppt.

Because the object of this system is to maximize throughput and the three receivers' link condition are symmetric, it seems that the optimal policy might be Φ_0. For example, when the channels are in state 3, if the broadcast rate is 1ppt, each UE can receive one packet, and the overall throughput is 3 packets. However, if the rate is selected to be 4ppt, although

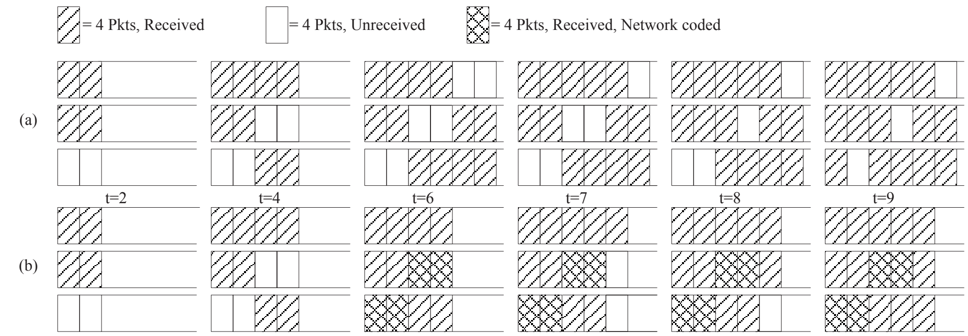

Fig. 2. Evolution of the receiver buffers at the beginning of the corresponding timeslot without (a) and with (b) network coding.

UE 1 can receive nothing, the other UEs can receive 4 packets each. The total throughput is 8 packets. It is easy to calculate that the average rate for each UE when the BS selects policy Φ_0 is $C_0 = \frac{73}{27}$ppt. However, we shall show that this optimal throughput cannot be achieved without network coding.

Assume that the channel state sequence is (6, 6, 5, 5, 3, 3, 4, 2, 1, $\cdots$). Fig. 2(a) illustrates the evolution process of the receivers' buffer without network coding. We can see that there is a 4-packet hole for each receiver at the beginning of timeslot 9. Unless sacrificing throughput, these holes cannot be filled because the average occurrence probability of states 3, 5, 6 are twice the probability of the states 1, 2, 4. However, these holes can be filled by performing network coding when needed, as illustrated in Fig. 2(b). At the beginning of timeslot 4, UE 2 possesses packet 1–8 and needs packet 9–16 while UE 3 possesses packet 9–16 and needs 1–8. If there is no network coding, the BS can only send packet 17–20 which are novel for both UE 2 and 3. When network coding is allowed, the BS can send the XOR of packet 1–4 and packet 9–12.

TABLE I
CHANNEL VARIATION PROBABILITY

Seq.	r_1	r_2	r_3	Probability
0	1	1	1	1/27
1	1	1	4	2/27
2	1	4	1	2/27
3	1	4	4	4/27
4	4	1	1	2/27
5	4	1	4	4/27
6	4	4	1	4/27
7	4	4	4	8/27

Since plain broadcast policy has to fill those holes, which results in throughput loss. We can calculate its maximal throughput $C_{pb} = \frac{65}{27} < C_0 = C_{nc}$. Note that the above throughput improvement of network coding is not related to channel error but due to multiuser diversity.

The above example illustrates that network coding is necessary to achieve maximum throughput in downlink wireless broadcast. However, the infinite backlog model hides the queue stability related issues. In this research, we carry out a thorough study of this problem using queueing model.

IV. Problem Formulation

A. System model

The downlink broadcast model used in this paper is illustrated in Fig. 1, where a BS broadcasts packets to N UEs, indexed from 1 to N. We assume the MAC layer encoding and decoding operation is transparent to the upper layers. Thus, when network coded packets are received by a UE, they are stored in the UE's decoder buffer. When the decoder is able to decode the buffered packets, it decodes and forwards those packets to upper layers immediately.

The system operates in discrete time. A time interval $[t, t+1)$, $t \in \mathbb{N}$ is called timeslot t. Denote by $a[t]$ the number of packets arrived in slot t. All arrivals in slot t happen at the time instant t and are available for service in slot t, but are not counted into the queue length at slot t. The channel states of different timeslots are i.i.d. The number of channel state is finite and its stationary distribution is denoted by $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_M)$. The sender can choose its sending rate from a bounded set $\mathcal{S} \in \mathbb{N}$. The sending rate is integer because it represents the number of packets broadcasted in that timeslot. Let $p_i^{(m)}(\mu)$ denote UE i's average packet delivery ratio when the channel is in state m and the BS broadcasts packets using rate $\mu \in \mathcal{S}$.

The BS knows exactly whether or not a particular packet has been received by UE i through feedback. Assume that each source packet is tagged with an integer sequence number starting from 0. $\mu[t]$ denotes the broadcast rate at timeslot t. Each timeslot t is divided into $\mu[t]$ sub timeslots, indexed by (t, l), $0 \le l < \mu[t]$, which represents the interval $[t + \frac{l}{\mu[t]}, t + \frac{l+1}{\mu[t]})$, $(t, \mu[t]) = (t+1, 0) = t+1$. In each sub timeslot, BS sends one packet. Let $\sigma_{i,\mu}^{(m)}[t, l]$ be a 0-1 indicator which represents whether or not the packet sent at sub timeslot (t, l) is successfully received by UE i when the channel is in state m with sending rate μ. We have $\mathrm{E}[\sigma_{i,\mu}^{(m)}[t, l]] = p_i^{(m)}(\mu)$.

Definition 1: [Convolutional Random Network Coding] The packet sent in sub timeslot (t, l) using CRNC is specified as:

$$\boldsymbol{P}[t, l] = \sum_{i=1}^{N} c_i[t, l] \cdot \boldsymbol{P}_{\mathrm{Next}_i[t,l]} \quad (1)$$

where $\boldsymbol{P}_{\mathrm{Next}_i[t,l]}$ is the source packet indexed by $\mathrm{Next}_i[t, l]$. $\mathrm{Next}_i[t, l]$ is the next sequence number expected by UE i at sub timeslot (t, l). The meaning of $\mathrm{Next}_i[t, l]$ will be clear later. $\boldsymbol{P}_{\mathrm{Next}_i[t,l]}$ is seen as a vector over Galois field $GF(2^D)$, $D \ge 1$ by grouping every D bits together to represent one symbol in $GF(2^D)$. $c_i[t, l]$ is randomly selected from $GF(2^D)$. $c_i[t, l] = 0$ if $p_i^{(m)}(\mu[t]) = 0$ or $\boldsymbol{P}_{\mathrm{Next}_i[t,l]}$ is unavailable. All addition and multiplication operations in

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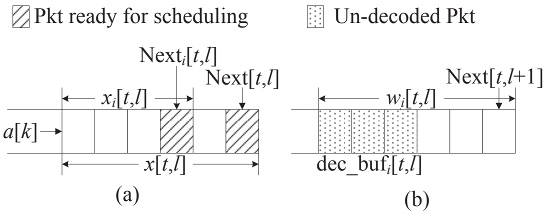

Fig. 3. Illustration of BS's queue (a) and UEs' queues (b).

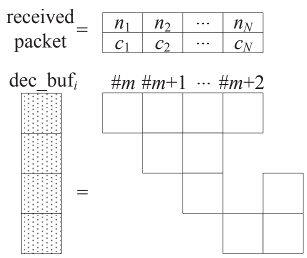

Fig. 4. Illustration of $\mathrm{dec\_buf}_i$.

(1) are over $GF(2^D)$. CRNC's dynamic coding operation is similar to the scheme in [4]. However, that scheme selects $c_i[t,l] \in GF(2)$, which may cause singularity problem and throughput penalty in multi-rate fading channels. $\boldsymbol{P}[t,l]$ is received at the end of sub timeslot (t,l).

Let $\boldsymbol{c}[t,l] = (c_1[t,l],\ c_2[t,l],\ \ldots,\ c_N[t,l])$ represent the coding coefficient vector and $\boldsymbol{n}[t,l] = (\mathrm{Next}_1[t,l],\ \mathrm{Next}_2[t,l],\ \cdots,\ \mathrm{Next}_N[t,l])$ represent the sequence vector, indicating which source packets are combined in the network coded packet. A network coded packet is characterized by $\boldsymbol{c}[t,l]$ and $\boldsymbol{n}[t,l]$. It needs to store these additional codeword information. This paper does not consider such overhead. In fact, $\boldsymbol{c}[t,l]$ can be dynamically generated using a seed and $\boldsymbol{n}[t,l]$ can be compressed using predictive coding techniques. The encoder buffer at the BS is illustrated in Fig. 3 (a). $\mathrm{Next}[t,l] := \min_{1\le i\le N}\mathrm{Next}_i[t,l]$ is the head of line packet in the sender's buffer. Let $x_i[t,l]$ be the number of packet which is not received by UE i in the BS's buffer. $x[t,l] := \max_{1\le i\le N} x_i[t,l]$ is the encoder's queue length.

The detail structure of UE i's decoder queue is illustrated in Fig. 3 (b). The length of the queue is denoted by $w_i[t,l]$. $\mathrm{dec\_buf}_i[t,l]$ is the set of undecoded packets. The packets in $\mathrm{dec\_buf}_i[t,l]$ are presented in echelon form, which is illustrated in Fig. 4. Each row represents the network codeword vector $\boldsymbol{c}[t,l]$. Each column represents the sequence number of combined packets. When a new packet is received, Gaussian elimination is employed if necessary to keep the echelon form. Using this expression, $\mathrm{Next}_i[t,l]$ is the first zero diagonal element's column index in $\mathrm{dec\_buf}_i$ (see Fig. 4). The decoding process uses Gaussian elimination.

B. Problem formulation

First, we shall define queue stability. Our definition conforms with [16].

Definition 2: [Queue length stability] Let overflow function for queue f be

$$g_f(V) = \limsup_{t\to\infty} \mathrm{E}\Big(\frac{1}{t}\sum_{i=0}^{t-1} 1_{f[t]>V}\Big) \tag{2}$$

where $1_X = 1$ if condition X satisfies and 0 otherwise. Then the system is stable if $g_x(V) \to 0$, $g_{w_i}(V) \to 0$, as $V \to \infty$.

Using the above symbol notations and assumptions, the core problem we try to solve can be formulated as follows.

Problem formulation: At each timeslot t, determine the sending rate $\mu[t]$ and the network code $(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l])$, so that the system is stable as long as the arrival rate is within the interior of network capacity region.

C. Symbol definitions and assumptions

$\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote the set of real, real nonnegative, and real positive numbers. $\mathbb{R}^N$, $\mathbb{R}^N_+$, $\mathbb{R}^N_{++}$ are their N-times product space. $\boldsymbol{A}^T$ is the transpose of matrix $\boldsymbol{A}$. $\|\boldsymbol{x}\| := \sqrt{\sum_{i=1}^N x_i^2}$. $\mathrm{conv}(\mathcal{X})$ is the convex hull of set $\mathcal{X}$.

1) $f_i(x)$: $\mathbb{R} \to [0,\infty)$, $1 \le i \le N$ are continuous, increasing functions. When $x \le 0$, $f_i(x) = 0$, $\lim_{x\to\infty} f_i(x) = \infty$;

2) Let $\boldsymbol{p}^{(m)}(\mu) = \left(p_1^{(m)}(\mu), p_2^{(m)}(\mu), \ldots, p_N^{(m)}(\mu)\right)^T$, define

$$\bar{\mathcal{S}} = \left\{ \sum_{m=1}^{M} \pi_m \cdot \mu^{(m)} \cdot \boldsymbol{p}^{(m)}(\mu^{(m)}), \mu^{(m)} \in \mathcal{S} \right\} \tag{3}$$

$$\mu_{\max} = \sup\left\{ \min_i\{\eta_i\} | \bar{\boldsymbol{\eta}} \in \mathrm{conv}(\bar{\mathcal{S}}) \right\} \tag{4}$$

$$\hat{\eta}_i = \sup\left\{ \eta_i | \bar{\boldsymbol{\eta}} \in \mathrm{conv}(\bar{\mathcal{S}}) \right\} \tag{5}$$

Both $\mathcal{S}$ and $\bar{\mathcal{S}}$ are compact sets, $\mu_{\max}$ characterizes the link layer capacity of the broadcast system. Let $\hat{\eta} = \sup \mathcal{S} < \infty$ be the maximum broadcast rate, $\lambda = \mathrm{E}(a[0])$, where $a[t]$ is the arrival process. Apparently, the encoder buffer will accumulate to infinity if $\lambda > \mu_{\max}$.

Similar to [15], some additional requirements are needed to guarantee the stability of the system. We summarize these constraints in Appendix A.

V. Joint Scheduling and Network Coding

A. Queue length evolution

At the beginning of timeslot t, the BS selects a sending rate $\mu[t] \in \mathcal{S}$ based on a scheduling rule. Let $a[t,0] = a[t]$ and $a[k,l] = 0, 1 \le l < \mu[t]$. $m[t]$ is the channel state at timeslot t. The queues evolve following (7)(8)(9)(10).

B. Stability

Let B_1, B_2, B_3 be constant integers, $B_1 > \hat{\eta}$, $B_2 > 0$, $B_3 \ge 0$. Besides the real queues $\boldsymbol{x}[t]$ and $\boldsymbol{w}[t]$, the system maintains a virtual queue $y_i[t]$ and an auxiliary variable $\zeta_i[t]$ for each UE, which is updated using the following rule:

$$\zeta_i[t+1] = \begin{cases} 1 & x_i[t+1] = x_{\min}[t+1]; \\ 0 & x_i[t+1] = x_{\max}[t+1], \\ & x_{\min}[t+1] \ne x_{\max}[t+1]; \\ \zeta_i[t] & \text{otherwise.} \end{cases} \tag{11}$$

$y_i[t]$ is updated according to the rule in Table II where $\Delta \in R_{++}$ is a constant parameter, $x_{\min}[t] = \min_{1\le i\le N} x_i[t]$,

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$$x_i[t,l+1] = x_i[t,l] + a[t,l] - \sigma^{(m[t])}_{i,\mu[t]}[t,l]\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \text{dec\_buf}_i[t,l]) \tag{7}$$

$$\text{Next}_i[t,l+1] = \text{Next}_i[t,l] + \sigma^{(m[t])}_{i,\mu[t]}[t,l]\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \text{dec\_buf}_i[t,l]) \tag{8}$$

$$\text{dec\_buf}_i[t,l+1] = \begin{cases} \text{dec\_buf}_i[t,l] \bigcup (\boldsymbol{n}[t,l], \boldsymbol{c}[t,l]) & \text{if } \sigma^{(m[t])}_{i,\mu[t]}[t,l] = 1; \\ \text{dec\_buf}_i[t,l] & \text{if } \sigma^{(m[t])}_{i,\mu[t]}[t,l] = 0. \end{cases} \tag{9}$$

$$\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \text{dec\_buf}_i[t,l]) = \begin{cases} 0 & \text{if } \text{rank}(\text{dec\_buf}_i[t,l] \bigcup (\boldsymbol{n}[t,l], \boldsymbol{c}[t,l])) = \text{rank}(\text{dec\_buf}_i[t,l]) \\ 1 & \text{if } \text{rank}(\text{dec\_buf}_i[t,l] \bigcup (\boldsymbol{n}[t,l]), \boldsymbol{c}[t,l])) = \text{rank}(\text{dec\_buf}_i[t,l]) + 1. \end{cases} \tag{10}$$

TABLE II
$y_i[t]$'S UPDATE RULE

$i=1$	0		
$i>1$	$\zeta_i[t]=\zeta_1[t]$		$y_i[t]$
	$\zeta_i[t]=1,\zeta_1[t]=0$		$y_i[t]-\Delta_1$ [a]
	$\zeta_i[t]=0,\zeta_1[t]=1$	$x_i[t]<\hat{\eta}$	$y_i[t]$
		$x_i[t]\geq\hat{\eta}$	$y_i[t]+\Delta_2$ [b]

[a] $\Delta_1 = \min(x_{\max}[t] - x_i[t], \Delta)$
[b] $\Delta_2 = \min(x_i[t] - x_{\min}[t], \Delta)$

$x_{\max}[t] = \max_{1\leq i\leq N} x_i[t]$, $\boldsymbol{y}[0] \in \mathbb{R}^N$, $y_1 \equiv 0$, $\boldsymbol{\zeta}[0] \in \{0,1\}^N$. Let $\mathcal{D}[t]$ be the set with maximum $\text{dec\_buf}_i[t,0]$:

$$\mathcal{D}[t] = \arg\max_{1\leq i\leq N}\{w_i[t] + x_i[t] - x[t]\} \tag{12}$$

Definition 3: [Bidirectional Pressure Policy] Let $\tau[0] = 0$, $w_{\max}[t] = \max_{1\leq i\leq N}(w_i[t])$. At the beginning of timeslot t:

1) If $\max_{1\leq i\leq N}(x_i[t] + y_i[t]) < B_1$ and $w_{\max}[t] > B_2$, or $\tau[t] > 0$, the BS selects its broadcast rate $\mu[t]$ as:

$$\mu[t] \in \text{agrmax}_{\eta\in\mathcal{S}}\{\eta \cdot p_i^{(m)}(\eta), i \in \mathcal{D}[t]\} \tag{13}$$

Let $\mathcal{D}'[t] = \{i | p_i^{(m)}(\mu[t]) = \max_{j\in\mathcal{D}[t]} p_j^{(m)}(\mu[t]), i \in \mathcal{D}[t]\}$, the broadcast packet is encoded as:

$$\boldsymbol{P}[t,l] = \sum_{i\in\mathcal{D}'[t]} c_i[t,l] \cdot \boldsymbol{P}_{\text{Next}_i[t,l]} \tag{14}$$

$$\tau[t+1] = \begin{cases} \tau[t] - 1 & \tau[t] > 0; \\ B_3 & \tau[t] = 0. \end{cases} \tag{15}$$

2) If $\max_{1\leq i\leq N}(x_i[t] + y_i[t]) \geq B_1$ or $w_{\max}[t] \leq B_2$, and $\tau[t] = 0$, $\mu[t]$ is determined by:

$$\mu[t] \in \arg\max_{\eta\in S}\{\sum_{i=1}^{N} f_i(x_i[t] + y_i[t]) \cdot \min(\eta \cdot p_i^{(m)}(\eta), x_i[t])\} \tag{16}$$

In each sub slot (t,l), CRNC (1) is used to construct the broadcasting packets, $\tau[t+1] = 0$. We call the combination of CRNC and BiP scheduling policy as JSNC.

The main result of this paper is summarized in Theorem 1, which characterizes the stability property of the JSNC policy.

Theorem 1: If JSNC is used in every timeslot t, $\lambda < \mu_{\max}(1-\epsilon)$, $D > \lceil log\frac{N}{\epsilon}\rceil$, $B_3 > \lceil\frac{\hat{\eta}}{\hat{\eta}_i - \lambda}\rceil - 1$, $\epsilon \in R_{++}$. Then $\exists\Delta_0$, as long as $0 < \Delta \leq \Delta_0$, both $x[t]$ and $\boldsymbol{w}[t]$ are stable.

Proof: Formal proof is included in Appendix A.

We give an intuitive explanation of the JSNC policy. We first discuss policy (16). Note that the UE corresponding to

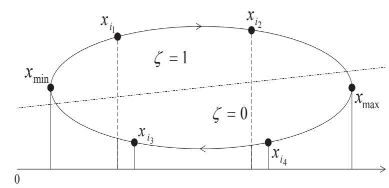

Fig. 5. Visualization of $y_i[t]$'s update rule.

$x_{\min}[t]$ has empty decoder buffer. If the scheduling policy can make each UE positive recurrently becomes $x_{\min}[t]$, the decoder buffer $\boldsymbol{w}[t]$ is stable. In fact, a combination of CRNC and classical backpressure, i.e. $y_i[t] = 0$ in (16), can make the encoder queue $x[t]$ stable. As $\lambda < \mu_{\max}(1-\epsilon)$, if the gap between λ and $\mu_{\max}$ can be well utilized, both $x[t]$ and $\boldsymbol{w}[t]$ can be stable. The upper bound constraint of Δ in Theorem 1 guarantees that the modification term $y_i[t]$ is conservative enough so that the stability of $x_i[t]$ is not violated. Fig. 5 visualizes how $y_i[t]$'s update rule is derived. When $\zeta_1[t] = 0$, UE 1 is in the lower half of the circle marked by $\zeta = 0$. From Table II, the $y_i[t]$ of those UEs on the upper half ($\zeta = 1$) decreases. Therefore, the scheduling policy biases towards the UEs in the lower ($\zeta = 0$) half and these UEs' $x_i[t]$ are more likely to becomes $x_{\min}[t]$. Similarly, when the $\zeta_1[t] = 1$, the $y_i[t]$ of those UEs on the lower half ($\zeta = 0$) increase. The scheduling policy also biases towards the UEs in the lower ($\zeta = 0$) half. The overall UEs tend to move clockwise around the circle and each $x_i[t]$ recurrently becomes $x_{\min}[t]$. Therefore, each UE's decoder buffer positive recurrently becomes full rank. As $\boldsymbol{y}[t]$ tends to equalize $x_i[t]$, $1 \leq i \leq N$, each $x_i[t]$ can becomes $x_{\min}[t]$ without changing very much, i.e. the time interval between UE i's decoder buffer becoming full rank is finite. Policy (13) highly biases towards the UE with the longest decoder queues, which can help reduce the average length of $\boldsymbol{w}[t]$.

Note that the field size bound $D > \lceil log\frac{N}{\epsilon}\rceil$ is loose. In practice, much smaller D is adequate for good performance. Furthermore, as $\lceil\frac{\hat{\eta}}{\hat{\eta}_i - \lambda}\rceil \leq \lceil\frac{\hat{\eta}}{\epsilon\mu_{\max}}\rceil$, B_3 can always choose a bounded value without knowing the source rate λ. In fact, B_3 is used to deal with boundary condition. In practical fading channels, due to its inherent randomness, B_3 can be selected to be very small. We will address these issues with simulations.

C. Realizations

Here, we address some practical issues. First, Theorem 1 shows that as long as the parameter Δ is small enough, JSNC can stabilize the system when the source rate is within the interior of $\mathcal{S}$. However, from the visual explanation of JSNC in Fig. 5, there is a tradeoff between the update speed Δ and the decoder buffer length $w_i[t]$. If Δ is small, the speed of the clockwise movement along the circle decreases, the time interval for a UE's decoder buffer to become full rank increases. Due to the weak connection between $w_i[t]$ and $y_i[t]$, the decoder buffers may become much larger than the encoder buffer. To mitigate this issue, B_1 and B_2 are introduced. When B_1 is big, B_2 characterizes the tradeoff between the encoder queue length and decoder queue lengths. When B_2 is small, the scheduler favors the decoder buffers so that the average decoder buffers are reduced. Otherwise, when B_2 is large, due to the conservative nature of Δ, in steady state, the decoder queues will be much longer than the encoder queue. By properly selecting B_2, without violating stability, the system's average queue size can be reduced.

However, how to choose Δ and B_2 are not explicitly specified in Theorem 1. In practice, we could dynamically adjust Δ and B_2 so that the system is stable and the average queue lengths are small.

VI. Simulation

In this section, we investigate the performance of JSNC through simulation. Unless explicitly specified, we use the settings as the example in Section III, so that we can compare our results with the theoretical value.

We primarily study two class of scheduling functions.

- Polynomial weighting (POLY),
$$f_i(x) = \begin{cases} x^\beta & x \geq 0 \\ 0 & x < 0 \end{cases}, \beta > 0;$$
- Exponential weighting (EXP),
$$f_i(x) = \begin{cases} e^{x^\kappa} - 1 & x \geq 0 \\ 0 & x < 0 \end{cases}, 0 < \kappa < 1.$$

To emphasize on the benefit of JSNC, we assume that there is no transmission error. The deliver probability satisfies:

$$p_i^{(m)}(\mu) = \begin{cases} 0, \mu \geq \mu_i^{(m)}; \\ 1, \mu < \mu_i^{(m)}. \end{cases} \quad (17)$$

where $\mu_i^{(m)}$ is UE i's supportable rate when its channel is in state m. The BS's and UEs' buffer limits are 2000. The default values of the parameters in JSNC policy are $\Delta = 0.4$, $\beta = 1$, $\kappa = 0.5$. The Galois field size is 256, $B_1 = 1000$, $B_2 = 750$, $B_3 = 0$, the arrival process is deterministic, i.e. $a[t] = \lambda$.

A. Minimum rate, pure scheduling, and JSNC

In this subsection, we compare the performance of JSNC with two other reference policies, named minimum rate scheduling and pure scheduling defined as follows.

1) *Minimum rate*: The BS chooses the minimum instaneous channel rate among all users as transmitting rate.

2) *Pure scheduling*: The base station chooses a transmitting rate by the following policy,

$$\mu[t] = \arg\max_{\eta \in \mathcal{S}} \sum_{i=1}^{N} f_i(x_i[t]) \cdot \eta \cdot p_i^{(m)}(\eta) \quad (18)$$

Then it transmits packets with the minimum $\text{Next}_i[t, l]$ among the users who can receive packets at that transmitting rate. Pure scheduling can be seen as a mimic of the backpressure scheduling to broadcast scenario.

In this subsection, we select POLY scheduling function. Besides the default configuration, we also adopt a more general setting: the number of user N is set to 5, the packet arrival rate is a constant, and the channel rate is chosen from a discrete set $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (ppt). For each UE, we generate 7 random numbers from $[0, 1]$, the corresponding 8 intervals are set as the UE's channel rate distribution.

The result is presented in Table III. EXA denotes the the channel rate distribution specified by the example in Section III. RND i, $1 \leq i \leq 4$ denotes the i^{th} group of randomly generated channel rate distribution.

TABLE III
Maximum arrival rate of different scheduling policy

	Min Rate	Pure scheduling	JSNC
EXA	2.1	2.3	2.7
RND 1	2.3	2.3	3
RND 2	2.6	2.5	3.4
RND 3	2.2	2.0	2.9
RND 4	1.8	1.7	2.3

From Section III, we know that the capacity of EXA is $\frac{73}{27} = 2.7037$ppt. The simulation results show that JSNC can support an arrival rate ≥ 2.65ppt (take the rounding error into account), which conforms to the conclusion of our analysis. On average, comparing with min rate policy and pure scheduling policy, JSNC can achieve significant throughput gains at about $20\% \sim 30\%$, respectively.

B. Impact of different parameters

This subsection discusses the impact of different form of scheduling function $f_i(\cdot)$, parameter Δ, and buffer threshold B_1, B_2, B_3.

1) Impact of scheduling functions: Fig. 6 shows CDF of $x_1[t]$ and $w_1[t]$ using different weighting functions. Other UE's corresponding CDF curve behaves similarly. To show the impact of different scheduling functions, we remove the effect of the thresholds by setting $B_1 = B_2 = 2000$. Then (16) in Theorem 1 is always used. The source rate is fixed at 2.5ppt, and we run 10^5 timeslots for each simulation. We can see that for POLY weighting function, the encoder maintains a small queue when the exponential coefficient β is small. The EXP weighting function approximately achieves the best encoder queue length distribution. We can also see from the figure that the decoder queue $w_i[t]$ is not sensitive to different weighting functions. This phenomenon is due to the weak connection between $w_i[t]$ and $y_i[t]$, which conforms to the intuitive explanation in the previous section.

2) Impact of Δ: Δ_0 depends on the arrival rate. Table IV presents Δ_0 as a function of the arrival rate.

We can see that Δ_0 increases when the arrival rate decreases. From Appendix A, we can see that a large Δ cannot support large throughput due to the fact that, in this case, the

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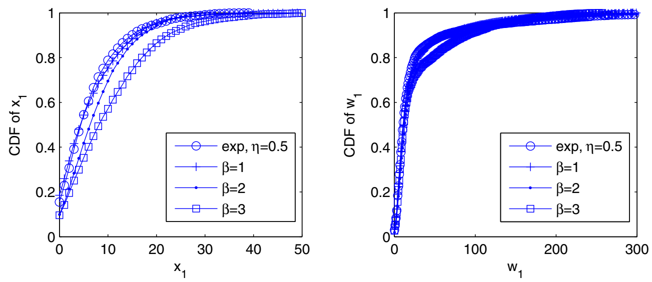

Fig. 6. CDF of $x_1[t]$ and $w_1[t]$ on different weighting coefficient.

TABLE IV
Δ_0 AS A FUNCTION OF λ.

λ	2.7	2.6	2.5	2.4
Δ_0	0.1	1	4	15

system spends too much time in the transient state and cannot converge to the optimal operation point. Table IV verifies these conclusions. From Section III, the capacity is $\frac{73}{27} = 2.704$ppt. The simulation conforms that JSNC can supports up to 2.7ppt when $\Delta \leq 0.1$. The loss of 0.004ppt is primarily due to finite buffer size and the finite field size (i.e. 2^8).

3) Impact of the encoder queue's threshold: We found that when B_1 is chosen properly large and B_3 is chosen properly small, the queue length distribution is not sensitive to these two values. This subsection investigates the relationship between the threshold B_2 and the queue length distributions by fixing $B_1 = 1000$ and $B_3 = 0$. The packet arrival rate is fixed at 2.5ppt. Fig. 7 shows the CDF of BS's and UEs' queue lengths on different threshold. When the threshold grows, the BS's queue decreases and the UEs' queues increase, i.e. there is a tradeoff between the encoder queue and the decoder queues. In fact, the policy (13)(14) bias to decoder queues while (16)(1) reacts slowly to the increase of the decoder queue. Therefore, B_2 can approximately be seen as the switch point of these two policies.

VII. Conclusion

In this research, we have developed a joint scheduling and network coding scheme to support stable maximum throughput broadcast in wireless fading channels. This scheme is fundamentally different from existing network coded broadcast systems in that maximum throughput broadcast is accomplished with multi-rate transmission and under the stable queue constraints at both base station and receivers. Both theoretical analysis and experimental simulations have been carried out to verify the performance of the proposed JSNC scheme.

The model used in this research requires the BS to be aware of each UE's status. We did not discuss the overhead of collecting this information. In practice, when the number of UE becomes large, such overhead cannot be omitted. To mitigate this difficulty, studying the impact of delayed feedback to network throughput is an important future work.

Appendix A

In this proof, we assume that the set $\bar{\mathcal{S}}$ is strictly convex. This assumption is not in essential, as long as $\bar{\mathcal{S}}$ is convex,

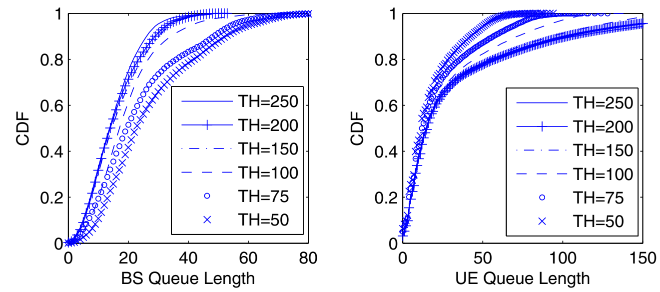

Fig. 7. CDF of UEs queue length and BS queue length on different threshold.

Theorem 1 holds. However, the strict convexity of $\bar{\mathcal{S}}$ simplifies the proof.

First we study deterministic model, and assume that the Galois field size is big enough so that $\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \text{dec\_buf}_i[t,l]) \equiv 1$. Both the arrival process and channel rate process are deterministic, i.e. $a[t] = \lambda$ and the sending rate is chosen from $\bar{\mathcal{S}}$. At the beginning of each timeslot t, if $\max_{1\leq i\leq N}(x_i[t] + y_i[t]) \geq B_1$ or $\max_{1\leq i\leq N} w_i[t] \leq B_2$, and $\tau[t] = 0$, the sender selects its rate $\boldsymbol{\mu}[t]$ according to:

$$\boldsymbol{\mu}[t] \in \arg\max_{\boldsymbol{\eta}} \{\sum_{i=1}^{N} f_i(x_i[t] + y_i[t]) \cdot \min(\eta_i[t], x_i[t]), \boldsymbol{\eta} \in \bar{\mathcal{S}}\} \tag{19}$$

and encode the packet using (1), $\tau[t+1] = 0$. Otherwise the scheduling rule is

$$\boldsymbol{\mu}[t] \in \arg\max_{\boldsymbol{\eta} \in \bar{\mathcal{S}}} \{\eta_i, i \in \mathcal{D}[t]\} \tag{20}$$

From the strictly convexity of $\bar{\mathcal{S}}$, if (20) is used, there is only one nonzero component in $\boldsymbol{\mu}[t]$. Denote the UE corresponding to this element as $I[t]$. The broadcasted packet is $\boldsymbol{P}_{\text{Next}_{I[t]}[t,l]}$, $\tau[t]$ is updated according to (15).

We now prove that the above policy is stable as long as $\lambda < \mu_{\max}$.

Let $\mathbf{1}_N = (1, 1, \ldots, 1)^T$ be a dim-N vector, $\dot{g}_i(z) = f_i(z)$, $L(\boldsymbol{z}) = \sum_{i=1}^{N} g_i(z_i)$. Define $\Delta_{\max} := \inf\{\|\lambda \cdot \mathbf{1}_N - \boldsymbol{\eta}\|, \boldsymbol{\eta} \in \bar{\mathcal{S}}\}$. As $\bar{\mathcal{S}}$ is convex and compact, λ is within the interior of $\bar{\mathcal{S}}$, we have $\Delta > 0$. From Taylor's theorem, $\exists z_i[t]$ between $x_i[t+1] + y_i[t+1]$ and $x_i[t] + y_i[t]$,

$$\begin{aligned} g_i(x_i[t+1] + y_i[t+1]) - g_i(x_i[t] + y_i[t]) \\ = f_i(z_i[t])(x_i[t+1] + y_i[t+1] - x_i[t] - y_i[t]) \end{aligned} \tag{21}$$

Further, from the strictly convex of $\bar{\mathcal{S}}$ and (19), $x_{\max}[t] \geq \hat{\eta} \Rightarrow x_i[t] \geq \mu_i[t]$. Select $\Delta < \frac{\Delta_{\max}}{2\sqrt{N}}$, when $\|\boldsymbol{f}(\boldsymbol{x}[t])\| > 0$

$$\begin{aligned} &\Delta L(\boldsymbol{x}[t] + \boldsymbol{y}[t]) \triangleq L(\boldsymbol{x}[t+1] + \boldsymbol{y}[t+1]) - L(\boldsymbol{x}[t] + \boldsymbol{y}[t]) \\ &= \sum_{i=1}^{N} f_i(z_i[t])(x_i[t+1] - x_i[t] + y_i[t+1] - y_i[t]) \\ &= \sum_{i=1}^{N} f_i(z_i[t])(\lambda - \mu_i[t]) + \sum_{i=1}^{N} f_i(z_i[t])(y_i[t+1] - y_i[t]) \end{aligned}$$

Let $\mathcal{E}[t] = \{i | 1 \leq i \leq N, x_i[t] < \hat{\eta}\}$, $\boldsymbol{\Delta}[t]$ is a N-dimension vector with $\Delta_i[t] = \begin{cases} \Delta & i \notin \mathcal{E}[t] \\ 0 & i \in \mathcal{E}[t] \end{cases}$, then the point $\lambda \cdot \mathbf{1}_N +$

$\boldsymbol{\Delta}[t]$ is strictly between the two hyper planes accross $\lambda \mathbf{1}_N$ and $\boldsymbol{\mu}[t]$ respectly with slop $\varphi_i[t] = \frac{f_i(z_i[t])}{\|\boldsymbol{f}(\boldsymbol{z}[t])\|}$. We have

$$\sum_{i=1}^{N} \varphi_i[t]\mu_i[t] - \sum_{i=1}^{N} \varphi_i[t]\lambda > \sum_{i=1}^{N} \varphi_i[t](\lambda + \Delta_i[t]) - \sum_{i=1}^{N} \varphi_i[t]\lambda$$

$$\Rightarrow \sum_{i=1}^{N} \varphi_i[t](\lambda + \Delta_i[t]) - \sum_{i=1}^{N} \varphi_i[t]\mu_i[t] < 0 \tag{22}$$

$$\begin{aligned} &\Delta L(\boldsymbol{x}[t] + \boldsymbol{y}[t]) \\ &\le \|\boldsymbol{f}(\boldsymbol{z}[t])\| \left(\sum_{i=1}^{N} \varphi_i[t](\lambda - \mu_i[t]) + \sum_{i=1}^{N} \varphi_i[t]\Delta_i[t] \right) < 0 \end{aligned}$$

Therefore, $\exists C_1 \in \mathbb{R}_{++}$,

$$L(\boldsymbol{x}[t] + \boldsymbol{y}[t]) = g_1(x_1[t]) + \sum_{i=2}^{N} g_i(x_i[t] + y_i[t]) < C_1 \tag{23}$$

We can draw the following three assertions.

Assertion 1. $\exists T_{\max} < \infty$, $\forall t$, $\exists 0 < T \le T_{\max}$, $\text{dec\_buf}_1[t+T]$ is full rank.

If $\forall t$, $\zeta_1[t] \equiv 0$, then unless $x_{\max}[t] = x_{\min}[t]$, $x_1[t] \neq x_{\min}[t]$, UE 1's decoder buffer is never full rank. If the scheduler never choose rule (19), there are two cases:

- If UE 1 is never scheduled, $x_1[t+1] - x_1[t] = \lambda$, i.e. $x_1[t] \to \infty$. From (23), $x_1[t] \le g_1^{-1}(C_1)$, we get a contradiction.
- $\exists t_0 < \infty$, UE 1 is scheduled. From (12), $\forall 2 \le i \le N$, UE 1's decoder buffer length at $t_0 + 1$ satisfies:

$$\begin{aligned} & w_1[t_0+1] + x_1[t_0+1] - x[t_0+1] \\ &= (w_1[t_0] + x_1[t_0] - x[t_0]) + \mu_1[t_0] \\ &> (w_i[t_0] + x_i[t_0] - x[t_0]) + \mu_i[t_0] \\ &= w_i[t_0+1] + x_i[t_0+1] - x[t_0+1] \end{aligned} \tag{24}$$

  Therefore, only UE 1 is scheduled for $\forall t > t_0$. As λ is within the interior of $\mathcal{S}$, $\exists T < \infty$, $x_1[t_0+T] = 0 = x_{\min}[t_0+T]$, we also get a contradiction.

Then (19) is recurrently trigged. The interval between successive trigger is bounded by

$$T_1 = \left\lceil g_1^{-1}(C_1) \left(\frac{1}{\lambda} + \frac{1}{\hat{\eta}_i - \lambda} \right) + 2 \right\rceil \tag{25}$$

When (19) is used, from Table II,

$$\sum_{i=1}^{N} y_i[t+1] - \sum_{i=1}^{N} y_i[t] < 0 \tag{26}$$

From the strictly convexity of $\bar{\mathcal{S}}$, if $x_i[t] + y_i[t] < 0$, $w_{\max}[t] > B_2$, and (19) is used, then $\mu_i[t] = 0$. From (26), $\exists 2 \le i_0 \le N$, $t_0 \in \mathbb{R}_+$, $y_{i_0}[t_0] < -g_1^{-1}(C_1) - \hat{\eta}$. So $\forall t \ge t_0$, $x_{i_0}[t] > g_1^{-1}(C_1) \ge x_1[t]$. This conclusion means that if $\zeta_i[t] = 1$, after a bounded time interval, $x_i[t] > x_1[t]$. $\forall t$, if $x_{\min}[t] \neq x_{\max}[t]$, $\{i | \zeta_i[t] = 1\} \neq \phi$, so $\exists t$, $x_1[t] = x_{\min}[t]$, we get a contradiction. From the above explanation, the time interval of $x_1[t]$ becomes $x_{\min}[t]$ is upper bounded by

$$T_2 = T_1 \cdot \left(\sum_{i=2}^{N} \left\lceil \frac{2g_i^{-1}(C_1) + \hat{\eta}}{\Delta} + 1 \right\rceil + \left\lceil \frac{B_2 + g_1^{-1}(C_1)}{\lambda} + 1 \right\rceil \right) \tag{27}$$

If $\forall t$, $\zeta_1[t] \equiv 1$, due to the same reason as illustrated before, the scheduler cannot continuously choose rule (20) more than T_1 timeslots. When (19) is used and $x_{\max}[t] \ge \hat{\eta}$,

$$\sum_{i=1}^{N} y_i[t+1] - \sum_{i=1}^{N} y_i[t] > 0 \tag{28}$$

Similar to the induction for the case of $\zeta_1[t] = 0$, the system cannot stay in the state $x_{\max}[t] \ge \hat{\eta}$ more than T_2 timeslots. If the system stays in the state with $x_{\max}[t] < \hat{\eta}$ and (19) is always selected, from (23) and the fact that when $\zeta_1[t] \equiv 1$, $y_i[t]$ are nondecreasing. Let

$$\mathcal{L}_1 = \left\{ \boldsymbol{x} + \boldsymbol{y} \;\middle|\; \begin{array}{c} 0 \le x_i < \hat{\eta}, y_i \ge -\hat{\eta}, L(\boldsymbol{x} + \boldsymbol{y}) < C_1, \\ \max_{1 \le i \le N}(x_i + y_i) \ge B_1 \end{array} \right\}$$

After no more than $|\mathcal{L}_1|$ timeslots, $x_1[t] = 0 = x_{\min}[t]$. Therefore, (20) is chosen recurrently. Once (20) is chosen, the rule will be used with at least B_3 continuous timeslots. As $B_3 + 1 > \lceil \frac{\hat{\eta}}{\hat{\eta}_i - \lambda} \rceil$, one UE's decoder buffer will become full rank. As N is finite, UE 1's decoder queue will becomes full rank. We get a contradiction. The time interval of $\zeta_1[t]$ to becomes 0 is upper bounded by

$$T_3 = T_2 \cdot |\mathcal{L}_1| \cdot N \cdot (B_3 + 1) \tag{29}$$

Let $T_{\max} = T_2 + T_3$, Assertion 1 holds.

Assertion 2. $\exists C_2 \in \mathbb{R}_{++}$, $\forall t$, $\max_{1 \le i \le N} |y_i[t]| < C_2$, $x_{\max}[t] < C_2$.

From Assertion 1, $x_1[t]$ recurrently becomes $x_{\max}[t]$ within a bounded period and $x_1[t] \le g_1^{-1}(C_1)$. Let

$$C_2 = \sum_{i=1}^{N} g_i^{-1}(C_1) + \lambda \cdot T_{\max} + \hat{\eta} \tag{30}$$

Assertion 2 satisfies.

Assertion 3. $\boldsymbol{w}[t]$ is bounded.

If $\exists w_i[t] \to \infty$, and (19) is never triggered, similar to the illustration in Assertion 1, UE i's decoder needs at most $T_{4,1} = \left\lceil C_2 \left(\frac{1}{\lambda} + \frac{1}{\hat{\eta}_i - \lambda} \right) + 1 \right\rceil$ timeslots to become full rank. So (19) is recurrently triggered. There are two cases:

- (20) is never used.
  If $\forall t$, $\zeta_i[t] \equiv 0$, after each time interval of UE 1 becomes $x_{\min}[t] \to x_{\max}[t] \to x_{\min}[t]$, unless $x_i[t] < \hat{\eta}$, $y_i[t]$ increase. However, $y_i[t]$ cannot increase to arbitrarily large due to (23). UE i can also not always stays within the state $x_i[t] < \hat{\eta}$ because in this case $L(\boldsymbol{x}[t] + \boldsymbol{y}[t])$ decrease. Let

$$\mathcal{L}_2 = \{\boldsymbol{x} + \boldsymbol{y} \,|\, |x_i| < C_2, |y_i| < C_2, \max_{1 \le i \le N}(x_i + y_i) \ge B_1\}$$

  Then $|\mathcal{L}_2|$ is finite. $x_i[t]$ becomes $x_{\min}[t]$ within a time interval of

$$T_{4,2} = \left(\left\lceil \frac{C_2 + B_2}{\lambda} + 1 \right\rceil + |\mathcal{L}_2| \right) \cdot \left\lceil \frac{2C_2}{\Delta} + 1 \right\rceil \cdot T_{\max} \tag{31}$$

If $\forall t$, $\zeta_i[t] \equiv 1$, after each round of UE 1, $y_i[t]$ decreases. Because $x_i[t] \geq -y_i[t] - \hat{\eta}$, $x_i[t] \to \infty$, which contradicts with Assertion 2. Therefore, each $\zeta_i[t]$ recurrently becomes 0 within a bounded interval

$$T_{4,3} = \left\lceil \frac{2C_2}{\Delta} + 1 \right\rceil \cdot T_{\max} \quad (32)$$

- Both (19) and (20) are recurrently triggered.
The induction of this case is similar to the above, UE i's decoder buffer needs at most

$$T_4 = T_{4,1} \cdot (T_{4,2} + T_{4,3}) \cdot N \cdot (B_3 + 1) \quad (33)$$

timeslots to become full rank.

From Assertion 2 and 3, Theorem 1 holds for deterministic model. Extending the deterministic model to stochastic model needs the following additional constraints.

1) $\forall M_1, M_2 \in \mathbb{R}_{++}$, $0 < \epsilon < 1$, $\exists B \in \mathbb{R}_+$, $\forall x > B$, $(1-\epsilon) f_i(x) \leq f_i(x - M_1) \leq f_i(x + M_2) \leq (1-\epsilon) f_i(x)$.

2) Let $\mu(m, \boldsymbol{x}, \boldsymbol{y})$ be the sending rate when channel is in state m with encoder queue vector $\boldsymbol{x}$ and virtual queue vector $\boldsymbol{y}$. $\forall A$, $\epsilon \in \mathbb{R}_{++}$, $\exists B \in \mathbb{R}_{++}$, such that for $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^N_+$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^N$ that satisfies $\|\boldsymbol{x}_1 + \boldsymbol{y}_1\|, \|\boldsymbol{x}_2 + \boldsymbol{y}_2\| < B$, and $|x_{1,i} - x_{2,i}|, |y_{1,i} - y_{2,i}| < A$, we have:

$$\Bigg| \sum_{i=1}^N f_i(x_{1,i} + y_{1,i}) \mu(m, \boldsymbol{x}_1, \boldsymbol{y}_1) p_i^{(m)}(\mu(m, \boldsymbol{x}_1, \boldsymbol{y}_1)) - \sum_{i=1}^N f_i(x_{2,i} + y_{2,i}) \mu(m, \boldsymbol{x}_2, \boldsymbol{y}_2) p_i^{(m)}(\mu(m, \boldsymbol{x}_2, \boldsymbol{y}_2)) \Bigg| < \epsilon \sum_{i=1}^N f_i(x_{1,i} + y_{1,i}) \mu(m, \boldsymbol{x}_1, \boldsymbol{y}_1) p_i^{(m)}(\mu(m, \boldsymbol{x}_1, \boldsymbol{y}_1))$$

3) The arrival process $a[t]$ is i.i.d. between different slots. It satisfies $\lim_{A \to \infty} A f_i(A) \mathrm{P}_r(a[t] > A) = 0$.

The proof can be done using the large number theory based technology in [15] or fluid limit technology in [12]. The detail is similar to [15] and we omit it due to page limit.

Now we analyze the throughput loss due to finite field size, i.e. $\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \mathrm{dec\_buf}_i[t,l]) = 0$. At each sub timeslot, the probability of sending a non-novel network coded packet to UE i is maximized when the dimension of $\mathrm{dec\_buf}_i[t,l]$ is K and its rank is $K-1$. The corresponding probability is

$$\frac{2^{D(K-1)}}{2^{DK}} = 2^{-D} \quad (34)$$

We have:

$$\Pr(\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \mathrm{dec\_buf}_i[t,l]) = 0) \leq 2^{-D} \quad (35)$$

Therefore, as long as $D > \left\lceil log_2 \frac{N}{\epsilon} \right\rceil$, i.e. $\epsilon > N 2^{-D}$, the rate region considering such loss, denoted by $\alpha\mathcal{S}$ satisfies:

$$\begin{aligned} \alpha &\geq 1 - \sum_{i=1}^N \Pr(\Phi(\boldsymbol{n}[t,l], \boldsymbol{c}[t,l], \mathrm{dec\_buf}_i[t,l]) = 0) \\ &\geq 1 - N 2^{-D} > 1 - \epsilon \end{aligned} \quad (36)$$

From the condition of $\lambda < (1-\epsilon)\mu_{\max}$, Theorem 1 holds. Note that in this scenario, $\Delta_{\max} = \inf \left\{ \|\lambda \cdot \mathbf{1}_N - \boldsymbol{\eta}\|, \boldsymbol{\eta} \in \alpha\bar{\mathcal{S}} \right\}$.

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