# Distributed Ranking in Networks with Limited Memory and Communication

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**Abstract– We consider the problem of information ranking in a network where each node has an individual preference over a set of candidates and the goal for each node is to correctly identify a list of top ranked candidates with respect to aggregate ranking scores. This is to be achieved by a distributed algorithm that uses limited memory per node and limited communication between each pair of nodes.**

**We show that this problem reduces to a plurality selection problem where the goal for each node is to identify one candidate with the largest aggregate ranking score and we provide an efficient randomized algorithm for this reduction.**

**We further study a plurality selection algorithm for the general case of $m > 1$ candidates that is lightweight in using only $\log_2(m) + 1$ bits of memory per node and the same amount of bits per node pair-wise communication. We establish correctness of the algorithm for the case of complete graphs with high probability as the number of nodes grows large, and establish tight convergence time bounds.**

## I. Introduction

In this paper, we consider information aggregation in network settings where each node in the network has an individual preference over a set of candidates and the goal for each node is to identify a set of candidates with the largest aggregate preference. This is to be achieved in a decentralized fashion based on limited information exchanged at pair-wise node interactions and limited amount of state stored by individual nodes. Such decentralized network information aggregation problems attracted considerable recent research interest, e.g. quantized distributed averaging [11], composite hypotheses testing [2] and consensus problem [2], [7], and are of interest in the context of sensor networks, distributed databases, and online social networks.

Specifically, we consider a *ranking problem* over a set of $m > 1$ candidates where each node has a preference score for each candidate and each candidate is associated with aggregate ranking score defined to be the sum of scores of all nodes for this candidate. The goal for each node is to identify a set of $l$ candidates with the largest ranking scores and sort them in decreasing order with respect to their ranking scores. For the special case $l = 1$, the problem boils down to a *plurality selection* where the goal for each node is to identify a candidate with the largest aggregate ranking score. Previously-studied consensus problem [7], [20] is a special case of plurality selection where the selection is over a set of two candidates. The challenge is to show how the ranking problem can be solved efficiently using a decentralized algorithm with limited memory and communication.

In this paper we show that the ranking problem can be reduced to a plurality selection problem over a set of rankings, and we provide a decentralized plurality selection algorithm. In particular, we propose a randomized algorithm where each node uses a simple random procedure to select a ranking of candidates in a way that allows inference of the top ranking using a plurality selection algorithm. For the top ranking set of size $l$, the state space is of size $O(m^l)$. We then introduce a plurality selection algorithm that uses a simple automata with only $2m$ states of per-node memory and node pair-wise communication for the problem of plurality selection over $m$ candidates. The state of a node indicates the identity of a candidate that the node would select as a plurality candidate according to the current belief of this node and one extra bit that indicates the strength of the belief of this node, which altogether can be encoded with $\log_2(m) + 1$ bits. For the case of majority selection over two candidates, this algorithm includes the ternary protocol studied in [20] as a special case. We provide analysis of the correctness and efficiency of the plurality selection algorithm for the case of complete graphs where each node communicates with any other node at some rate. Albeit a special case, the complete graphs are a good approximation for networks where peering between nodes are pseudo-random, e.g. various peer-to-peer networks, and have been considered in prior work, e.g. [18], [20].

### A. Related Work

Our work relates to that on information aggregation in networks where a variety of problems were studied [1], [2], [4], [5], [10]–[13], [17]. Our work is also related to the selection[1] and sorting problems considered in the context of distributed systems, e.g. [15], [18], and some early work on hypothesis testing with limited memory in the context of information theory [9], [14] but in a network setting, as well as that of interacting particle systems, e.g. standard voter model [16], [19]. In this section we focus on the work that is most closely related to ours.

The most closely related work to ours is that on the consensus problem [7], [20] which is equivalent to the majority selection over a set of two candidates. In [20], the authors proposed a ternary algorithm for the consensus problem which uses three states for per-node memory and three states per pair-wise communication. It was showed that for the case of the complete graph of $n$ nodes, the algorithm fails to correctly identify the majority candidate with probability of error that diminishes to zero exponentially fast with $n$ and that the convergence time is $\Theta(\log(n))$. A quaternary algorithm for two candidates under a two-way communication model was proposed in [7] which guarantees correct selection with probability 1 for any finite connected graph and whose expected convergence time was showed to be $O(\frac{1}{\beta(n)}\log(n))$ where $\beta(n) > 0$ is a lower bound on the absolute eigenvalues of some matrices that characterize the rates of node pair-wise interactions; for the case of the complete graph of $n$ nodes, $\beta(n)$ is independent of $n$ and is equal to the fractional margin by which the majority candidate is preferred over the other candidate. Our work is different from this prior work in that we consider the general problem of plurality selection.

[1]In the distributed $l$-selection, the goal is to identify the $l$-th largest (or smallest) element of a distributed set of elements with respect to values associated to elements.

In [3], Benezit et al considered the problem of plurality selection over a set of $m > 1$ candidates, assuming existence of a majority candidate. They showed that for arbitrary connected networks and under a two-way communication model, convergence to correct selection can be guaranteed for the cases $m = 2, 3$ and $4$ using 4, 15, and 100 states, respectively. For the binary case, the algorithm is the same as that studied in [7]. On the one hand, our work is more general as we do not assume uniqueness of a plurality candidate. On the other hand, our algorithm is more restrictive as it cannot guarantee correctness for arbitrary graphs, but it guarantees correctness with high probability for large complete graphs. It is noteworthy that our algorithm is lightweight in requiring only $2m$ states of memory and communication where the algorithm in [3] requires exponentially many states on $m$. Our work could be seen as a first step towards understanding what are necessary and sufficient memory and communication requirements for solving the plurality selection problem for general networks.

### B. Summary of our Contributions

Our contributions can be summarized in the following points:

• We propose a simple randomized algorithm for distributed ranking which reduces the problem to plurality selection over a set of rankings (Section III). We show that the plurality candidate over this set of rankings corresponds to the correct top ranking with probability of error that is diminishing exponentially to zero with the number of nodes. The algorithm applies to arbitrary connected networks provided a plurality selection algorithm is used which ensures correctness for arbitrary connected networks.

• We introduce a plurality selection algorithm for the general case of $m > 1$ candidates that is simple and lightweight in using only $2m$ states of memory per node and communicating one of $2m$ states at each communication instance between a pair of nodes (Section IV). We establish correctness of the algorithm with high probability and study its convergence properties for large complete graphs.

• For our plurality selection algorithm, we identify the large-system limit dynamics and show that under this limit dynamics, the convergence time of the plurality selection algorithm is logarithmic in both $1/\delta$ and $1/\gamma$ where $\delta$ is given fraction of nodes in a non-plurality state at the stopping time, and $\gamma$ is the gap between the aggregate ranking score of a plurality candidate and the largest ranking score of a non-plurality candidate. Specifically, we show that the convergence time $t_\delta$ is such that

$$t_\delta \leq (2m-1)\left[\log(1/\delta) + \log(1/\gamma)\right] + C_m,$$

where $C_m$ is a positive constant that depends only on $m$ and show that this bound is tight up to a constant factor (Section IV-A).

• We establish a general convergence result that enables us to relate the original stochastic system with the large-system limit dynamics over time horizons that are allowed to grow logarithmically with the system size $n$ (Section IV-B). Using this result, we show that the fraction of nodes that are in a non-plurality state at time $t_\delta$, for $\delta = 1/n^\alpha$, is $O(n^{-\alpha})$, with high probability, for every sufficiently small $\alpha > 0$.

• We provide simulation results to demonstrate the main correctness properties of the plurality selection algorithm (Section V).

## II. Problem Formulation

In this section, we introduce the ranking problem and present our algorithms.

### A. Basic Setup

We consider a network represented by a graph $G = (V, E)$ where the set of vertices $V = [n] = \{1, 2, \dots, n\}$ corresponds to a set of $n > 1$ nodes and edges corresponds to links between nodes. For each node $i \in V$, a node $j \in V$ is said to be a neighbor of node $i$ if $(i, j) \in E$. For our analysis of plurality selection algorithm we will assume that the graph is complete and in this case $(i, j) \in E$ for every node $i, j \in V$ such that $i \neq j$. Each node is assumed to communicate with each of its neighbors at some time instances. Specifically, we admit the standard asynchronous communication model (e.g. [4], [7], [20]) where each edge between a pair of nodes $(i, j) \in E$ is activated at instances of a Poisson process of rate $\lambda_{i,j} \geq 0$. In an instance of such a communication, we will call $i$ an *observer* node and $j$ *a contacted* node. At each communication, the observer node changes its state based on the state of the contacted node and its own. As a special case, we consider the complete graph and assume that contact rates by individual nodes are identical (without loss of generality assumed to be equal to 1), $\lambda_{i,j} = 1/(n-1)$, for every $i, j \in V$ such that $i \neq j$.

We denote with $C = [m] = \{1, 2, \dots, m\}$ a set of $m \geq 2$ candidates. The preference of each node $i \in V$ over candidates is described by a vector of ranking scores $\vec{v}_i = (v_{i,1}, v_{i,2}, \dots, v_{i,m})$ such that $v_{i,j} \geq 0$ quantifies the preference of node $i$ for candidate $j$ and we assume the preference vector is normalized such that $\sum_{j=1}^m v_{i,j} = 1$, for every $i$. In particular, if each node prefers exactly one candidate so that for each node $i$, $v_{i,j} = 1$ for some $j \in C$ and $v_{i,l} = 0$ for every $l \neq j$, then each vector of ranking scores $\vec{v}_i$ can be expressed as the preferred candidate $j \in C$. In this case, we call $j$ a *single candidate* of node $i$. We let $v_j$ denote the *aggregate preference* for candidate $j \in C$ over all nodes defined by $v_j = \frac{1}{n}\sum_{i=1}^n v_{i,j}$. Without loss of generality we assume that candidates are enumerated in decreasing order of aggregate preference so that $v_1 \geq v_2 \geq \cdots \geq v_m$.

A *top $l$ ranking* is defined as a tuple of $1 \leq l \leq m$ candidates sorted in decreasing order of the aggregate ranking scores, so that each candidate from this tuple has an aggregate ranking score that is at least as large as the aggregate ranking score of any candidate that is not in the tuple, i.e. according to the assumed enumeration, a candidate $j$ is in a top $l$ ranking only if $v_j \geq v_l$. A top $l$ ranking system includes positional voting systems such as Borda count [6] as a special case.

We denote with $k$ the number of candidates with the largest aggregate preference, i.e. $k$ is an integer $1 \leq k \leq m$ such that $v_1 = \cdots = v_k > v_{k+1} \geq \cdots \geq v_m$. A candidate $i$ is said to be a *plurality candidate* for $1 \leq i \leq k$ and is said to be a *non-plurality candidate* for $k < i \leq m$. Notice that if there is a unique plurality candidate, i.e. $k = 1$, then this candidate is necessarily preferred by a majority of nodes which indeed does not necessarily hold if $1 < k \leq m$.

In the remainder of this section we introduce our algorithms for top $l$ ranking and plurality selection.

### *B. Randomized Distributed Ranking*

We consider randomized distributed ranking algorithm that is defined as follows. Given $1 \leq l \leq m$, let $p$ be a decreasing probability distribution on the set of values $[l] = \{1, 2, \ldots, l\}$. For example, one may choose $p_j = \frac{2}{l(l+1)}(l + 1 - j)$, $j = 1, 2, \ldots, l$. Then the algorithm is defined as given in the following.

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**Randomized Distributed Ranking**

1) Each node $i$ that holds a vector of ranking scores $\vec{v}_i$ selects candidate $c_i \in C$ with probability $\mathbb{P}(c_i = j) = v_{i,j}$, for $j \in C$. After this assignment, we consider $c_i$ as the single preferred candidate of node $i$.
2) Each node $i$ that prefers candidate $c_i$ assigns a set of distinct candidates to ranks $1, 2, \ldots, l$ as follows.
   - Candidate $c_i$ is assigned rank $r$ where $r$ is a random sample from the distribution $p$.
   - Other ranks are assigned randomly by random sampling without replacement from the set of candidates $C \setminus \{c_i\}$.
3) Run a plurality selection algorithm over the set of $l$ rankings of individual nodes.

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Notice that each node constructs a ranking of $l$ candidates using only the knowledge of identities of candidates in the set $C$ and the identity of the most preferred candidate of this node; in particular, the node does not use any knowledge that involves preference over candidates by other nodes. The set of $l$ rankings contains $m(m-1)\cdots(m-l+1) = O(m^l)$ elements. The ranking of candidates by the node is constructed by a random assignment of candidates to ranks in a biased way so that the preferred candidate of the node is assigned a higher rank position with larger probability than any other candidate. This biased random assignment is the key idea that allows us to infer a top $l$ ranking by running a plurality selection algorithm over the ranking states of nodes, which we will show in the next section.

If the goal is to identify a set of $l$ distinct candidates with the largest aggregate preferences (and not a ranking amongst them), then the algorithm presented above is slightly modified as follows: the most preferred candidate of the node is assigned rank $r$ that is sampled uniformly at random from the set of ranks $1, 2, \ldots, l$, i.e. $p_j = 1/l$, for $j = 1, 2, \ldots, l$.

### *C. A Plurality Selection Algorithm*

We introduce a plurality selection algorithm over a set of $m > 1$ candidates where at every time instant each node is in one of $2m$ states and each communication between a pair of nodes amounts to communicating one of these $2m$ states. Each node is assumed to be in either strong or weak state that corresponds to a candidate in $C$. Initially each node assigns strong state of its preferred candidate. The algorithm is specified by the state update rules presented in the following.

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**Plurality Selection**

At each communication instance between two nodes:

1) If the observer node is in strong state $j$ and the contacted node is in a different strong state, then the observer node switches to weak state $j$.
2) If the observer node is in weak state $j$, it switches to the state of the contacted node.

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We will provide analysis of this algorithm for the case of complete graphs and show that the algorithm ensures identifying a plurality candidate by nodes with high probability for large $n$ and establish tight characterizations of the convergence time. The algorithm is similar in spirit to those proposed earlier based on adding extra states. For example, in [20] a ternary protocol was proposed for majority selection over two candidates by adding an extra "neutral" state $e$ to states 0 and 1 that correspond to the two candidates. Similarly, for the same binary setting, a quaternary protocol for two-way communication was considered in [7] where states 0 and 1 are added extra states $e_0$ and $e_1$, respectively.

The essence of our plurality selection algorithm is for a node to adopt a state that corresponds to a different candidate than the candidate indicated by the current state only if the node is in a weak state and switching into a weak state whenever the node is in a strong state and observes a different strong state from the contacted node. The weak state essentially serves to remember the last strong state in which the node was. The state of the node can be encoded by keeping the identity of the candidate that corresponds to the current state and keeping an extra bit to distinguish whether the state of a node is strong or weak. Therefore, the algorithm requires $\log_2(m)+1$ bits of memory per node and communication at every communication of a pair of nodes.

One would note that in the plurality selection algorithm the update rules for a node in a weak state is independent on the identity of the weak state, which may suggest that the memory and communication requirements of the algorithm could be further compressed by lumping all different weak states into a single weak state. However, we will show that for this algorithm it is crucial to distinguish different weak states, as otherwise the algorithm cannot guarantee identification of a plurality candidate. The lumping can be done if there exists a majority candidate, i.e. $l = 1$, as we will see that in this case the large system limit is such that the fraction of nodes in a weak state diminishes to zero. The plurality selection algorithm can be seen as a generalization of the ternary protocol in [20] as for the case of two candidates we can lump the two weak states into one single state and in this case the algorithm is exactly the ternary protocol in [20]. Otherwise, for more than two candidates, we may have multiple plurality candidates, i.e. $1 < l \leq m$, and we will see that in this case the fraction of nodes in a weak state converges to a positive constant, and, thus, it is needed to maintain identities of weak states in order to allow these nodes to decide upon a plurality candidate.

**State Evolution**. Under the assumed asynchronous communication model, the system state evolves according to a continuous-time Markov process which we introduce in the following. Let $S_i(t)$ and $W_i(t)$ be the number of nodes that at time $t$ are in strong $i$ and weak $i$ state, respectively. The state evolution $(\vec{S}(t), \vec{W}(t),\ t \geq 0)$ is a continuous-time Markov process specified by the following transition rates:

$$(\vec{S}, \vec{W}) \to \begin{cases} (\vec{S}, \vec{W}) + (-\vec{e}_i, \vec{e}_i) & : \quad S_i \frac{\sum_{l \neq i} S_l}{n-1} \\ (\vec{S}, \vec{W}) + (0, -\vec{e}_i + \vec{e}_j) & : \quad W_i \frac{W_j}{n-1} \\ (\vec{S}, \vec{W}) + (\vec{e}_j, -\vec{e}_i) & : \quad W_i \frac{S_j}{n-1} \end{cases} \quad (1)$$

where $\vec{e}_i$ is a vector of dimension $m$ whose all elements are equal to 0 but the $i$-th element is equal to 1.

It is clear from the definition of the algorithm that the evolution of the number of nodes in each strong state is independent of the number of nodes in individual weak states which is evident from the fact that $\vec{S}(t)$ evolves as an autonomous continuous-time Markov process with the following transition rates, for each candidate $i$,

$$S_i \to \begin{cases} S_i + 1 & : \quad S_i \frac{n - \sum_l S_l}{n-1} \\ S_i - 1 & : \quad S_i \frac{\sum_{l \neq i} S_l}{n-1}. \end{cases} \quad (2)$$

The number of nodes per weak state evolves according to

$$W_i \to \begin{cases} W_i + 1 & : \quad S_i \frac{\sum_{l \neq i} S_l}{n-1} + W_i \frac{\sum_{l \neq i} W_l}{n-1} \\ W_i - 1 & : \quad W_i \frac{\sum_{l \neq i} W_l}{n-1} + W_i \frac{\sum_l S_l}{n-1} \end{cases} . \quad (3)$$

We note that the state evolution can be equivalently represented by a Markov process $(\vec{S}(t), \vec{U}(t),\ t \geq 0)$ where $U_i(t)$ denotes the number of nodes that at time $t$ are in either strong or weak state $i$, i.e. $U_i(t) = S_i(t) + W_i(t)$, for $i \in C$ and $t \geq 0$. We will find it convenient to use this alternative representation in some cases later in the paper.

**The Limit O.D.E.**. The Markov process $(\vec{S}(t), \vec{W}(t),\ t \geq 0)$ with transition rates (1) is a density-dependent Markov process whose scaled version $(\vec{s}^n(t), \vec{w}^n(t)) = (\frac{1}{n}\vec{S}(t), \frac{1}{n}\vec{W}(t))$ such that $\lim_{n\to\infty}(\vec{s}^n(0), \vec{w}^n(0)) = (\vec{s}(0), \vec{w}(0))$, for some fixed $(\vec{s}(0), \vec{w}(0))$, uniformly converges over any compact time interval to the solution of the following system of ordinary differential equations:

$$\frac{d}{dt}s_i(t) = (1 - 2s(t) + s_i(t))s_i(t) \quad (4)$$

$$\frac{d}{dt}w_i(t) = s_i(t)(s(t) - s_i(t)) - s(t)w_i(t) \quad (5)$$

where $i = 1, 2, \ldots, m$, $s(t) = \sum_j s_j(t)$ and $t \geq 0$.

We will use this limit system to establish convergence of the algorithm to a state where only states that correspond to plurality candidates prevail and characterize convergence time in Section IV-A. We will then establish convergence of the original stochastic system $(\vec{S}(t), \vec{W}(t),\ t \geq 0)$ to the limit O.D.E., and provide its error bound in Section IV-B. We will also use an equivalent alternative representation by defining $u_i(t) = s_i(t) + w_i(t)$, for $i \in C$ and $t \geq 0$, which corresponds to the fraction of nodes in either strong or weak state $i$ in the limit dynamics. This alternative representation amounts to the following system of ordinary differential equations: for $1 \leq i \leq m$ and $t \geq 0$,

$$\frac{d}{dt}s_i(t) = (1 - 2s(t) + s_i(t))s_i(t) \quad (6)$$

$$\frac{d}{dt}u_i(t) = s_i(t) - s(t)u_i(t). \quad (7)$$

**Convergence Time**. We will consider convergence time defined for the limit dynamics as follows.

*Definition 1:* For given $0 < \delta < 1$, $t_\delta \geq 0$ is said to be $\delta$-convergence time if

$$\sum_{i=1}^{k} u_i(t_\delta) = 1 - \delta.$$

This is a natural definition of the convergence time which ensures that at time $t_\delta$ in the limit dynamics the fraction of nodes that are in a non-plurality state (either strong or weak) is less than or equal to an arbitrarily chosen $0 < \delta < 1$. In Section IV-B we will characterize the fraction of nodes in a non-plurality state at time $t_\delta$ in the original stochastic system.

## III. Randomized Distributed Ranking

In this section we show that the randomized assignment of rankings to nodes, as introduced in the previous section, guarantees correct inference of a top ranking with large probability, under some mild assumptions on the aggregate ranking scores and the distribution $p$ used in the definition of the algorithm.

First we show that finding a top ranking over vectors of ranking scores is asymptotically equivalent to finding top ranking over single candidates under our single candidate assignment rule. Let

$$Y_{i,j} = \begin{cases} 1, & \text{node } i \text{ assigns } j \text{ as its single candidate} \\ 0, & \text{otherwise.} \end{cases}$$

We define $y_j$ to be the fraction of nodes that hold $j$ as their single candidates after the random assignment, i.e.

$$y_j = \frac{1}{n}\sum_{i=1}^{n} Y_{i,j},\ j \in [m].$$

Since each node $i$ selects single candidate randomly using the vector of ranking scores $\vec{v}_i$ as a probability distribution, we obtain

$$\mathbb{E}(y_j) = v_j.$$

We say that aggregate ranking scores are *$\epsilon$-separated*, if there exists $\epsilon > 0$ such that for every $1 \leq i < m$, either $v_i = v_{i+1}$ or $v_i \geq v_{i+1} + \epsilon$.

We show that the random assignment of single candidate to each node enables us to rank the candidates with respect to their aggregate ranking scores with probability of error that decays exponentially with $n$ in the following theorem.

***Theorem** 1:* For every $\epsilon$-separated aggregate ranking scores, there exists $C_\epsilon > 0$ such that $\mathbb{P}(\cup_{j\in[m]}\{|y_j - v_j| \geq \frac{\epsilon}{3}\}) \leq 2m \exp(-C_\epsilon n)$.

Proof of the theorem is provided in Appendix.

Now we prove the correctness of our randomized rank assignment. Given $1 \leq l \leq m$, let us define the set $C_l$ to contain all tuples of length $l$ of distinct candidates from the set $C$. Now, for every node $i$ and $S \in C_l$, let

$$X_i(S) = \begin{cases} 1, & \text{node } i \text{ assigns } S \\ 0, & \text{otherwise.} \end{cases}$$

We define $x(S)$ to be the fraction of nodes that select tuple $S \in C_l$, i.e.

$$x(S) = \frac{1}{n}\sum_{i=1}^{n} X_i(S).$$

By definition of the random assignment, each node $i$ whose preferred candidate is $c_i$ selects tuple $S \in C_l$ with probability

$$\mathbb{E}(X_i(S)) = \begin{cases} \eta p_{r_S(c_i)}, & \text{if } c_i \in S \\ 0, & \text{otherwise,} \end{cases}$$

where $r_S(c_i)$ is the position of candidate $c_i$ in tuple $S$ and

$$\eta = \frac{1}{m(m-1)\cdots(m-(l-2))}.$$

Furthermore, we observe that the expected fraction of nodes that select tuple $S = (s_1, s_2, \ldots, s_l) \in C_l$ is given by:

$$\mathbb{E}(x(S)) = \eta \sum_{j=1}^{l} p_j v_{s_j}.$$

In the following lemma we note the fact that the expected aggregate ranking scores $\mathbb{E}(x(S))$, $S \in C_l$, under suitable choice of the distribution $p$, enable us to correctly identify a top ranking tuple.

***Lemma** 1:* For every non-decreasing distribution $p$, it holds

$$\mathbb{E}(x(S)) \leq \eta \sum_{j=1}^{l} p_j v_j, \text{ for every } S \in C_l.$$

Furthermore, if $p$ is decreasing and $\min_j p_j > 0$, then the equality holds only if $S = (s_1, s_2, \ldots, s_l)$ is such that $v_{s_1} = v_1$, $v_{s_2} = v_2$, $\ldots$, $v_{s_l} = v_l$.

Proof is a direct application of the rearrangement inequality [8] and is thus omitted.

We next show the main result of this section. The result identifies conditions under which the randomized assignment guarantees identification of a top ranking from the random aggregate ranking scores $x(S)$, $S \in C_l$, with large probability. To this end, let us define $x^* = \max_{S\in C_l} \mathbb{E}(x(S))$ and let $T_l$ and $B_l$ be collections of top and bottom rankings defined as follows:

$$\begin{aligned} T_l &= \{S \in C_l : \mathbb{E}(x(S)) = x^*\} \\ B_l &= \{S \in C_l : \mathbb{E}(x(S)) < x^*\}. \end{aligned}$$

***Theorem** 2:* Suppose that the aggregate ranking scores are $\epsilon$-separated. Then, for every $1 \leq l \leq m$ and decreasing distribution $p$ such that $\min_j p_j > \kappa$, for $\kappa > 0$, there exists $\xi \geq \kappa\epsilon/3$ and $C_\xi > 0$ such that

$$\begin{aligned} &\mathbb{P}(x(T) \geq x(B) + \xi, \text{ for all } T \subset T_l, B \subset B_l) \\ \geq\ & 1 - 2\frac{m!}{(m-l)!}\exp(-C_\xi n). \end{aligned}$$

Proof of the theorem is provided in Appendix.

The theorem tells us that under the given assumptions, the random assignment guarantees that a ranking $T$ such that $x(T) \geq x(S)$, for every $S \in C_l$ is not a correct ranking with probability that diminishes exponentially with the number of nodes $n$. The assumptions for this to hold are rather mild. It is only assumed that the aggregate ranking scores are separated enough and that the distribution $p$ is suitably chosen. In fact, it follows from the proof that the random assignment guarantees correct identification of a top $l$ ranking with high probability provided that $\epsilon = \Omega(\sqrt{\frac{\log(n)}{n}})$.

Combining the results of Theorem 1 and Theorem 2, we obtain that the overall probability of error decreases exponentially with the number of nodes $n$.

## IV. Plurality Selection

In this section we establish convergence results for our plurality selection algorithm introduced in Section II. We will first characterize the limit points of the large-system limit dynamics, (4)-(5) in Section II, which establishes the correctness of the algorithm under the limit dynamics. We will then establish lower and upper bounds on the convergence time under the limit dynamics which reveal how the speed of convergence relate to the key system parameters, including the number of candidates, the fraction of nodes in a non-plurality state, and the gap between the fraction of nodes that prefer a plurality candidate and the fraction of nodes that prefer a non-plurality candidate. We will then establish a convergence result in Section IV-B that relates the original stochastic system with the large-system limit dynamics.

### A. Convergence of the Limit System

**The Limit Point**. We first present the main result which establishes that the limit dynamics is such that the fraction of nodes that are in a state that corresponds to a non-plurality candidate diminishes to zero with time.

***Theorem** 3:* Suppose $u_1(0) = \cdots u_k(0) > u_{k+1}(0) \geq \cdots \geq u_m(0)$ for some $1 \leq k \leq m$ and $\sum_l s_l(0) = 1$. Then, for every $t \geq 0$,

$$u_1(t) = \cdots = u_k(t) > u_{k+1}(t) \geq \cdots \geq u_m(t).$$

Moreover,

$$\lim_{t\to\infty} u_i(t) = \begin{cases} \frac{1}{k}, & i = 1, 2, \ldots, k \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Proof of the theorem relies on the following lemma.

***Lemma** 2:* Suppose $s_1(0) = \cdots s_k(0) > s_{k+1}(0) \geq \cdots \geq s_m(0)$ and $\sum_j s_j(0) = 1$, then, for every $t \geq 0$,

$$s_1(t) = \cdots = s_k(t) > s_{k+1}(t) \geq \cdots \geq s_m(t). \tag{9}$$

Moreover,

$$s(t) \geq \frac{1}{2 - \frac{1}{m}}, \text{ for every } t \geq 0, \tag{10}$$

and

$$\lim_{t\to\infty} s_i(t) = \begin{cases} \frac{1}{2k-1}, & i = 1, 2, \ldots, k \\ 0, & \text{otherwise} \end{cases}. \tag{11}$$

The lemma tells us that for every initial value, the system preserves the order of candidates with respect to the fractions of nodes in the corresponding strong states. We also observe that if initially each node is in a strong state, then at least half of the nodes are in a strong state at any given time. Moreover, in case there is a majority candidate, i.e. $k = 1$, then all nodes converge to the strong state corresponding to the majority candidate. We further observe that the limit point is such that the fraction of nodes $(k-1)/(2k-1)$ converge to weak states. Thus, in case of non-unique plurality candidate, i.e. $k > 1$, the weak states corresponding to different candidates cannot be lumped into a single weak state as, otherwise, a strictly positive fraction of nodes that converge to the weak state would not be able to decide upon the plurality candidate. For large $k$, about half of nodes converge to a strong state of a plurality candidate. In the following lemma we characterize how nodes occupy individual strong and weak states as time grows large.

***Lemma** 3:* For each state $i$, the portion of nodes in the strong $i$ state among nodes in state $i$, as time $t$ grows large, is given as follows

$$\lim_{t\to\infty} \frac{s_i(t)}{u_i(t)} = \begin{cases} \frac{k}{2k-1}, & i = 1, 2, \ldots, k \\ \frac{k-1}{2k-1}, & i = k+1, \ldots, m. \end{cases} \tag{12}$$

Proof of the lemma is provided in Appendix.

For the case when there exists a majority candidate, i.e. the case $k = 1$, we have that all the nodes converge to the strong state corresponding to the majority candidate while for each other state, the number of nodes in the strong state diminishes faster than the number of nodes in the weak state with time. Thus, asymptotically, the number of nodes in the strong state is dominated by the number of nodes in the weak state. On the contrary, for the case when there is no majority winner, i.e. the case $k > 1$, we have that that for each state the number of nodes in the corresponding strong state is a constant factor of the number of nodes in the given state, asymptotically for large $t$. Moreover, if the number $k$ of plurality candidates is large, then for each state approximately half of the nodes in the given state is in the strong state and the other half is in the weak state.

**Convergence Time**. In this section we consider the speed of convergence of the limit system to a state where only some given fraction $0 < \delta < 1$ of nodes is in a state that corresponds to a non-plurality candidate. We will consider the rate of convergence and provide lower and upper bounds for the convergence time, introduced in Definition 1. We will use the convergence time of the limit system to derive bounds for the original stochastic system in Section IV-B (Corollary 2).

We first characterize the rate of convergence at which the fraction of nodes in either strong or weak state that corresponds to a non-plurality candidate diminishes to zero asymptotically for large $t$.

***Lemma** 4:* For every non-plurality candidate $i = k + 1, \ldots, m$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log(s_i(t)) = \lim_{t\to\infty} \frac{1}{t} \log(w_i(t)) = -\frac{1}{2k-1}.$$

The lemma implies that the fraction of nodes that are in a state that corresponds to a non-plurality candidate diminishes to zero exponentially with rate $1/(2k-1)$, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \log(u_B(t)) = -\frac{1}{2k-1}.$$

Hence, this reveals that the larger the number of plurality candidates, the slower the rate of convergence to the state where only plurality states prevail. In particular, this rate of convergence is inversely proportional to the number $k$ of plurality candidates, asymptotically for large $k$.

We next show an upper bound on the $\delta$-convergence time that holds for any fixed $m > 1$ and any given number $1 \leq k < m$ of plurality candidates and any initial state such that each node is in a strong state and the gap between the initial fraction of nodes that prefer a plurality candidate and the fraction of nodes that prefer a non-plurality candidate is at least $0 < \gamma < 1/k$, i.e. $s_1(0) - s_{k+1}(0) > \gamma$.

***Theorem** 4:* For every fixed $m > 1$ and initial state such that $s_{k+1}(0) \leq s_1(0) - \gamma$ for $\gamma > 0$ and $\sum_i s_i(0) = 1$, there exists a constant $C_m > 0$ such that the $\delta$-convergence time $t_\delta$

is such that

$$t_\delta \le (2m-1)\left[\log\left(\frac{1}{\delta}\right)+\log\left(\frac{1}{\gamma}\right)\right]+C_m.$$

Proof of the theorem is provided in Appendix.

The theorem establishes that the convergence time is at most logarithmic in both $1/\delta$ and $1/\gamma$, i.e.

$$t_\delta = O(\log(1/\delta)+\log(1/\gamma)).$$

It follows from the proof that one may choose $C_m = (2m)\log(2m)$ which is of the same order as the running time for sorting a set of $m$ elements. For the binary case $m=2$ and $\delta = 1/n$, note that $t_\delta \le 3\log(n)+O(1)$ which is exactly of the same order as the upper bound established for the binary case in [20]. Even for the binary case, the result provides new information by showing how the convergence time upper bound depends on the gap $\gamma$.

A lower bound on the convergence time follows from the rate of convergence of the fraction of nodes in a non-plurality state to zero that was established in Lemma 4.

***Theorem** 5:* For every fixed $m>1$ and initial state $\vec{u}(0)\in \mathbb{R}_+^m$ such that $u_1(0)=\dots=u_k(0)>u_{k+1}(0)\ge\dots\ge u_m(0)>0$ the $\delta$-convergence time $t_\delta$ is

$$t_\delta = \Omega\left((2k-1)\log\left(\frac{1}{\delta}\right)\right).$$

We conclude consideration of the convergence time by showing the following result, which demonstrates that the worst-case $\delta$-convergence time is $\Omega(m(\log(1/\delta)+\log(1/\gamma)))$ for the gap $0<\gamma<1/k$, $s_1(0)-s_{k+1}(0)\ge\gamma$.

***Theorem** 6:* For every even $m>1$ there exists an initial state with the gap at least $\gamma>0$ and constant $C_m>0$ such that the $\delta$-convergence time satisfies, for every sufficiently small $\delta,\gamma>0$,

$$t_\delta \ge (m-1)\log\left(\frac{1}{\delta}\right)+(2m-1)\log\left(\frac{1}{\gamma}\right)+C_m.$$

Proof of the theorem is in Appendix. The proof is based on considering the following symmetric case: suppose the number of candidates $m$ is even, $k=m/2$, and for $0<\gamma<2/m$ the initial state is given as follows

$$s_i(0)=\begin{cases}\frac{1}{m}+\frac{\gamma}{2}, & i=1,2,\dots,\frac{m}{2}\\ \frac{1}{m}-\frac{\gamma}{2}, & i=\frac{m}{2}+1,\dots,m.\end{cases}$$

### *B. Convergence to the Limit O.D.E.*

In this section we consider the convergence of the stochastic system (1) to the solution of the limit differential system (4)-(5), or (6)-(7), as the number of nodes $n$ grows large. Since the stochastic system is a density-dependent Markov process, by Kurtz's convergence theorem [21], the scaled stochastic system $(\frac{1}{n}\vec{S}(t), \frac{1}{n}\vec{W}(t),\ t\ge 0)$ uniformly converges on every time interval $[0,T]$, for fixed $T>0$, to the solution of the limit differential system (4)-(5), where this convergence is exponential with $n$. We will extend this result to time intervals $[0,T_n]$ where $T_n$ is allowed to grow with $n$ logarithmically fast, i.e. $T_n = O(\log(n))$. This will suffice to relate the convergence time estimates derived for the limit differential system in the previous section to the convergence of the original stochastic system.

We first present a general convergence result for density-dependent Markov processes which we then use to derive convergence results for our problem; the proof of the general convergence result follows that of Kurtz's theorem in [21]. Suppose $\vec{Z}$ is a Markov process on a countable state space $\mathcal{X}\subset\mathbb{R}^d$, for $d\ge 1$, with transition rates specified as follows: let $(\vec{e}_i,\ i=1,2,\dots,k)$, for positive integer $k\ge 1$, denote a collection of vectors in $\mathbb{R}^d$ such that $\vec{e}_i$ specifies the direction of a jump of the Markov process that occurs with rate $\lambda_i(\vec{z})$ when the Markov process is in state $\vec{z}$, i.e.

$$\vec{Z}\to\vec{Z}+\vec{e}_i \text{ with rate } \lambda_i(\vec{Z}),\ i=1,2,\dots,k.$$

Let $\bar{e}\ge 0$ and $\bar{\lambda}\ge 0$ be such that

$$\max_{1\le i\le k}|\vec{e}_i|\le\bar{e} \text{ and } \max_{1\le i\le k}|\vec{\lambda}_i(z)|\le\bar{\lambda},\ \text{every } \vec{z}\in\mathbb{R}^d.$$

Notice that here and later in this section $|\cdot|$ is for Euclidean distance.[2]

Let $\vec{z}_n(t)$ be the scaled process defined by $\vec{z}_n(t)=\frac{1}{n}Z(t)$, for $t\ge 0$. We define $\vec{z}_\infty(t)$ to be the unique solution, for given initial value $\vec{z}_\infty(0)=\vec{x}$, of the following system of ordinary differential equations:

$$\frac{d}{dt}\vec{z}_\infty(t)=\sum_{i=1}^{I}\lambda_i(\vec{z}_\infty(t))\vec{e}_i. \tag{13}$$

In the following theorem, we show that given that $\lim_{n\to\infty}\vec{z}_n(0)=\vec{x}$, the scaled process $\vec{z}_n(t)$ converges to $\vec{z}_\infty(t)$ uniformly as $n$ grows large over any time interval $[0,T_n]$ such that $T_n=O(\log(n))$. This theorem justifies our use of the limit ordinary differential system to characterize the convergence time in the stochastic system.

***Theorem** 7:* Assume that $\lambda_i:\mathbb{R}^d\to\mathbb{R}_+$ is uniformly bounded and Lipschitz continuous with constant $K>0$, for every $i=1,2,\dots,I$, and let $\vec{z}_\infty$ be the unique solution of the system of ordinary differential equations (13) with initial value $\vec{z}_\infty(0)$. Then, for every $\epsilon>0$, $0<C<1/(2K)$ and $T_n=C\log(n)$ there exists $C_1>0$ such that for large enough $n$, it holds

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T_n}|\vec{z}_n(t)-\vec{z}_\infty(t)|\ge\epsilon\right)\\ \le\ & C_1\exp\left(-\frac{\epsilon^2}{d\bar{e}^2\bar{\lambda}IC}\frac{n^{1-2KC}}{\log(n)}\right).\end{aligned}$$

Proof of the theorem is provided in Appendix.

[2] That is, for vector $\vec{x}\in\mathbb{R}^d$, $|\vec{x}|=\left(\sum_{j=1}^d x_j^2\right)^{1/2}$.

Notice that the theorem implies that $\sup_{0\le t\le T_n} |\vec{z}_n(t) - \vec{z}_\infty(t)| < \epsilon$ holds with high probability for large $n$ and any constant $\epsilon$, i.e. $\mathbb{P}\left(\sup_{0\le t\le T_n} |\vec{z}_n(t) - \vec{z}_\infty(t)| \ge \epsilon\right) = O(n^{-1})$.

We next show that the Markov process (1) verifies assumptions of Theorem 7 and we instantiate the bound for the error probability. First note that the dimension of our Markov process is $d = 2m$. Then note that for the Markov process (1) there are $m$ directions of type $(-\vec{e}_i, \vec{e}_i)$, $m(m-1)$ directions of type $(-\vec{e}_i + \vec{e}_j, 0)$ and $m(m-1)$ directions of type $(\vec{e}_j, -\vec{e}_i)$, for $i, j = 1, 2, \ldots m$ and $i \neq j$. The transition rates of the directions are given as follows

$$\lambda_l((\vec{s}, \vec{w})) = \begin{cases} s_i(s - s_i), & l = (-\vec{e}_i, \vec{e}_i) \\ w_i w_j, & l = (-\vec{e}_i + \vec{e}_j, 0) \\ w_i s_j, & l = (\vec{e}_j, -\vec{e}_i). \end{cases}$$

Note that the total number of distinct directions is $I = (2m-1)m$.

In the following lemma we show that the Markov process verifies the assumptions of Theorem 7.

***Lemma** 5:* For the Markov process (1), the following properties hold

1) $\lambda_l$ is bounded and $\bar{\lambda} = 1$;
2) $\bar{e} = \sqrt{2}$;
3) $\lambda_l$ is Lipschitz continuous with constant 2.

Using this lemma, it is not difficult to note the following corollary of Theorem 7.

***Corollary** 1:* For the Markov process (1), it holds that for every $\epsilon > 0$, $0 < C < 1/4$ and $T_n = C\log(n)$, there exists $C_1 > 0$ such that for sufficiently large $n$,

$$\begin{aligned} & \mathbb{P}\left(\sup_{0\le t\le T_n} |(\vec{s}_n(t), \vec{w}_n(t)) - (\vec{s}_\infty(t), \vec{w}_\infty(t))| \ge \epsilon\right) \\ \le \quad & C_1 \exp\left(-\frac{\epsilon^2}{(2m)^3 C}\frac{n^{1-4C}}{\log(n)}\right). \end{aligned}$$

Furthermore, we consider the fraction of nodes at time $t$ that are either in strong or weak state $k < i \le m$ as

$$u_B^n(t) = \sum_{i=k+1}^{m}\left(\frac{1}{n}S_i(t) + \frac{1}{n}W_i(t)\right).$$

We have the following result that characterizes the fraction of nodes in a non-plurality state of the stochastic system at the $\delta$-convergence time.

***Corollary** 2:* Suppose $t_\delta$ is the convergence time with $\delta = 1/n^\alpha$ for some $0 < \alpha < 1/(8m+2)$. Then,

$$u_B^n(t_\delta) = O(n^{-\alpha}) \text{ with high probability.}$$

Proof of the corollary is provided in Appendix.

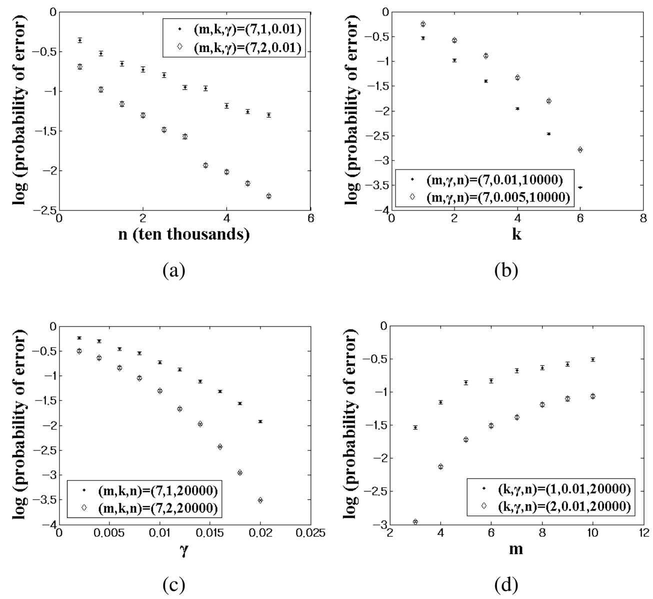


Figure 1. Probability of error vs. (a) the number of nodes $n$ with $(m, \gamma, \delta) = (7, 0.01, 0.01)$ and $k = 1$ and 2, (b) the number of plurality candidates $k$ with $(m, \delta, n) = (7, 0.01, 10000)$ and $\gamma = 0.005$ and 0.01, (c) the gap $\gamma$ with $(m, \delta, n) = (7, 0.01, 20000)$ and $k = 1$ and 2, (d) the number of candidates $m$ with $(\gamma, \delta, n) = (0.01, 0.01, 20000)$ and $k = 1$ and 2.

## V. Simulation Results

In this section we provide experimental results for our plurality selection algorithm. For each simulation, we vary one parameter while keeping other parameters fixed and obtain results that demonstrate the effect of the parameter on the probability of error and the convergence time. We did 1000 times of simulations for each setup, and confidence intervals are for 95% of confidence. We consider the case where each node prefers a single candidate with aggregate ranking scores $v_1 = \cdots = v_k > v_{k+1} = \cdots = v_m$ with $v_k - v_{k+1} = \gamma > 0$.

### *A. Probability of Error for Plurality Selection*

We observe the effect of the parameters on the probability of error, which is estimated by observing the state at the first time at which 99% of nodes are in the same state, either weak or strong. Figure 1-(a) shows that the probability of error decays approximately exponentially with the number of nodes $n$, which conforms to the result of Theorem 7. In Figure 1-(b) and (c), we observe that the probability of error seems to decay somewhat faster than exponential with $k$ and $\gamma$, respectively. Finally, from Figure 1-(d), we observe that the probability of error exhibits a diminishing returns increase with the number of candidates $m$.

### *B. Convergence Time for Plurality Selection*

We observe the effect of the parameters on the convergence time. Figure 2-(a), (b), and (c) support the result of Theorem 4, which provides an upper bound for the convergence time which is linear in $\log(\frac{1}{\delta})$, $\log(\frac{1}{\gamma})$, and $m$, respectively.

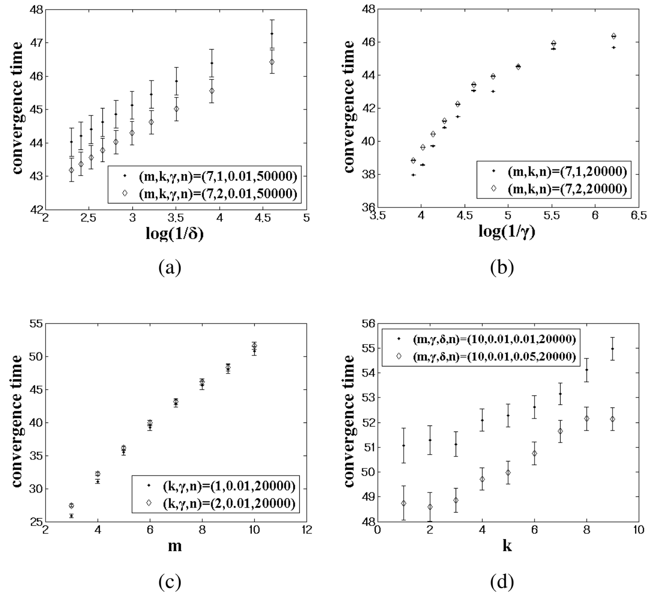


Figure 2. Convergence time vs. (a) $\log(1/\delta)$ with $(m, \gamma, n) = (7, 0.01, 50000)$ and $k = 1$ and 2, (b) $\log(1/\gamma)$ with $(m, \delta, n) = (7, 0.01, 20000)$ and $k = 1$ and 2, (c) the number of candidates $m$ with $(\gamma, \delta, n) = (0.01, 0.01, 20000)$ and $k = 1$ and 2, (d) the number of plurality candidates $k$ with $(m, \gamma, \delta, n) = (10, 0.01, 0.01, 20000)$.

Specifically, Figure 2-(a) exhibits linear growth of the convergence time with $\log(\frac{1}{\delta})$; Figure 2-(b) exhibits a growth of the convergence time with $\log(1/\gamma)$, for $\delta = 0.01$, which is approximately linear for large enough $\gamma$; in Figure 2-(c) the convergence time increases linearly with the number of candidates $m$ for $\delta = 0.01$. Lastly, in Figure 2-(d), we note that the convergence time approximately follows a linear growth with the number of plurality candidates $k$ which is suggested by the rate of convergence result in Lemma 4.

## VI. Conclusion

In this paper, we showed that the distributed ranking of candidates reduces to a plurality selection problem and proposed a simple yet efficient randomized algorithm that realizes this reduction. We studied a plurality selection algorithm for an arbitrary number of candidates that is lightweight in the amount of memory used per node and the amount of state communicated between nodes, which is a natural generalization of some previously-studied algorithms for plurality selection over two candidates. We found that the algorithm has desirable convergence properties for the case of complete graphs and derived tight characterizations of the convergence time.

For future work, one may further analytically characterize the error exponents of the plurality selection algorithm. The open research problem remains on the design of lightweight distributed algorithms for the plurality selection problem with guaranteed correctness for arbitrary connected graphs.

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## Appendix

### A. Proof of Theorem 1

We use the well known Chernoff's bounds for a sequence of independent binary random variables $X_1, X_2, \ldots, X_n$ that take values 0 or 1 and such that for some $\mu \geq 0$, $\mathbb{E}(S_n) = \mu$ where $S_n = \sum_{i=1}^n X_i$. The bounds read as follows:

$$\begin{aligned}\mathbb{P}(S_n \geq (1+\delta)\mu) &\leq e^{-\frac{1}{3}\min(\delta^2,\delta)\mu}\\ \mathbb{P}(S_n \leq (1-\delta)\mu) &\leq e^{-\frac{1}{2}\delta^2\mu}.\end{aligned}$$

Therefore, for every $0 \leq \delta \leq 1$,

$$\mathbb{P}(|S_n - \mu| \geq \mu\delta) \leq 2e^{-\frac{1}{3}\delta^2\mu}. \tag{14}$$

For each $j$, $y_j = \frac{1}{n}\sum_{i=1}^n Y_{i,j}$ where $Y_{1,j}, Y_{2,j}, \ldots, Y_{n,j}$ is a sequence of independent binary random variables such that $\mathbb{P}(Y_{i,j} = 1) = 1 - \mathbb{P}(Y_{i,j} = 0) = v_{i,j}$. Notice that $\mathbb{E}(y_j) = v_j$.

Using the union bound, we have

$$\mathbb{P}(\cup_{j\in[m]}\{|y_j - v_j| \geq \frac{\epsilon}{3}\}) \leq \sum_{j=1}^m \mathbb{P}(|y_j - v_j| \geq \frac{\epsilon}{3}).$$

Now, using Chernoff's bound (14), it holds

$$\mathbb{P}(|y_j - v_j| \geq \frac{\epsilon}{3}) \leq 2e^{-n\frac{\epsilon^2}{27v_j}},\ j \in [m].$$

Therefore,

$$\mathbb{P}(\cup_{j\in[m]}\{|y_j - v_j| \geq \frac{\epsilon}{3}\}) \leq 2me^{-n\frac{\epsilon^2}{27v_1}}$$

which completes the proof.

### B. Proof of Theorem 2

First, we note that for every $S \subset B_l$,

$$x^* - \mathbb{E}(x(S)) \geq (\min_j p_j)\epsilon.$$

Let us introduce the notation $\xi = (\min_j p_j)\epsilon$. The error event implies $|x(S) - \mathbb{E}(x(S))| \geq \frac{\xi}{3}$, for some $S \in C_l$. Therefore,

$$\begin{aligned} & \mathbb{P}(x(T) \leq x(B) + \xi,\ \text{for some } T \subset T_l, B \subset B_l) \\ \leq\ & \mathbb{P}(\cup_{S\in C_l}|x(S) - \mathbb{E}(x(S))| \geq \frac{\xi}{3}) \\ \leq\ & |C_l| \max_{S\in C_l} \mathbb{P}(|x(S) - \mathbb{E}(x(S))| \geq \frac{\xi}{3}) \qquad (15) \end{aligned}$$

where $|C_l| = \frac{m!}{(m-l)!}$.

Now, note that $x(S) = \frac{1}{n}\sum_{i=1}^n X_i(S)$ where for each $S \in C_l$, $X_1(S)$, $X_2(S)$, ..., $X_n(S)$ is a sequence of independent binary random variables taking values 0 or 1 such that $\mathbb{E}(x(S)) = \eta\sum_{j=1}^l p_j v_{s_j}$.

Applying Chernoff's bound (14), we have for every $S \in C_l$,

$$\begin{aligned} \mathbb{P}(|x(S) - \mathbb{E}(x(S))| \geq \frac{\xi}{3}) &\leq 2\exp\left(-\frac{1}{27}\frac{\xi^2}{\mathbb{E}(x(S))}\right) \\ &\leq 2\exp\left(-\frac{1}{27}\frac{\xi^2}{x^*}\right) \\ &\leq 2\exp\left(-\frac{1}{27}\frac{\xi^2}{v_1}\right). \end{aligned}$$

Combining this with (15) and the fact $\xi = (\min_j p_j)\epsilon$, we have

$$\begin{aligned} & \mathbb{P}(x(T) \leq x(B) + \xi,\ \text{for some } T \subset T_l, B \subset B_l) \\ \leq\ & \frac{m!}{(m-l)!}2\exp\left(-\frac{[(\min_j p_j)\epsilon]^2}{27v_1}\right) \end{aligned}$$

which completes the proof.

### C. Proof of Lemma 5

It is straightforward to show the first two properties, so we provide the proof only for the third one. Note that for every $i = 1, 2, \ldots, m$ and $t \geq 0$,

$$-1 \leq \frac{d}{dt}s_i(t) \leq 1 \text{ and } -1 \leq \frac{d}{dt}w_i(t) \leq 1. \qquad (16)$$

Furthermore, we note that for $t \geq 0$,

$$\frac{d}{dt}s(t) = 1 - s(t) + \sum_i s_i(t)^2 - s(t)^2$$

and, thus,

$$-1 \leq \frac{d}{dt}s(t) \leq 1,\ \text{for every } t \geq 0. \qquad (17)$$

Now for $l = (-e_i, e_i)$ and $t \geq 0$ we have

$$\frac{d}{dt}\lambda_l(\vec{s}(t), \vec{w}(t)) = (s(t) - 2s_i(t))\frac{d}{dt}s_i(t) + s_i(t)\frac{d}{dt}s(t).$$

Using (16) and (17), it follows $-2 \leq \frac{d}{dt}\lambda_l(\vec{s}(t), \vec{w}(t)) \leq 2$, for every $t \geq 0$. Similar analysis follows for $l = (-e_i + e_j, 0)$ and $(e_j, -e_i)$.

### D. Proof of Lemma 2

From (6) it is evident that for every $i$ and $j$ such that $s_i(0) = s_j(0)$, we have $s_i(t) = s_j(t)$ for every $t \geq 0$. It thus suffices to consider $i$ and $j$ such that $s_i(0) \neq s_j(0)$.

We first note the following fact which we will use in this proof and also invoke at some instances later.

***Lemma** 6:* Let $\Lambda(t) = \int_0^t (2s(x) - 1)dx$, $t \geq 0$. For every $i$ and $j$ and $t \geq 0$, the following identity holds:

$$\frac{1}{s_j(t)} - \frac{1}{s_i(t)} = \left(\frac{1}{s_j(0)} - \frac{1}{s_i(0)}\right)e^{\Lambda(t)}. \qquad (18)$$

*Proof:* From (6) observe that

$$d\log(s_l(t)) = (1 + s_l(t) - 2s(t))dt,\ \text{for every } l.$$

Hence,

$$d\log(s_i(t)s_j(t)) = (2 + s_i(t) + s_j(t) - 4s(t))dt.$$

By integrating, it follows

$$\begin{aligned} & \log\left(\frac{s_i(t)s_j(t)}{s_i(0)s_j(0)}\right) \\ =& 2t + \int_0^t (s_i(x) + s_j(x))dx - 4\int_0^t s(x)dx. \end{aligned} \qquad (19)$$

Furthermore, from (6), we have

$$\frac{d}{dt}(s_i(t) - s_j(t)) = (s_i(t) - s_j(t))[1 - 2s(t) + s_i(t) + s_j(t)].$$

Hence,

$$d\log(s_i(t) - s_j(t)) = [1 - 2s(t) + s_i(t) + s_j(t)]dt.$$

By integrating, it follows

$$\begin{aligned} & \log\frac{s_i(t) - s_j(t)}{s_i(0) - s_j(0)} \\ =& t + \int_0^t (s_i(x) + s_j(x))dx - 2\int_0^t s(x)dx. \end{aligned} \qquad (20)$$

Summing (19) and (20) we obtain

$$\log\frac{\frac{1}{s_j(t)} - \frac{1}{s_i(t)}}{\frac{1}{s_j(0)} - \frac{1}{s_i(0)}} = \int_0^t (2s(x) - 1)dx$$

from which the asserted identify follows. ■

We claim that $s(t) > 1/2$ for every $t \geq 0$, i.e. (10) holds. Suppose that $s_i(0) > s_j(0)$, we then have that the right-hand side in (18) is increasing with $t$, and hence by (18), $s_i(t) > s_j(t)$ for every $t \geq 0$. It remains only to prove the claim to establish the first part of the lemma.

From (6), we have

$$\frac{d}{dt}s(t) = s(t)(1-2s(t)) + \sum_l s_l(t)^2.$$

By Jensen's inequality, for every $t \geq 0$, $\frac{1}{m}\sum_l s_l(t)^2 \geq \left(\frac{1}{m}\sum_l s_l(t)\right)^2 = \frac{1}{m^2}s(t)^2$. Then, by standard comparison argument for ordinary differential equations, we have $s(t) \geq \underline{s}(t)$, for $t \geq 0$ given that $\underline{s}(0) = s(0)$ and for $t \geq 0$,

$$\frac{d}{dt}\underline{s}(t) = \underline{s}(t)(1-(2-1/m)\underline{s}(t)).$$

The latter has the following solution:

$$\underline{s}(t) = \frac{s(0)}{2-1/m-((2-1/m)s(0)-1)e^{-t}}, \text{ for every } t \geq 0.$$

Since $s(0) = 1$, we have

$$s(t) \geq \frac{1}{2-1/m-(1-1/m)e^{-t}}, \text{ for every } t \geq 0.$$

Hence, (10) follows.

The second part of the lemma can be established as follows. From (6), the fixed points are such that for each $i$,

$$\text{either } s_i(+\infty) = 0 \text{ or } s_i(+\infty) = 2s(+\infty) - 1.$$

Combined with the first part of the lemma, we have that $\lim_{t\to\infty} s_i(t) = 0$ for $i = k+1, \ldots, m$. Furthermore, from

$$s(+\infty) = \sum_{i=1}^k s_i(+\infty) = k(2s(+\infty)-1)$$

we have $s(+\infty) = k/(2k-1)$. Since $s_i(t) = s_1(t)$ for every $i = 1, 2, \ldots, k$, it follows $s_i(+\infty) = s(+\infty)/k = 1(2-1/k)$, for $i = 1, 2, \ldots, k$. This completes the proof.

### E. Proof of Theorem 3

The proof of the theorem is based on Lemma 2. Notice that (7) is a linear differential equation whose solution is

$$u_i(t) = s_i(0)e^{-\int_0^t s(x)dx} + \int_0^t s_i(y)e^{-\int_y^t s(x)dx}dy.$$

Combined with (9) it follows $u_1(t) = \cdots = u_k(t) > u_{k+1}(t) \geq \cdots \geq u_m(t)$ for every $t \geq 0$, which establishes the first part of the theorem. From (7), the fixed points are such that $s_i(+\infty) = s(+\infty)u_i(+\infty)$. By (10), $s(+\infty) > 0$ and thus for every $i$, $u_i(+\infty) > 0$ if and only if $s_i(+\infty) > 0$. Therefore, (8) follows in view of (11).

### F. Proof of Lemma 12

For the case $1 \leq i \leq k$, the asserted limit follows directly from (8) and (11). In the remainder we consider the case $k < i \leq m$. Note, for $t \geq 0$,

$$\frac{d}{dt}\frac{s_i(t)}{u_i(t)} = \frac{1}{u_i(t)}\frac{d}{dt}s_i(t) - \frac{s_i(t)}{u_i(t)^2}\frac{d}{dt}u_i(t).$$

Using (6) and (7), we have for $t \geq 0$,

$$\frac{d}{dt}\frac{s_i(t)}{u_i(t)} = \frac{s_i(t)}{u_i(t)}\left(1 - s(t) + s_i(t) - \frac{s_i(t)}{u_i(t)}\right).$$

Any limit point must satisfy $\lim_{t\to\infty}\frac{d}{dt}\frac{s_i(t)}{u_i(t)} = 0$, therefore,

$$\lim_{t\to\infty}\frac{s_i(t)}{u_i(t)}\left(1 - s(t) + s_i(t) - \frac{s_i(t)}{u_i(t)}\right) = 0.$$

From Theorem 3, $\lim_{t\to\infty} s(t) = \frac{k}{2k-1}$ and $\lim_{t\to\infty} s_i(t) = 0$, thus

$$\lim_{t\to\infty}\frac{s_i(t)}{u_i(t)}\left(\frac{k-1}{2k-1} - \frac{s_i(t)}{u_i(t)}\right) = 0.$$

It follows that either

$$\lim_{t\to\infty}\frac{s_i(t)}{u_i(t)} = 0 \text{ or } \lim_{t\to\infty}\frac{s_i(t)}{u_i(t)} = \frac{k-1}{2k-1}.$$

We show that the latter case holds. The two cases are the same for $k = 1$, thus it suffices to consider the case $k > 1$. Using (4) and (5), it is easy to obtain

$$\frac{d}{dt}\frac{w_i(t)}{s_i(t)} = s(t) - s_i(t) - (1 - s(t) + s_i(t))\frac{w_i(t)}{s_i(t)}.$$

From this, we obtain $\lim_{t\to\infty}\frac{d}{dt}\frac{w_i(t)}{s_i(t)} = \frac{k}{2k-1} - \frac{k-1}{2k-1}\lim_{t\to\infty}\frac{w_i(t)}{s_i(t)} = 0$, thus

$$\lim_{t\to\infty}\frac{w_i(t)}{s_i(t)} = \frac{k}{k-1}.$$

Using the fact $\frac{s_i(t)}{u_i(t)} = \frac{1}{1+w_i(t)/s_i(t)}$, it follows that

$$\lim_{t\to\infty}\frac{s_i(t)}{u_i(t)} = \frac{k-1}{2k-1}.$$

### G. Proof of Lemma 4

From (4), we have that for every $i$ and $t \geq 0$,

$$\frac{d}{dt}\log(s_i(t)) = 1 - 2s(t) + s_i(t). \tag{21}$$

Now, from Theorem 3, $\lim_{t\to\infty} s(t) = k/(2k-1)$ and $\lim_{t\to\infty} s_i(t) = 0$, for every $k < i \leq m$. Therefore,

$$\lim_{t\to\infty}\frac{d}{dt}\log(s_i(t)) = -\frac{1}{2k-1}.$$

Combining this with

$$\frac{1}{t}\log(s_i(t)) = \frac{1}{t}\log(s_i(0)) + \frac{1}{t}\int_0^t \frac{d}{dx}\log(s_i(x))dx,$$

the result for $s_i(t)$ follows.

Similarly, from (5),

$$\frac{d}{dt}\log(w_i(t)) = (s(t) - s_i(t))\frac{s_i(t)}{w_i(t)} - s(t).$$

Combining with the facts

$$\lim_{t\to\infty} s(t) = \frac{k}{2k-1}$$

and, for $k < i \leq m$,

$$\lim_{t\to\infty} s_i(t) = 0$$
$$\lim_{t\to\infty} \frac{s_i(t)}{w_i(t)} = 1 - \frac{1}{k}$$

we obtain

$$\lim_{t\to\infty} \frac{d}{dt}\log(w_i(t)) = -\frac{1}{2k-1}.$$

The result follows by similar argument as for $s_i(t)$.

### H. Proof of Theorem 4

From Lemma 6, we have for every $1 \leq i \leq m$ and $t \geq 0$,

$$s_i(t) = \frac{s_i(0)e^{-\Lambda(t)}}{1 - \frac{s_i(0)}{s_1(0)} + \frac{s_i(0)}{s_1(t)}e^{-\Lambda(t)}}.$$

Hence,

$$s_i(t) \leq \frac{s_i(0)s_1(0)}{s_1(0)-s_i(0)}e^{-\Lambda(t)}.$$

Combining with $\Lambda(t) \geq -\frac{1}{2m-1}t$ and summing over $k < i \leq m$, we have

$$s_B(t) \leq \bar{s}_B(t) = \bar{s}_B(0)e^{-\frac{1}{2m-1}t}, \text{ every } t \geq 0, \tag{22}$$

where

$$\bar{s}_B(0) = \sum_{i=k+1}^{m} \frac{s_i(0)s_1(0)}{s_1(0)-s_i(0)}.$$

Now, from (7) and Lemma 2, for every $1 \leq i \leq m$ and $t \geq 0$ it holds

$$\frac{d}{dt}u_i(t) \leq s_i(t) - \frac{m}{2m-1}u_i(t).$$

Therefore, for every $t \geq 0$,

$$u_B(t) \leq u_B(0)e^{-\frac{m}{2m-1}t} + e^{-\frac{m}{2m-1}t}\int_0^t s_B(x)e^{\frac{m}{2m-1}x}dx.$$

Combining with (22), we have

$$\begin{aligned} u_B(t) &\leq u_B(0)e^{-\frac{m}{2m-1}t} + e^{-\frac{m}{2m-1}t}\bar{s}_B(0)\int_0^t e^{\frac{m-1}{2m-1}x}dx \\ &\leq C_0 e^{-\frac{1}{2m-1}t} \end{aligned}$$

where

$$C_0 = u_B(0) + \frac{2m-1}{m-1}\bar{s}_B(0).$$

By definition $u_B(t_\delta) = \delta$, thus for this to hold it suffices that $t_\delta$ is such that

$$t_\delta \geq (2m-1)\left[\log\left(\frac{1}{\delta}\right) + \log(C_0)\right]. \tag{23}$$

Now, note $u_B(0) \leq 1$ and

$$\begin{aligned} \bar{s}_B(0) &\leq (m-k)\frac{1}{\frac{1}{s_{k+1}(0)} - \frac{1}{s_1(0)}} \\ &\leq (m-k)\frac{1}{\frac{1}{s_1(0)-\gamma} - \frac{1}{s_1(0)}} \\ &\leq (m-k)\frac{s_1(0)^2}{\gamma} \\ &\leq \frac{m-k}{k^2}\frac{1}{\gamma}. \end{aligned}$$

Hence,

$$\begin{aligned} \log(C_0) &\leq \log\left(1 + \frac{(2m-1)(m-k)}{(m-1)k^2}\frac{1}{\gamma}\right) \\ &= \log\left(\frac{1}{\gamma}\right) + \log\left(\gamma + \frac{(2m-1)(m-k)}{(m-1)k^2}\right) \\ &\leq \log\left(\frac{1}{\gamma}\right) + \log\left(\frac{1}{k} + \frac{(2m-1)(m-k)}{(m-1)k^2}\right) \\ &\leq \log\left(\frac{1}{\gamma}\right) + \log(2m) \end{aligned} \tag{24}$$

where the second inequality follows from the fact $\gamma < \frac{1}{k}$.

The proof is completed by noting that with (24), the inequality of the theorem is sufficient for (23) to hold for every constant $C_m$ such that $C_m \geq (2m-1)\log(2m)$.

### I. Proof of Theorem 6

We consider the symmetric case introduced in the text following Theorem 6. Suppose $0 < \delta < 1/4$.

Using the identities (19) and (20), we have for every $t \geq 0$,

$$\log\left(\frac{s_1(t)s_m(t)}{s_1(0)s_m(0)}\right) = 2t - (2m-1)f(t)$$
$$\log\left(\frac{s_1(t)-s_m(t)}{s_1(0)-s_m(0)}\right) = t - (m-1)f(t)$$

where we use the notation $f(t) = \int_0^t (s_1(x)+s_2(x))dx$.

From this, we obtain that for every $t \geq 0$,

$$\begin{aligned} t = {} & (m-1)\log\left(\frac{s_1(0)s_m(0)}{s_1(t)s_m(t)}\right) \\ & + (2m-1)\log\left(\frac{s_1(t)-s_m(t)}{s_1(0)-s_m(0)}\right). \end{aligned} \tag{25}$$

First, we note

$$\begin{aligned} \log\left(\frac{s_1(0)s_m(0)}{s_1(t_\delta)s_m(t_\delta)}\right) &= \log\left(\frac{(\frac{1}{m}+\frac{\gamma}{2})(\frac{1}{m}-\frac{\gamma}{2})}{s_1(t_\delta)s_m(t_\delta)}\right) \\ &\geq \log\left(\frac{m(\frac{1}{m}+\frac{\gamma}{2})(\frac{1}{m}-\frac{\gamma}{2})}{2\delta}\right) \\ &= \log\left(\frac{1}{\delta}\right) + \log\left(\frac{m}{2}\left(\frac{1}{m^2}-\frac{\gamma^2}{4}\right)\right) \end{aligned}$$

where the inequality follows from the facts $s_1(t_\delta) \leq 1$ and $\frac{m}{2}s_m(t_\delta) \leq \delta$.

Second, we observe

$$\begin{aligned}\log\left(\frac{s_1(t_\delta)-s_m(t_\delta)}{s_1(0)-s_m(0)}\right) &= \log\left(\frac{\frac{2}{m}s(t_\delta)-2s_m(t_\delta)}{\gamma}\right)\\ &\geq \log\left(\frac{\frac{2}{2m-1}-2s_m(t_\delta)}{\gamma}\right)\\ &\geq \log\left(\frac{\frac{2}{2m-1}-\frac{2\delta}{m}}{\gamma}\right)\\ &\geq \log\left(\frac{1}{\gamma}\right)+\log\left(\frac{1-4\delta}{m}\right)\end{aligned}$$

where the first inequality follows from $s(t) \geq \frac{m}{2m-1}$, every $t \geq 0$ (Lemma 2), and the second inequality is by the fact $\frac{m}{2}s_m(t_\delta) \leq \delta$.

The result follows from the above inequalities and (25).

### J. Proof of Theorem 7

The proof follows the same steps as in the proof of Kurtz's theorem in [21] wherein an exponential bound with respect to $n$ is established for $T_n$ a fixed positive constant. In the following we outline the main the steps of the proof which suffice to establish the asserted result.

From Lemma 5.9 [21], for every $\vec{\theta} \in \mathbb{R}^d$ such that $|\vec{\theta}| = 1$, we have

$$\mathbb{P}(\sup_{0\leq t\leq T_n}\{\langle \vec{z}_n(t)-\vec{y}_n(t),\vec{\theta}\rangle\} \geq a) \leq \exp\left(-C_{n,\epsilon}(a)\right) \quad (26)$$

where

$$\vec{y}_n(t) = \vec{z}_\infty(t) + \int_0^t (\lambda_i(\vec{z}_n(s)) - \lambda_i(\vec{z}_\infty(s)))\vec{e}_i ds$$

and

$$C_{n,\epsilon}(x) = \sup_{\rho>0}\left\{\rho\left(x - \frac{\bar{\lambda}\bar{e}^2 I}{2}\rho\frac{T_n e^{\frac{\bar{e}\rho}{n}}}{n}\right)\right\}. \quad (27)$$

Now, we use the following lemma from [21].

***Lemma** 7:* Let $\vec{y}$ be a random vector in $\mathbb{R}^d$. Suppose there are numbers $a$ and $\delta$ such that, for each $\vec{\theta} \in \mathbb{R}^d$ with $|\vec{\theta}| = 1$,

$$\mathbb{P}(\langle \vec{y},\vec{\theta}\rangle \geq a) \leq \delta.$$

Then

$$\mathbb{P}(|\vec{y}| \geq a\sqrt{d}) \leq 2d\delta.$$

Combining this lemma with (26), we have

$$\mathbb{P}\left(\sup_{0\leq t\leq T_n}|\vec{z}_n(t)-\vec{y}_\infty(t)| \geq \epsilon\right) \leq 2d\exp\left(-C_{n,\epsilon}(\epsilon/\sqrt{d})\right).$$

Using same arguments as in the proof of Theorem 5.3 [21], we have

$$|\vec{z}_n(t)-\vec{y}_n(t)| \geq |\vec{z}_n(t)-\vec{z}_\infty(t)| - \int_0^t K|\vec{z}_n(s)-\vec{z}_\infty(s)|ds.$$

Furthermore, by Gronwall's inequality, if for some $t \geq 0$,

$$|\vec{z}_n(t)-\vec{z}_\infty(t)| > \epsilon e^{Kt}$$

then

$$|\vec{z}_n(t)-\vec{z}_\infty(t)| - \int_0^t K|\vec{z}_n(s)-\vec{z}_\infty(s)|ds \geq \epsilon.$$

Therefore,

$$\mathbb{P}\left(\sup_{0\leq t\leq T_n}|\vec{z}_n(t)-\vec{z}_\infty(t)| \geq \epsilon\right) \leq 2d\exp\left(-C_{n,\epsilon}(\frac{\epsilon e^{-KT_n}}{\sqrt{d}})\right).$$

By definition of the function $C_{n,\epsilon}(x)$ in (27), we have

$$C_{n,\epsilon}(x) \geq C_3(x)$$

where $C_3(x) = \rho\left(x - \frac{\bar{\lambda}\bar{e}^2 I}{2}\rho\frac{T_n e^{\frac{\bar{e}\rho}{n}}}{n}\right)$ with $\rho = \frac{xn}{\bar{\lambda}\bar{e}^2 IT_n}$, i.e.

$$C_3(x) = \frac{x^2}{\bar{\lambda}\bar{e}^2 IT_n}\left(1 - \frac{1}{2}e^{\frac{x}{\bar{e}\bar{\lambda}IT_n}}\right).$$

Hence

$$\begin{aligned}&C_3\left(\frac{\epsilon e^{-KT_n}}{\sqrt{d}}\right)\\ =\ & \frac{\epsilon^2}{d\bar{\lambda}\bar{e}^2 IC}\frac{n^{1-KC}}{\log(n)}\left(1 - \frac{1}{2}e^{\frac{\epsilon}{n^{KC}\log(n)\sqrt{d}\bar{e}\bar{\lambda}IC}}\right)\\ =\ & \frac{\epsilon^2}{d\bar{\lambda}\bar{e}^2 IC}\frac{n^{1-KC}}{\log(n)}\left(\frac{1}{2} + O(n^{-KC})\right).\end{aligned}$$

The result follows.

### K. Proof of Corollary 2

Suppose that for $\epsilon > 0$,

$$\sup_{0\leq t\leq t_\delta}|(\vec{s}_n(t),\vec{w}_n(t)) - (\vec{s}_\infty(t),\vec{w}_\infty(t))| < \epsilon \quad (28)$$

Then, it is easily concluded that

$$u_B^n(t_\delta) \leq \delta + 2k\epsilon.$$

Now, let $\epsilon = n^{-\alpha}$ and note from Corollary 1 that (28) holds with high probability provided that

$$1 - 4C - 2\alpha > 0$$

where $C > 0$ is such that $t_\delta \leq C\log(n)$.

Using the result in Theorem 4,

$$t_\delta \leq 2m\log(1/\delta) + O(1)$$

we have $C \leq 2m\alpha$.

Therefore, for $0 < \alpha < 1/(8m+2)$, it holds with high probability

$$u_B^n(t_\delta) \leq \frac{1+2k}{n^\alpha}$$

which completes the proof.