Combinatorial Agency

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Abstract

We study a combinatorial variant of the classical principal-agent problem. In our setting a principal must motivate a team of strategic agents to exert costly effort on his behalf, but their actions are hidden from him. Our focus is on cases where arbitrary *combinations* of the efforts of the different agents probabilistically determine the outcome, and the principal's problem is to decide *which set of agents* to motivate to exert effort and to what extent. The principal motivates the agents by offering to them a set of contracts, which together should put the agents in an equilibrium point of the induced game. We present formal models for this setting, embark on a comprehensive analysis of the basic issues, but leave many questions open.

1 Introduction

1.1 Background

The well studied principal-agent problem deals with how a "principal" can motivate a rational "agent" to exert costly effort towards the welfare of the principal. The difficulty in this model is that the agent's action (i.e. whether he exerts effort or not) is invisible to the principal and only the final outcome, which is probabilistic and also influenced by other factors, is visible. "Invisible" here is meant in a wide sense that includes "not precisely measurable", "costly to determine", or "non-contractible" (meaning that it can not be upheld in "a court of law"). This problem is well studied in many contexts in classical economic theory and we refer the readers to introductory texts on economic theory such as [4] Chapter 14. The solution is based on the observation that a properly designed contract, in which the payments are contingent upon the final outcome, can influence a rational agent to exert the required effort.

In this paper we initiate a general study of handling *combinations* of agents rather than a single agent. While much work was already done on motivating teams of agents [3, 10], our emphasis is on dealing with the complex combinatorial structure of dependencies between agents' actions. In the general case, each combination of efforts exerted by the n different agents may result in a different expected gain for the principal. The general question asks, given an exact specification of the expected utility of the principal for each combination of agents' actions, which conditional payments should the principal offer to which agents as to maximize his net utility? In our setting and unlike in previous work (see, e.g., [11]), the main challenge is to decide *which set of agents* to offer contracts to; once this set is determined, the optimal contracts themselves can be easily determined.

Our study may be viewed as part of a general research agenda stemming from the fact that all types of economic activity are increasingly being handled with the aid of sophisticated computer systems. In general, in such computerized settings, complex scenarios involving multiple agents and goods can naturally occur, and they need to be algorithmically handled. This calls for the study of the standard issues in economic theory in new complex settings. The principal-agent problem is a prime example where such complex settings introduce new challenges.

Applications of such complex multi-agent scenarios include those of firms that wish to hire many individuals (or firms) to collectively perform a task such as finding information, testing a system, or promoting a product. In many such cases, the exact contribution of each sub-contractor cannot be fully monitored (as the contribution is hidden or too costly to monitor). A more extreme point of view may envision this as the first step towards a game-theoretic foundation for designing profit-based collaborative projects like open-source software or "Wikis" which are now only done voluntarily: a principal that wishes such a project to "emerge" and is willing to pay for it, need only put the optimal set of contracts in place – no further monitoring is necessary[1].

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[1]No doubt that there is still a significant gap between our models and this type of application – e.g. where do we get the specification of the expected success for each set of agents?

Another class of applications we have in mind is the incentive-based management of the Internet infrastructure – addressing the difficulty that various parts of the infrastructure are owned by different entities who have their own "selfish" goals. While the field of *algorithmic mechanism design* [5] deals with how to extract private information from the different "selfish" sub-systems, it assumes that the *actions* of these sub-systems are publicly known (or at least may be verified). This paper deals with the complementary point of view: there is no private information, but the actions of the different, privately owned, pieces of the network are not publicly observable. An example that was discussed in [2] is Quality of Service (QoS) routing in a network: every intermediate link or router may exert different amount of effort when attempting to pass a packet of information. How can we assure that the "right" combination of efforts is exerted? Similarly, a task that runs on a collection of shared servers may be allocated, by each server, an unknown percentage of the CPU's processing power or of the physical memory.

This paper suggest models for and provides some interesting initial results about this "combinatorial agency" problem. We believe that we have only scratched the surface and leave many open questions, conjectures, and directions for further research.

1.2 Our Models

We start by presenting a general model: in this model each of n agents has a set of possible *actions*, the combination of actions by the players results in some *outcome*, where this happens probabilistically. The main part of the specification of a problem in this model is a function that specifies this distribution for each n-tuple of agents' actions. Additionally, the problem specifies the principal's utility for each possible outcome, and for each agent, the agent's cost for each possible action. The principal motivates the agents by offering to each of them a *contract* that specifies a payment for each possible outcome of the whole project[2]. Key here is that the actions of the players are non-observable and thus the contract cannot make the payments directly contingent on the actions of the players, but rather only on the outcome of the whole project.

Given a set of contracts, the agents will each optimize his own utility: i.e. will choose the action that maximizes his expected payment minus the cost of his action. Since the outcome depends on the actions of all players together, the agents are put in a game and are assumed to reach a Nash equilibrium[3]. The principal's problem, our problem in this paper, is of designing an optimal set of contracts: i.e. contracts that maximize his expected utility from the outcome, minus his expected total payment. The main difficulty is that of determining the required Nash equilibrium point.

In order to focus on the main issues, the rest of the paper deals with the basic binary case: each agent has only two possible actions "exert effort" and "shirk" and there are only two possible outcomes "success" and "failure". It seems that this case already captures the main interesting ingredients[4]. In this case, each agent's problem boils down to whether to exert effort or not, and the principal's problem boils down to which agents should be contracted to exert effort. This model is still pretty abstract, and every problem description contains a complete table specifying the success probability for each subset of the agents who exert effort.

We then consider a more concrete model which concerns a subclass of problem instances where this exponential size table is succinctly represented. This subclass will provide many natural types of problem instances. In this subclass every agent performs a subtask which succeeds with a low probability γ if the agent does not exert effort and with a higher probability $\delta > \gamma$, if the agent does exert effort. The whole project succeeds as a deterministic Boolean function of the success of the subtasks. This Boolean function can now be represented in various ways. Two basic examples are the "AND" function in which the project succeeds only if all subtasks succeed, and the "OR" function which succeeds if any of the subtasks succeeds. A more complex example considers a communication network, where each agent controls a single edge, and success of the subtask means that a message is forwarded by that edge. "Effort" by the edge increases this success probability. The complete project succeeds if there is a complete path of successful edges between a given source and sink. Complete definitions of the models appear in Section 2.

1.3 Our Results

We address a host of questions and prove a large number of results. We believe that despite the large amount of work that appears here, we have only scratched the surface. In many cases we were not able to achieve the

[2] One could think of a different model in which the agents have intrinsic utility from the outcome and payments may not be needed, as in [8, 9].

[3] In this paper our philosophy is that the principal can "suggest" a Nash equilibrium point to the agents, thus focusing on the "best" Nash equilibrium. One may alternatively study the worst case equilibrium as in [11], or alternatively, attempt modeling some kind of an extensive game between the agents, as in [7, 8, 9].

[4] However, some of the more advanced questions we ask for this case can be viewed as instances of the general model.

general characterization theorems that we desired and had to settle for analyzing special cases or proving partial results. In many cases, simulations reveal structure that we were not able to formally prove. We present here an informal overview of the issues that we studied, what we were able to do, and what we were not. The full treatment of most of our results appears only in the appendix, and only some are discussed, often with associated simulation results, in the body of the paper.

Our first object of study is the structure of the class of sets of agents that can be contracted for a given problem instance. Let us fix a given function describing success probabilities, fix the agent's costs, and let us consider the set of contracted agents for different values of the principal's associated value from success. For very low values, no agent will be contracted since even a single agent's cost is higher than the principal's value. For very high values, all agents will always be contracted since the marginal contribution of an agent multiplied by the principal's value will overtake any associated payment. What happens for intermediate principal's values?

We first observe that there is a finite number of "transitions" between different sets, as the principal's project value increases. These transitions behave very differently for different functions. For example, we show that for the AND function only a single transition occurs: for low enough values no agent will be contracted, while for higher values all agents will be contracted − there is no intermediate range for which only some of the agents are contracted. For the OR function, the situation is opposite: as the principal's value increases, the set of contracted agents increases one-by-one. We are able to fully characterize the types of functions for which these two extreme types of transitions behavior occur. However, the structure of these transitions in general seems quite complex, and we were not able to fully analyze them even in simple cases like the "Majority function" (the project succeeds if a majority of subtasks succeeds) or very simple networks. We do have several partial results, including a construction with an exponential number of transitions.

During the previous analysis we also study what we term "the price of unaccountability": How much is the social utility achieved under the optimal contracts worse than what could be achieved in the non-strategic case[5], where the socially optimal actions are simply dictated by the principal? We are able to fully analyze this price for the "AND" function, where it is shown to tend to infinity as the number of agents tends to infinity. More general analysis remains an open problem.

Our analysis of these questions sheds light on the difficulty of the various natural associated algorithmic problems. In particular, we observe that the optimal contract can be found in time polynomial in the representation of the probability functions. We prove a communication lower bound that shows that the optimal contract can not be found in time that is polynomial just in the number of agents, in a general black-box model. We also show that when the probability function is succinctly represented as a read-once network, the problem becomes $\#P$-complete. The status of some algorithmic questions remains open, in particular that of finding the optimal contract for technologies defined by serial-parallel networks.

In a follow-up paper [1] we deal with equilibria in mixed strategies and show that the principal can gain from inducing a mixed-Nash equilibrium between the agents rather than a pure one. We also show cases where the principal can gain by asking agents to reduce their effort level, even when this effort comes for free. Both phenomena can not occur in the non-strategic setting.

2 Model and Preliminaries

2.1 The General Setting

A principal employs a set of agents N of size n. Each agent $i \in N$ has a possible set of actions A_i, and a cost (effort) $c_i(a_i) \geq 0$ for each possible action $a_i \in A_i$ ($c_i : A_i \to \Re_+$). The actions of all players determine, in a probabilistic way, a "contractible" outcome $o \in O$, according to a success function $t : A_1 \times ... \times A_n \to \Delta(O)$ (where $\Delta(O)$ denotes the set of probability distributions on O). A technology is a pair, (t, c), of a success function, t, and a cost function, c. The principal has a certain value for each possible outcome, given by the function $v : O \to \Re$. As we will only consider risk-neutral players in this paper[6], we will also treat v as a function on $\Delta(O)$, by taking simple expected value. Actions of the players are invisible, but the final outcome o is visible to him and to others (in particular the court), and he may design enforceable contracts based on the final outcome. Thus the contract for agent i is a function (payment) $p_i : O \to \Re$; again, we will also view p_i as a function on $\Delta(O)$.

Given this setting, the agents have been put in a game, where the utility of agent i under the vector of actions $a = (a_1, \ldots, a_n)$ is given by $u_i(a) = p_i(t(a)) - c_i(a_i)$. The agents will be assumed to reach Nash equilibrium, if such equilibrium exists. The principal's problem (which is our problem in this paper) is

[5]The non-strategic case is often referred to as the case with "contractible actions" or the principal's "first-best" solution.

[6]The risk-averse case would obviously be a natural second step in the research of this model, as has been for non-combinatorial scenarios.

how to design the contracts p_i as to maximize his own expected utility $u(a) = v(t(a)) - \sum_i p_i(t(a))$, where the actions $a_1, \ldots, a_n$ are at Nash-equilibrium. In the case of multiple Nash equilibria we let the principal choose the equilibrium, thus focusing on the "best" Nash equilibrium. A variant, which is similar in spirit to "strong implementation" in mechanism design would be to take the worst Nash equilibrium, or even, stronger yet, to require that only a single equilibrium exists.

2.2 The Binary-Outcome Binary-Action Model

We wish to concentrate on the complexities introduced by the combinatorial structure of the success function t, we restrict ourselves to a simpler setting that seems to focus more clearly on the structure of t. A similar model was used in [11]. We first restrict the action spaces to have only two states (binary-action): 0 (low effort) and 1 (high effort). The cost function of agent i is now just a scalar $c_i > 0$ denoting the cost of exerting high effort (where the low effort has cost 0). The vector of costs is $\vec{c} = (c_1, c_2, \ldots, c_n)$, and we use the notation $(t, \vec{c})$ to denote a technology in such a binary-outcome model. We then restrict the outcome space to have only two states (binary-outcome): 0 (project failure) and 1 (project success). The principal's value for a successful project is given by a scalar $v > 0$ (where the value of project failure is 0). We assume that the principal can pay the agents but not fine them (known as the *limited liability* constraint). The contract to agent i is thus now given by a scalar value $p_i \geq 0$ that denotes the payment that i gets in case of project success. If the project fails, the agent gets 0. When the lowest cost action has zero cost (as we assume), this immediately implies that the participation constraint holds.

At this point the success function t becomes a function $t : \{0,1\}^n \to [0,1]$, where $t(a_1, \ldots, a_n)$ denotes the probability of project success where players with $a_i = 0$ do not exert effort and incur no cost, and players with $a_i = 1$ do exert effort and incur a cost of c_i.

As we wish to concentrate on motivating agents, rather than on the coordination between agents, we assume that more effort by an agent always leads to a better probability of success. I.e. that the success function t is strictly monotone. Formally, if we denote by a_{-i} the $(n-1)$-dimensional vector of the actions of all agents excluding agent i. i.e., $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, then a success function must satisfy:

$$\forall i \in N,\ \forall a_{-i} \in A_{-i} \quad t(1, a_{-i}) > t(0, a_{-i})$$

Additionally, we assume that $t(a) > 0$ for any $a \in A$ (or equivalently, $t(0, 0, \ldots, 0) > 0$).

Definition 1. *The marginal contribution of agent i, denoted by Δ_i, is the difference between the probability of success when i exerts effort and when he shirks.*

$$\Delta_i(a_{-i}) = t(1, a_{-i}) - t(0, a_{-i})$$

Note that since t is monotone, Δ_i is a strictly positive function. At this point we can already make some simple observations. The best action, $a_i \in A_i$, of agent i can now be easily determined as a function of what the others do, $a_{-i} \in A_{-i}$, and his contract p_i.

Claim 1. *Given a profile of actions a_{-i}, agent i's best strategy is $a_i = 1$ if $p_i \geq \frac{c_i}{\Delta_i(a_{-i})}$, and is $a_i = 0$ if $p_i \leq \frac{c_i}{\Delta_i(a_{-i})}$. (In the case of equality the agent is indifferent between the two alternatives.)*

As $p_i \geq \frac{c_i}{\Delta_i(a_{-i})}$ if and only if $u_i(1, a_{-i}) = p_i \cdot t(1, a_{-i}) - c_i \geq p_i \cdot t(0, a_{-i}) = u_i(0, a_{-i})$, i's best strategy is to choose $a_i = 1$ in this case.

This allows us to specify the contracts that are the principal's optimal, for inducing a given equilibrium.

Observation 1. *The best contracts (for the principal) that induce $a \in A$ as an equilibrium are $p_i = 0$ for agents who exert no effort ($a_i = 0$), and $p_i = \frac{c_i}{\Delta_i(a_{-i})}$ for agents who exert effort ($a_i = 1$).*

In this case, the expected utility of agents who exert effort is $c_i \cdot \left(\frac{t(1, a_{-i})}{\Delta_i(a_{-i})} - 1 \right)$, and 0 for agents who shirk. The principal's expected utility is given by $u(a, v) = (v - P) \cdot t(a)$, where P is the total payment in case of success, given by $P = \sum_{i | a_i = 1} \frac{c_i}{\Delta_i(a_{-i})}$.

We say that the principal *contracts with* agent i if $p_i > 0$ (and $a_i = 1$ in the equilibrium $a \in A$). The principal's goal is to maximize his utility given his value v, i.e. to determine the profile of actions $a^* \in A$, which gives the highest value of $u(a, v)$ in equilibrium. We call the set of agents S^* that the principal contracts with in a^* ($S^* = \{i | a^*_i = 1\}$) an *optimal contract* for the principal at value v.

A natural yardstick by which to measure this decision is the non-strategic case, i.e. when the agents need not be motivated but are rather controlled directly by the principal (who also bears their costs). In this case the principal will simply choose the profile $a \in A$ that optimizes the social welfare (global efficiency), $t(a) \cdot v - \sum_{i | a_i = 1} c_i$. The ratio between the social welfare in this non-strategic case and the social welfare for the profile $a \in A$ chosen by the principal in the agency case, may be termed the *price of unaccountability*.

Given a technology $(t, \vec{c})$, let $S^*(v)$ denote the optimal contract in the agency case and let $S^*_{na}(v)$ denote an optimal contract in the non-strategic case, when the principal's value is v. The social welfare for value v when the set S of agents is contracted is $t(S) \cdot v - \sum_{i \in S} c_i$ (in both the agency and non-strategic cases).

Definition 2. *The price of unaccountability $POU(t, \vec{c})$ of a technology $(t, \vec{c})$ is defined as the ratio between the total social welfare in the non-strategic case and the agency case:*

$$POU(t, \vec{c}) = Sup_{v>0} \frac{t(S^*_{na}(v)) \cdot v - \sum_{i \in S^*_{na}(v)} c_i}{t(S^*(v)) \cdot v - \sum_{i \in S^*(v)} c_i}$$

In cases where several sets are optimal in the agency case, we take the worst set (i.e., the set that yields the lowest social welfare).

When the technology $(t, \vec{c})$ is clear in the context we will use POU to denote the price of unaccountability for technology $(t, \vec{c})$. Note that the POU is at least 1 for any technology, and is exactly 1 for some technology $(t, \vec{c})$ if and only if the socially efficient outcome is achieved (in the strategic case) for all values.

As we would like to focus on results that derived from properties of the success function, in most of the paper we will deal with the case where all agents have the same cost c of exerting effort, that is $c_i = c$ for all $i \in N$. We denote a technology $(t, \vec{c})$ with *identical costs* by (t, c). For the simplicity of the presentation, we sometimes use the term *technology function* to refer to the success function of the technology.

2.3 Structured Technology Functions

In order to be more concrete, we will especially focus on technology functions whose structure can be described easily as being derived from independent agent tasks – we call these *structured technology functions*. This subclass will first give us some natural examples of technology function, and will also provide a succinct and natural way to represent the technology functions.

In a structured technology function, each individual succeeds or fails in his own "task" independently. The project's success or failure depends, in a complex way, on the set of successful sub-tasks. Thus we will assume a monotone Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ which denotes whether the project succeeds as a function of the success of the n agents' tasks, and constants $0 < \gamma_i < \delta_i < 1$, where γ_i denotes the probability of success for agent i if he does not exert effort, and δ_i ($> \gamma_i$) denotes the probability of success if he does exert effort. In order to reduce the number of parameters, we will restrict our attention to the case where $\gamma_1 = \ldots = \gamma_n = \gamma$ and $\delta_1 = \ldots = \delta_n = 1 - \gamma$ thus leaving ourselves with a single parameter γ s.t. $0 < \gamma < \frac{1}{2}$.

Under this structure, the technology function t is defined by $t(a_1, \ldots, a_n)$ being the probability that $f(x_1, \ldots, x_n) = 1$ where the bits $x_1, \ldots, x_n$ are chosen according to the following distribution: if $a_i = 0$ then $x_i = 1$ with probability γ and $x_i = 0$ with probability $1 - \gamma$; otherwise, i.e. if $a_i = 1$, then $x_i = 1$ with probability $1 - \gamma$ and $x_i = 0$ with probability γ. We denote $x = (x_1, \ldots, x_n)$.

The question of the representation of the technology function is now reduced to that of representing the underlying monotone Boolean function f. In the most general case, the function f can be given by a general monotone Boolean circuit. An especially natural sub-class of functions in the structured technologies setting would be functions that can be represented as a *read-once network* – a graph with a given source and sink, where every edge is labeled by a player. The project succeeds if the edges that belong to player's whose task succeeded form a path between the source and the sink[7].

A few simple examples should be in order here:

1. The "AND" technology: $f(x_1, \ldots, x_n)$ is the logical conjunction of x_i ($f(x) = \bigwedge_{i \in N} x_i$). Thus the project succeeds only if all agents succeed in their tasks. This is shown graphically as a read-once network in Figure 1(a). If m agents exert effort ($\sum_i a_i = m$), then $t(a) = t_m = \gamma^{n-m}(1-\gamma)^m$. E.g. for two players, the technology function $t(a_1 a_2) = t_{a_1+a_2}$ is given by $t_0 = t(00) = \gamma^2$, $t_1 = t(01) = t(10) = \gamma(1-\gamma)$, and $t_2 = t(11) = (1-\gamma)^2$.
2. The "OR" technology: $f(x_1, \ldots, x_n)$ is the logical disjunction of x_i ($f(x) = \bigvee_{i \in N} x_i$). Thus the project succeeds if at least one of the agents succeed in their tasks. This is shown graphically as a read-once network in Figure 1(b). If m agents exert effort, then $t_m = 1 - \gamma^m(1-\gamma)^{n-m}$.E.g. for two players, the technology function is given by $t(00) = 1 - (1-\gamma)^2$, $t(01) = t(10) = 1 - \gamma(1-\gamma)$, and $t(11) = 1 - \gamma^2$.
3. The "Or-of-Ands" (OOA) technology: $f(x)$ is the logical disjunction of conjunctions. In the simplest case of equal-length clauses (denote by n_c the number of clauses and by n_l their length), $f(x) = \bigvee_{j=1}^{n_c}(\bigwedge_{k=1}^{n_l} x_k^j)$. Thus the project succeeds if in at least one clause all agents succeed in their tasks. This is shown graphically as a read-once network in Figure 2(a). If m_i agents on path i exert effort, then $t(m_1, ..., m_{n_c}) = 1 - \prod_i (1 - \gamma^{n_l - m_i}(1-\gamma)^{m_i})$. E.g. for four players, the technology function $t(a_1^1\ a_2^1, a_1^2\ a_2^2)$ is given by $t(00, 00) = 1 - (1-\gamma^2)^2$,

[7] One may view this representation as directly corresponding to the project of delivering a message from the source to the sink in a real network of computers, with the edges being controlled by selfish agents.

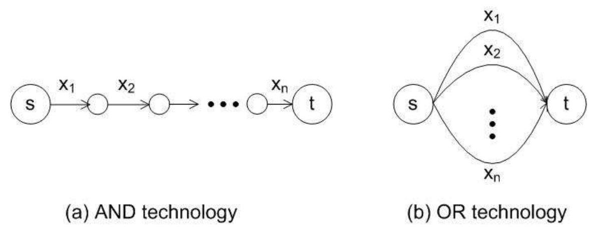

Figure 1: Graphical representations of (a) AND and (b) OR technologies.

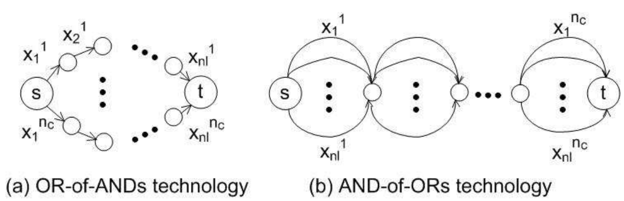

Figure 2: Graphical representations of (a) OOA and (b) AOO technologies.

$t(01,00) = t(10,00) = t(00,01) = t(00,10) = 1-(1-\gamma(1-\gamma))(1-\gamma^2)$, and so on.

4. The "And-of-Ors" (AOO) technology: $f(x)$ is the logical conjunction of disjunctions. In the simplest case of equal-length clauses (denote by n_l the number of clauses and by n_c their length), $f(x) = \bigwedge_{j=1}^{n_l}(\bigvee_{k=1}^{n_c} x_k^j)$. Thus the project succeeds if at least one agent from each disjunctive-form-clause succeeds in his tasks. This is shown graphically as a read-once network in Figure 2(b). If m_i agents on clause i exert effort, then $t(m_1, ..., m_{n_c}) = \prod_i (1-\gamma^{m_i}(1-\gamma)^{n_c - m_i})$. E.g. for four players, the technology function $t(a_1^1\ a_2^1, a_1^2\ a_2^2)$ is given by $t(00,00) = (1-(1-\gamma)^2)^2$, $t(01,00) = t(10,00) = t(00,01) = t(00,10) = (1-\gamma(1-\gamma))(1-(1-\gamma)^2)$, and so on.

5. The "Majority" technology: $f(x)$ is "1" if a majority of the values x_i are 1. Thus the project succeeds if most players succeed. The majority function, even on 3 inputs, can not be represented by a read-once network, but is easily represented by a monotone Boolean formula $maj(x,y,z) = xy + yz + xz$. In this case the technology function is given by $t(000) = 3\gamma^2(1-\gamma)+\gamma^3$, $t(001) = t(010) = t(100) = \gamma^3 + 2(1-\gamma)^2\gamma + \gamma^2(1-\gamma)$, etc.

3 Analysis of Some Anonymous Technologies

A success function t is called *anonymous* if it is symmetric with respect to the players. I.e. $t(a_1, \ldots, a_n)$ depends only on $\sum_i a_i$. A technology (t, c) is *anonymous* if t is anonymous and the cost c is identical to all agents. Of the examples presented above, the AND, OR, and majority technologies were anonymous (but not AOO and OOA). For an anonymous t we denote $t_m = t(1^m, 0^{n-m})$ and $\Delta_m = t_{m+1} - t_m$.

3.1 AND and OR Technologies

Let us start with a direct and full analysis of the AND and OR technologies for two players for the case $\gamma = 1/4$ and $c = 1$.

Example 1. AND **technology with two agents,** $c = 1$**,** $\gamma = 1/4$**:** *we have* $t_0 = \gamma^2 = 1/16$*,* $t_1 = \gamma(1-\gamma) = 3/16$*, and* $t_2 = (1-\gamma)^2 = 9/16$ *thus* $\Delta_0 = 1/8$ *and* $\Delta_1 = 3/8$*. The principal has 3 possibilities: contracting with 0, 1, or 2 agents. Let us write down the expressions for his utility in these 3 cases:*

- **0 Agents:** *No agent is paid thus and the principal's utility is* $u_0 = t_0 \cdot v = v/16$*.*
- **1 Agent:** *This agent is paid* $p_1 = c/\Delta_0 = 8$ *on success and the principal's utility is* $u_1 = t_1(v - p) = 3v/16 - 3/2$*.*
- **2 Agents:** *each agent is paid* $p_2 = c/\Delta_1 = 8/3$ *on success, and the principal's utility is* $u_2 = t_2(v - 2p_2) = 9v/16 - 3$*.*

Notice that the option of contracting with one agent is always inferior to either contracting with both or with none, and will never be taken by the principal. The principal will contract with no agent when $v < 6$*, with both agents whenever* $v > 6$*, and with either non or both for* $v = 6$*.*

This should be contrasted with the non-strategic case in which the principal completely controls the agents (and bears their costs) and thus simply optimizes globally. In this case the principal will make both agents exert effort whenever $v \geq 4$*. Thus for example, for* $v = 6$ *the globally optimal decision (non-strategic case) would give a global utility of* $6 \cdot 9/16 - 2 = 11/8$ *while the principal's decision (in the agency case) would give a global utility of* $3/8$*, giving a ratio of* $11/3$*.*

It turns out that this is the worst price of unaccountability in this example, and it is obtained exactly at the transition point of the agency case, as we show below.

Example 2. OR **technology with two agents,** $c = 1$**,** $\gamma = 1/4$**:** *we have* $t_0 = 1-(1-\gamma)^2 = 7/16$*,* $t_1 =$

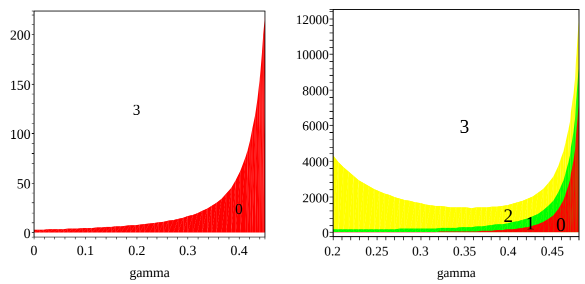

Figure 3: Number of agents in the optimal contract of the AND (left) and OR (right) technologies with 3 players, as a function of γ and v. AND technology: either 0 or 3 agents are contracted, and the transition value is monotonic in γ. OR technology: for any γ we can see all transitions.

$1-\gamma(1-\gamma) = 13/16$, *and* $t_2 = 1-\gamma^2 = 15/16$ *thus* $\Delta_0 = 3/8$ *and* $\Delta_1 = 1/8$. *Let us write down the expressions for the principal's utility in these three cases:*

- **0 Agents:** *No agent is paid and the principal's utility is* $u_0 = t_0 \cdot v = 7v/16$.
- **1 Agent:** *This agent is paid* $p_1 = c/\Delta_0 = 8/3$ *on success and the principal's utility is* $u_1 = t_1(v - p_1) = 13v/16 - 13/6$.
- **2 Agents:** *each agent is paid* $p_2 = c/\Delta_1 = 8$ *on success, and the principal's utility is* $u_2 = t_2(v - 2p_2) = 15v/16 - 15$.

Now contracting with one agent is better than contracting with none whenever $v > 52/9$ *(and is equivalent for* $v = 52/9$*), and contracting with both agents is better than contracting with one agent whenever* $v > 308/3$ *(and is equivalent for* $v = 308/3$*), thus the principal will contract with no agent for* $0 \leq v \leq 52/9$, *with one agent for* $52/9 \leq v \leq 308/3$, *and with both agents for* $v \geq 308/3$.

In the non-strategic case, in comparison, the principal will make a single agent exert effort for $v > 8/3$, *and the second one exert effort as well when* $v > 8$.

It turns out that the price of unaccountability here is 19/13, and is achieved at $v = 52/9$, which is exactly the transition point from 0 to 1 contracted agents in the agency case. This is not a coincidence that in both the AND and OR technologies the POU is obtained for v that is a transition point (see proof in Appendix A).

Lemma 1. *For any given technology* $(t, \vec{c})$ *the price of unaccountability* $POU(t, \vec{c})$ *is obtained at some value* v *which is a transition point, of either the agency or the non-strategic cases.*

We already see a qualitative difference between the AND and OR technologies (even with 2 agents): in the first case either all agents are contracted or none, while in the second case, for some intermediate range of values v, exactly one agent is contracted. Figure 4 shows the same phenomena for AND and OR technologies with 3 players.

Theorem 1. *For any anonymous* AND *technology*[8]*:*

- *there exists a value*[9] $v_* < \infty$ *such that for any* $v < v_*$ *it is optimal to contract with no agent, for* $v > v_*$ *it is optimal to contract with all* n *agents, and for* $v = v_*$, *both contracts* $(0, n)$ *are optimal.*
- *the price of unaccountability is obtained at the transition point of the agency case, and is* $POU = \left(\frac{1}{\gamma} - 1\right)^{n-1} + (1 - \frac{\gamma}{1-\gamma})$.

The findings of Theorem 1 are special cases of the characterization presented in Appendix B.1 and the POU result of Lemma 2.

The property of a single transition occurs in both the agency and the non-strategic cases, where the transition occurs at a smaller value of v in the non-strategic case. Notice that the POU is not bounded across the AND family of technologies (for various n, γ) as $POU \to \infty$ either if $\gamma \to 0$ (for any given $n \geq 2$) or $n \to \infty$ (for any fixed $\gamma \in (0, \frac{1}{2})$).

Next we consider the OR technology and show that it exhibits all n transitions.

Theorem 2. *For any anonymous* OR *technology, there exist finite positive values* $v_1 < v_2 < \ldots < v_n$ *such that for any* v *s.t.* $v_k < v < v_{k+1}$, *contracting with exactly* k *agents is optimal (for* $v < v_1$, *no agent is contracted, and for* $v > v_n$, *all* n *agents are contracted). For* $v = v_k$, *the principal is indifferent between contracting with* $k-1$ *or* k *agents.*

The same behavior occurs in both the agency and the non-strategic case. This characterization is a direct corollary of a more general characterization given in Appendix B.1.

For OR technology with $n = 2$ we can bound the POU by 2 (see Lemma 15 in Appendix B.3). Based on the lemma, we already observe a qualitative difference between the POU in the AND and OR technologies.

Observation 2. *While in the* AND *technology the POU for* $n = 2$ *is not bounded from above (for* $\gamma \to 0$*), for* OR *technology the POU is bounded by 2.*

[8] AND technology with any number of agents n and any γ, and any identical cost c.

[9] v_* is a function of n, γ, c.

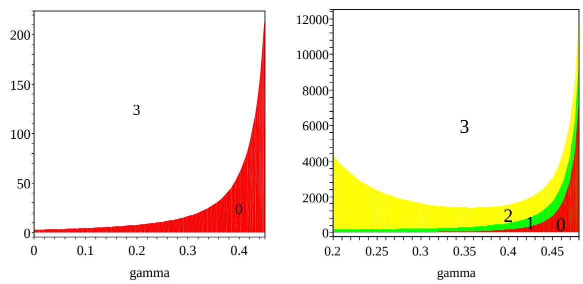

Figure 4: Number of agents in the optimal contract of the AND (left) and OR (right) technologies with 3 players, as a function of γ and v. AND technology: either 0 or 3 agents are contracted, and the transition value is monotonic in γ. OR technology: for any γ we can see all transitions.

3.2 What Determines the Transitions?

Theorems 1 and 2 say that both the AND and OR technologies exhibit the same transition behavior (changes of the optimal contract) in the agency and the non-strategic cases. However, this is not true in general. In Appendix B.1 we provide a full characterization of the sufficient and necessary conditions for general anonymous technologies to have a single transition and all n transitions. We find that the conditions in the agency case are different than the ones in the non-strategic case.

We are able to determine the POU for any anonymous technology that exhibits a single transition in both the agency and the non-strategic cases (see proof in Appendix B.3).

Lemma 2. *For any anonymous technology that has a single transition in both the agency and the non-strategic cases, the POU is given by:*
$POU = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$.

3.3 The MAJORITY Technology

The project under the MAJORITY function succeeds if the majority of the agents succeed in their tasks (see Section 2.3). We are unable to characterize the transition behavior of the MAJORITY technology analytically. Figure 5 presents the optimal number of contracted agents as a function of v and γ, for $n = 5$. The phenomena that we observe in this example (and others that we looked at) leads us to the following conjecture.

Conjecture 1. *For any Majority technology (any n, γ and c), there exists l, $1 < l \leq \lceil n/2 \rceil$ such that the first transition is from 0 to l agents, and then all the remaining $n - l$ transitions exist.*

Moreover, for any fixed c, n, $l = 1$ when γ is close enough to $\frac{1}{2}$, l is a non-decreasing function of γ (with image $\{1, \ldots, \lceil n/2 \rceil\}$), and $l = \lceil n/2 \rceil$ when γ is close enough to 0.

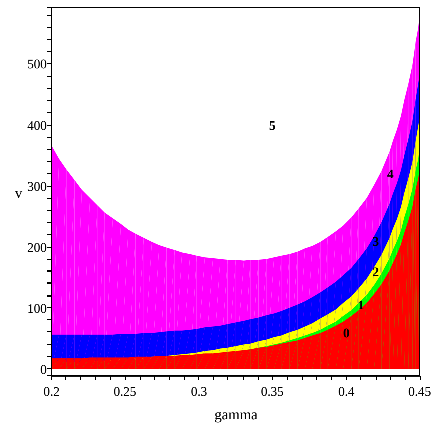

Figure 5: Number of agents in the optimal contract of the $MAJORITY$ technology with 5 players, as a function of γ and v. As γ decreases the first transition is at a lower value and to a higher number of agents. For any sufficiently small γ, the first transition is to $3 = \lceil 5/2 \rceil$ agents, and for any sufficiently large γ, the first transition is to 1 agents. For any γ, the first transition is never to more than 3 agents, and after the first transition we see all following possible transitions.

4 Non-Anonymous Technologies

In non-anonymous technologies (even with identical costs), we need to talk about the contracted *set* of agents and not only about the number of contracted agents. In this section, we identify the sets of agents that can be obtained as the optimal contract for some v. These sets construct the *orbit* of a technology.

Definition 3. *For a technology t, a set of agents S is in the* orbit of t *if for some value v, the optimal contract is exactly with the set S of agents (where ties between different S's are broken according to a lexicographic order*[10]*). The* k-orbit *of t is the collection of sets of size exactly k in the orbit.*

An important observation is that the orbit of a technology is actually an ordered list of sets of agents, where the order is determined by the following lemma (see proof in Appendix A).

Lemma 3. *(***Monotonicity lemma***) For any technology $(t, \vec{c})$, in both the agency and the non-strategic*

[10] This implies that there are no two sets with the same success probability on the orbit.

cases, the utility of the principal at the optimal contracts, the success probability of the optimal contracts, and the expected payment of the optimal contract, are all monotonically non-decreasing with the value.

4.1 AOO and OOA Technologies

We begin our discussion of non-anonymous technologies with two examples; the And-of-Ors (*AOO*) and Or-of-Ands (*OOA*) technologies.

The *AOO* technology (see figure 2) is composed of multiple *OR-components* that are "And"ed together.

Theorem 3. *Let h be an anonymous OR technology, and let $f = \bigwedge_{j=1}^{n_c} h$ be the AOO technology that is obtained by a conjunction of n_c of these OR-components on disjoint inputs. Then for any value v, an optimal contract contracts with the same number of agents in each OR-component. Thus, the orbit of f is of size at most $n_l + 1$, where n_l is the number of agents in h.*

Part of the proof of the theorem (for the complete proof see Appendix C.2), is based on such *AOO* technology being a special case of a more general family of technologies, in which disjoint anonymous technologies are "And"-ed together, as explained in the next section. We conjecture that a similar result holds for the *OOA* technology.

Conjecture 2. *In an OOA technology which is a disjunction of the same anonymous paths (with the same number of agents, γ and c, but over disjoint inputs), for any value v the optimal contract is constructed from some number of fully-contracted paths. Moreover, there exist $v_1 < \ldots < v_{n_l}$ such that for any v, $v_i \leq v \leq v_{i+1}$, exactly i paths are contracted.*

We are unable to prove it in general, but can prove it for the case of an *OOA* technology with two paths of length two (see Appendix C.2).

4.2 Orbit Characterization

The *AOO* is an example of a technology whose orbit size is linear in its number of agents. If conjecture 2 is true, the same holds for the *OOA* technology. What can be said about the orbit size of a general non-anonymous technology?

A collection of sets of k elements (out of n) is "admissible", if every two sets in the collection differ by at least 2 elements (e.g. for k=3, 123 and 234 can not be together in the collection, but 123 and 345 can be).

Theorem 4. *Every admissible collection can be obtained as the $k - orbit$ of some t.*

In Appendix C.1 we present the proof of the theorem, as well as the proofs of all other claims presented in this section. We next show that there exist very large admissible collections.

Lemma 4. *For any $n \geq k$, there exists an admissible collection of k-size sets of size $\Omega(\frac{1}{n} \cdot \binom{n}{k})$.*

Corollary 1. *There exists a technology (t, c) with orbit of size $\Omega(\frac{2^n}{n\sqrt{n}})$.*

Thus, we are able to construct a technology with exponential orbit, but this technology is not a network technology or a structured technology.

Open Question 1. *Is there a Read Once network with exponential orbit? Is there a structured technology with exponential orbit?*

Nevertheless, for some network technologies, we believe that the orbit size cannot be very large.

Conjecture 3. *The orbit size of serial-parallel networks is $O(n)$.*

We are unable to fully prove this, but we are able to show that it holds for networks that are "AND"-ed together, as characterized below.

Let g and h be two Boolean functions on disjoint inputs and let $f = g \bigwedge h$ (i.e., take their networks in series). The optimal contract for f for some v, denoted by S, is composed of some agents from the h-part and some from the g-part, call them T and R respectively.

Lemma 5. *Let S be an optimal contract for $f = g \bigwedge h$ on v. Then, T is an optimal contract for h on $v \cdot t_g(R)$, and R is an optimal contract for g on $v \cdot t_h(T)$.*

Lemma 6. *Let g and h be two Boolean functions on disjoint inputs and let $f = g \bigwedge h$ (i.e., take their networks in series). Suppose x and y are the respective orbit sizes of g and h; then, the orbit size of f is less or equal to $x + y - 1$.*

By induction we get the following corollary.

Corollary 2. *Assume that $\{(g_j, c_j)\}_{j=1}^m$ is a set of anonymous technologies on disjoint inputs, each with identical agent cost (all agents of technology g_j has the same cost c_j). Then the orbit of $f = \bigwedge_{j=1}^m g_j$ is of size at most $(\sum_{j=1}^m n_j) - 1$, where n_j is the number of agents in technology g_j (the orbit is linear in the number of agents).*

In particular, this holds for *AOO* technology where each OR-component is anonymous.

Open Question 2. *Does Lemma 6 hold also for the Boolean function $f = g \bigvee h$ (i.e., when the networks g, h are taken in parallel)?*

We conjecture that this is indeed the case, and that the corresponding Lemmas 5 and 6 exist for the OR case as well. If this is true, we will be able to prove conjecture 3.

We also conjecture that IRS ensures small orbit.

Conjecture 4. *Any IRS technology has orbit of size* $O(n)$.

5 Algorithmic Aspects

Our analysis throughout the paper sheds some light on the algorithmic aspects of computing the best contract. In this section we state these implications (for the proofs see Appendix D). We first consider the general model where the technology function is given by an arbitrary monotone function t (with rational values), and we then consider the case of structured technologies given by a network representation of the underlying Boolean function.

5.1 Binary-Outcome Binary-Action Technologies

Here we assume that we are given a technology and value v as the input, and our output should be the optimal contract, i.e. the set S^* of agents to be contracted and the contract p_i for each $i \in S^*$. In the general case, the success function t is of size exponential in n, the number of agents, and we will need to deal with that. In the special case of anonymous technologies, the description of t is only the $n+1$ numbers $t_0, \ldots, t_n$, and in this case our analysis in section 3 completely suffices for computing the optimal contract.

Proposition 1. *Given as input the full description of a technology (the values* $t_0, \ldots, t_n$ *and the identical cost* c *for an anonymous technology, or the value* $t(S)$ *for all the* 2^n *possible subsets* $S \subseteq N$ *of the players, and a vector of costs* $\vec{c}$ *for non-anonymous technologies), the following can all be computed in polynomial time:*

- *The orbit of the technology in both the agency and the non-strategic cases.*
- *An optimal contract for any given value* v*, for both the agency and the non-strategic cases.*
- *The price of unaccountability* $POU(t, \vec{c})$.

A more interesting question is whether if given the function t as a black box, we can compute the optimal contract in time that is polynomial in n. We can show that, in general this is not the case:

Theorem 5. *Given as input a black box for a success function* t *(when the costs are identical), and a value* v*, the number of queries that is needed, in the worst case, to find the optimal contract is exponential in* n.

5.2 Structured Technologies

In this section we will consider the natural representation of read-once networks for the underlying Boolean function. Thus the problem we address will be:

The Optimal Contract Problem for Read Once Networks:

Input: A read-once network $G = (V, E)$, with two specific vertices s, t; rational values γ_e for each player $e \in E$ (and $c_e = 1$), and a rational value v.

Output: A set S of agents who should be contracted in an optimal contract.

We first notice that even computing the value $t(E)$ is a hard problem: it is called the network reliability problem is is known to be $\#P-complete$ [6]. Just a little effort will reveal that our problem is no easier:

Theorem 6. *The Optimal Contract Problem for Read Once Networks is #P-complete (under Turing reductions).*

A special case which find of interest is the case of series-parallel networks for which the network reliability problem (computing the value of t) is easy. We conjecture that finding the optimal contract is also easy:

Conjecture 5. *The optimal contract problem for Read Once series-parallel networks can be solved in polynomial time.*

We can only handle the first non-trivial level, AND-of-OR networks:

Lemma 7. *Given a Read Once* $AND-of-OR$ *network such that each OR-component is an anonymous technology, the optimal contract problem can be solved in polynomial time.*

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A General

Let $Q(S)$ be the expected total payment to all agents in S in the case that the principal contracts with the set S and the project succeeds (we use this notation in claims that are true for both the agency and the non-strategic cases). Let $Q_{ns}(S)$ be the above expected payment in the non-strategic case, that is $Q_{ns}(S) = \sum_{i\in S} c_i$. Let $Q_a(S)$ be the above expected payment in the agency case, that is $Q_a(S) = t(S)\cdot\sum_{i\in S}\frac{c_i}{t(S)-t(S\setminus i)}$.

Lemma 3 *(***Monotonicity lemma***) For any technology $(t,\vec{c})$, in both the agency and the non-strategic cases, the utility of the principal at the optimal contracts, the success probability of the optimal contracts, and the expected payment of the optimal contract, are all monotonically non-decreasing with the value.*

Proof. Suppose the sets of agents S_1 and S_2 are optimal in v_1 and $v_2 < v_1$, respectively. The utility is a linear function of the value, $u(S,v) = t(S)\cdot v - Q(S)$. As S_1 is optimal at v_1, $u(S_1,v_1) \geq u(S_2,v_1)$, and as $t(S)\geq 0$ and $v_1 > v_2$, $u(S_2,v_1)\geq u(S_2,v_2)$. We conclude that $u(S_1,v_1)\geq u(S_2,v_2)$, thus the utility is monotonic non-decreasing in the value.

Next we show that the success probability is monotonic non-decreasing in the value. S_1 is optimal at v_1, thus:

$$t(S_1)\cdot v_1 - Q(S_1) \geq t(S_2)\cdot v_1 - Q(S_2)$$

S_2 is optimal at v_2, thus:

$$t(S_2)\cdot v_2 - Q(S_2) \geq t(S_1)\cdot v_2 - Q(S_1)$$

Summing these two equations, we get that $(t(S_1) - t(S_2))\cdot(v_1 - v_2) \geq 0$, which implies that if $v_1 > v_2$ than $t(S_1)\geq t(S_2)$.

Finally we show that the expected payment is monotonic non-decreasing in the value. As S_2 is optimal at v_2 and $t(S_1)\geq t(S_2)$, we observe that:

$$t(S_2)\cdot v_2 - Q(S_2) \geq t(S_1)\cdot v_2 - Q(S_1) \geq t(S_2)\cdot v_2 - Q(S_1)$$

or equivalently, $Q(S_2)\leq Q(S_1)$, which is what we wanted to show. □

The above implies that the social welfare is monotonically non-decreasing with the value in the non-strategic case (as it equals to the principal utility).

Lemma 1 *For any given technology $(t,\vec{c})$ the price of unaccountability $POU(t,\vec{c})$ is obtained at some value v which is a transition point, of either the agency or the non-strategic cases.*

Proof. For high enough values, all agents are contracted in both the agency and the non-strategic case. As the payments are independent of the value, the ratio decreases when v increases, thus if the POU is obtained, it happens for a finite positive value, at which an optimal set in the agency case is not N (all the agents). Let v^* be the infimum over values for which contracting with all the agents is optimal in the agency case (v^* is finite). Up to the first transition point $\underline{v} > 0$ of the non-strategic case, the ratio is 1, thus if the POP is obtained, it is obtained for some value $v \geq \underline{v}$.

We can assume w.l.o.g. that ties between optimal sets are broken in a consistent way (as we only care about the welfare of the principal). By Lemma 3, we can partition the $[\underline{v}, v^*]$ interval to at most $2^n - 1$ segments, in each the optimal contract in the agency case is fixed (and is not all agents). Similarly, each of these segments can be partitioned to at most 2^n segments, in each the optimal contract for the non-strategic case is fixed. We conclude that there is a finite partition of the $[\underline{v}, v^*]$ interval such that at each segment, the optimal contracts of the agency and non-strategic cases are fixed.

Let $f(v) = \frac{t(S^*_{na})\cdot v - \sum_{i\in S^*_{na}} c_i}{t(S^*)\cdot v - \sum_{i\in S^*} c_i}$. On each of the segments mentioned above, f satisfies the conditions of Lemma 8, thus its suprimum is obtained at an end point of the segment. The global suprimum (over all segments) is obtained as the maximum of finitely many maximal numbers obtained, one in each segment. □

Lemma 8. *Let $f(x) = \frac{a\cdot x - b}{c\cdot x - d}$ be a function for $c > 0$. Let $\overline{x} \geq \underline{x} > 0$ be two points for which $c\underline{x} - d > 0$. Then the supremum of f on the range $[\underline{x}, \overline{x}]$ is obtained at either $\underline{x}$ or $\overline{x}$.*

Proof. As f is a continuous function (recall that $cx - d > 0$ on the range as $c > 0$ and $c\underline{x} - d > 0$) on a compact range, its supremum is obtained.

In order to find the maximum of f, we take the first derivative and equate to zero:

$$\frac{\partial f}{\partial x} = \frac{a(cx-d) - c(ax-b)}{(cx-d)^2} = \frac{bc - ad}{(cx-d)^2} = 0$$

which holds if and only if $bc = ad$. As this equality is independent of x, it either hold for any x (and in particular for $\overline{x}$ and $\underline{x}$), or for no x. If it holds for no

x, then the maximum must be obtained at either $\underline{x}$ or $\overline{x}$. □

B Analysis of Some Anonymous Technologies

B.1 What Determines the Transitions?

In this appendix we characterize anonymous technologies with a single transition, and with all n transitions for both the agency and the non-strategic cases. The proofs for the claims presented below appear in Appendix B.2.

We begin by an example that shows that transitions are not necessarily the same in the agency and the non-strategic cases.

Example 3. *Consider the following anonymous technology with two agents: $t_0 = 0$, $t_1 = 0.3$, and $t_2 = 0.61$. In the agency case, the principal will contract with no agent for $0 \leq v \leq 3.33..$, with one agent for $3.33.. \leq v \leq 9.47..$, and with both agents for $v \geq 9.47..$, thus either 0, 1, or 2 agents can be obtained as the optimal contract. In contrast, in the non-strategic case, the principal contracts with no agent for $0 \leq v \leq 3.33..$ and with both agents for $v \geq 3.33...$ Contracting with a single agent is never optimal.*

A technology (t, c) has *all transitions* if for each $k \in \{0, 1, \ldots, n\}$, there exists v for which $\forall k' \neq k, u(k) > u(k')$.[11]

What determines the number of transitions in the agency and non-strategic cases? It turns out that in the non-strategic case, the conditions are simple. We denote $a_{-i} < b_{-i}$ if for any agent $j \neq i$ it holds that $a_j \leq b_j$, and for some $j \neq i$ it holds that $a_j < b_j$.

Definition 4. *A technology success function*[12] *t exhibits*

- *(strictly)* **increasing returns to scale (IRS)** *if for every i and every $a_{-i} < b_{-i}$ it holds that $\Delta_i(a_{-i}) < \Delta_i(b_{-i})$*
- *(strictly)* **decreasig returns to scale (DRS)** *if for every i and every $a_{-i} < b_{-i}$ it holds that $\Delta_i(a_{-i}) > \Delta_i(b_{-i})$*
- *(strictly)* **under-proportional contribution (UPC)** *if the technology is anonymous, and for every k it holds that*

$$\frac{k}{n} > \frac{t_k - t_0}{t_n - t_0}$$

Note that every success function that exhibits IRS also exhibits UPC (see Lemma 13).

Theorem 7. *In the non-strategic model, an anonymous technology (t, c) has*

- *all n transitions if and only if it exhibits DRS.*
- *a single transition if and only if it exhibits UPC.*

The analysis of the agency case is more complex and involves less intuitive conditions.

Definition 5. *For an anonymous technology (t, c) let $Q_k = \frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$ be the total expected payment in the best contract for which there exist an equilibrium with k agents exerting effort.*

An anonymous technology[13] *(t, c) exhibits*

- *(strictly)* **over-payment (OP)** *if for any k, it holds that*
$$\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$$
- *(strictly)* **increasing relative marginal payment (IRMP)** *if for any k, it holds that*
$$\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$

Intuitively, the over-payment condition compares the proportional increase in success probability when moving from k agents to n agents ($\frac{t_k - t_0}{t_n - t_0}$), to the proportional expected payment to k agents with respect to the expected payment to n agents ($\frac{Q_k}{Q_n}$). If any set of k agents ($k \neq 0, k \neq n$) needs to be paid "over proportionally", the principal will contract either with 0 agents or with all n agents.

The IRMP condition looks at the proportional increase in payment ($Q_k - Q_{k-1}$) with respect to the increase in success probability ($t_k - t_{k-1}$), when increasing the number of contracting agents by one, from $k-1$ to k. The IRMP condition requires that any additional agent has a larger effect than its predecessor.

Notice that if we adjust Q_k to the non-strategic case, OP is equivalent to UPC, and IRMP is equivalent to DRS.

Theorem 8. *In the agency model, an anonymous technology (t, c) has*

- *all n transitions if and only if it exhibits IRMP.*
- *a single transition if and only if it exhibits OP.*

[11] A technology (t, c) has *all transitions* in the *weak* sense if the inequality holds only weakly.

[12] If the success function of the technology (t, c) exhibits some property, we will also say that the technology exhibits the same property.

[13] Note that these are conditions on the success function, independent of the identical cost c

Notice that, in general, none of the conditions in Theorem 7 imply any of the conditiond in Theorem , and vice versa. Also note that Theorems 7 and B.1 can both be derived from a more general analysis, which can be found in Appendix B.2.

While we are unable to determine the POU in general, for technologies that have a single transition, we are able to fully characterize the POU.

Lemma 2 *For any anonymous technology that exhibits both UPC and OP, and $t_0 > 0$, the POU is*

$$POU = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$$

B.2 Phase Transitions: Proofs

The following characterization holds for both the agency and the non-strategic cases. Let Q_k be the total expected payment to all agents in the best (minimal payment) contract in which k agents exert effort (for the non-strategic case $Q_k = c \cdot k$ and for the agency case $Q_k = \frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$). Note that Q_k is only a function of the technology t.

Theorem 9. *An anonymous technology (t, c) has*

- *a single transition if and only if it exhibits Over-Payment, that is for any $k \in \{1, \ldots, n\}$ it holds that*
$$\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$$
- *all n transitions at different values if and only if it exhibits IRMP, that is for any $k \in \{1, 2, \ldots, n-1\}$ it holds that*
$$\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$

Proof. We begin by a lemma that characterize the two cases by some properties of the principal utilities, and than show that these properties are equivalent to the properties presented by the theorem.

We use the following corollary from Lemma 3.

Observation 3. *For any anonymous technology (t, c), assume that contracting with k_1 agents is optimal for v_1, and contracting with k_2 agents is optimal for v_2. If $v_1 > v_2$ then $k_1 \geq k_2$.*

Given an anonymous technology (t, c), let $u(k, v)$ be the utility at value v, when optimally contracting with k agents[14], and let $v_{i,j}$ be the value v in which the principal is indifferent between contracting with either i agents or with j agents (by the definition of $v_{i,j}$, $u(j, v_{i,j}) = u(i, v_{i,j})$. That is $t_i \cdot v_{i,j} - Q_i = t_j \cdot v_{i,j} - Q_j$, or equivalently $v_{i,j} = \frac{Q_j - Q_i}{t_j - t_i}$).

Lemma 9. *An anonymous technology (t, c) has*

1. *all n transitions at different values if and only if $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for all $k \in \{1, 2, \ldots, n-1\}$, and*
2. *a single transition (from 0 agents to n agents) if and only if $u(n, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$.*

Proof. First we show that a technology t has all n transitions at different values if and only if for all $k \in \{1, 2, \ldots, n-1\}$ it holds that $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$.

case if: Assume that $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for all $k \in \{1, 2, \ldots, n-1\}$. By Lemma 11 this condition is equivalently to the IRMP condition, that is $\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$ for any $k \in \{1, 2, \ldots, n-1\}$. As $v_{i,j} = \frac{Q_j - Q_i}{t_j - t_i}$, IRMP is equivalent to $v_{k,k+1} > v_{k-1,k}$ for any $k \in \{1, 2, \ldots, n-1\}$. We next show that at any value $v \in (v_{k-1,k}, v_{k,k+1})$ for some $k \in \{1, 2, \ldots, n-1\}$, contracting with k agents is optimal for the principal, and this is the only optimal contract for him. Together with the fact that 0 is optimal for value of 0, and n is optimal for values larger that $v_{n-1,n}$, we conclude that all transitions occur.

We first show by induction that at any value $v \in (v_{k-1,k}, v_{k,k+1})$ for some $k \in \{1, 2, \ldots, n-1\}$, contracting with k agents has higher utility than contracting with $j < k$ agents. Clearly the claim holds for $k = 0$. Assume that we have proven the claim of to $k-1$, which means that at $v_{k-1,k}$ contracting with $k-1$ agents has higher utility than contracting with $j < k-1$ agents. By definition of $v_{k-1,k}$, contracting with k agents is better than contracting with $k-1$ agents for any value larger than $v_{k-1,k}$, thus contracting with k agents is better than contracting with any $j < k$ agents.

A similar argument shows by induction that at any value $v \in (v_{k-1,k}, v_{k,k+1})$ for some $k \in \{1, 2, \ldots, n-1\}$, contracting with k agents has higher utility than contracting with $j > k$ agents. This is proven by starting from $k = n$ agents and going backwards. Combining the two claims we derive that contracting with k agents at a value $v \in (v_{k-1,k}, v_{k,k+1})$ achieves higher utility for the principal than contracting with any other number of agents.

case only if: Assume that t has all n transitions, and at different values. For all $k \in \{1, 2, \ldots, n-1\}$, by Observation 3 k is not optimal for $v < v_{k-1,k}$ and is an optimal contract at $v_{k-1,k}$, thus $u(k, v_{k-1,k}) \geq u(k+1, v_{k-1,k})$. If $u(k, v_{k-1,k}) = u(k+1, v_{k-1,k})$, then (again

[14] Note that in the strategic model, $u(k, v)$ denote the utility of the principal, and in the non-strategic model, the principal's utility coincides with the social welfare.

by the same Observation), k is not optimal for any $v > v_{k-1,k}$, contradicting the transitions in different values. We conclude that $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for all $k \in \{1, 2, \ldots, n-1\}$.

Next we show that a technology t has a single transition (from 0 agents to n agents) if and only if $u(n, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$.

case if: Assume that $u(n, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$. By Observation 3, since n is optimal contract at $v_{0,n}$, for any $v > v_{0,n}$, n is the only optimal contract. On the other hand, as 0 is optimal at $v_{0,n}$, by Observation 3, if $k > 0$ was optimal for any $v < v_{0,n}$ then 0 was not optimal for $v_{0,n}$. Thus for any $v < v_{0,n}$, 0 is the only optimal contract. As at $v_{0,n}$ the only optimal contracts are 0 and n, this imlies that any $k \in \{1, 2, \ldots, n-1\}$ is never optimal, thus t has a single transition.

case only if: Assume that t has a single transition from 0 to n at $v_{0,n}$. This implies that for any $v \leq v_{0,n}$, 0 is the optimal contract, thus $u(0, v) > u(k, v)$ for all $k \in \{1, 2, \ldots, n-1\}$. for any $v \geq v_{0,n}$, n is the optimal contract, thus $u(n, v) > u(k, v)$ for all $k \in \{1, 2, \ldots, n-1\}$. We conclude that at $v = v_{0,n}$, $u(n, v_{0,n}) = u(0, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$. □

Next we show that Condition 1 in Lemma 9 is equivalent to the Over-Payment condition.

Lemma 10. *$u(n, v_{0,n}) > u(k, v_{0,n})$ for any $k \in \{1, 2, \ldots, n-1\}$ if and only if $\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$ for any $k \in \{1, 2, \ldots, n-1\}$.*

Proof. For all $k \in \{1, 2, \ldots, n-1\}$

$$u(n, v_{0,n}) > u(k, v_{0,n}) \Leftrightarrow t_n \cdot v_{0,n} - Q_n > t_k \cdot v_{0,n} - Q_k$$
$$\Leftrightarrow (t_n - t_k) \cdot v_{0,n} > Q_n - Q_k$$

As $v_{0,n} = \frac{Q_n}{t_n - t_0}$, the above happens if and only if

$$(t_n - t_k) \cdot \frac{Q_n}{t_n - t_0} > Q_n - Q_k \quad \Leftrightarrow \quad \frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$$

which is what we wanted to prove. □

Next we show that Condition 2 in Lemma 9 is equivalent to the IRMP condition.

Lemma 11. *$u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for any $k \in \{1, 2, \ldots, n-1\}$ if and only if $\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$ for any $k \in \{1, 2, \ldots, n-1\}$.*

Proof. For all $k \in \{1, 2, \ldots, n-1\}$

$$u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$$
$$\Leftrightarrow \quad t_k \cdot v_{k-1,k} - Q_k > t_{k+1} \cdot v_{k-1,k} - Q_{k+1}$$

As $v_{k-1,k} = \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$, the above happens if and only if

$$Q_{k+1} - Q_k > (t_{k+1} - t_k) \cdot v_{k-1,k} = (t_{k+1} - t_k) \cdot \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$
$$\Leftrightarrow \quad \frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$

which is what we wanted to prove. □

the theorem is now a direct result from the claims above. □

We now turn to the non-strategic case and show that the Over-Payment and IRMP conditions are equivalent to DRS and UPC, respectively.

Observation 4. *For the non-strategic case, an anonymous technology (t, c):*

1. *exhibits Over-Payment if and only if it exhibits UPC.*
2. *exhibits IRMP if and only if it exhibits DRS.*

Proof. As for the non-strategic case the total expected payment to all agents in the best (minimal payment) contract in which k agents exert effort is $c \cdot k$, this means that $Q_k = c \cdot k$ for any k. Thus, for the non-strategic case $\frac{Q_k}{Q_n} = \frac{k}{n}$, which implies the first claim, and $Q_{k+1} - Q_k = Q_k - Q_{k-1} = c$ which implies the second claim. □

Observation 5. *For the agency case, an anonymous technology (t, c) it holds that $Q_k = \frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$. Thus technology (t, c)*

1. *has all n transitions if and only if for any $k \in \{1, \ldots, n\}$ it holds that $\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$ (exhibits Over-Payment), for Q_k as defined above.*
2. *has a single transition if and only if for any $k \in \{1, 2, \ldots, n-1\}$ it holds that $\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$ (exhibits IRMP) for Q_k as defined above.*

Proof. As for the agency case the total expected payment to all agents in the best (minimal payment) contract in which k agents exert effort is $\frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$, this observation is derived directly from Theorem 9. □

The following holds for the symmetric case (in γ and c). Let $AND(n, \gamma, c)$ be the AND technology with n symmetric agents, each with cost c. Fixing n and c, let $v(\gamma)$ be the transition value of the optimal contract from the 0 contract to the n contract, when we use the parameter γ. We next show that the function $v(\gamma)$ is a monotonic function of γ. This means that if the success probability in case that an agent exert effort increases, the principal will move to the n contract earlier.

Lemma 12. *For $AND(n,\gamma_1,c)$ and $AND(n,\gamma_2,c)$, where $\gamma_1 < \gamma_2$ it holds that $v(\gamma_1) < v(\gamma_2)$.*

Proof. Recall that at $v(\gamma)$ the utility of the agent is the same for the 0 and n contracts. Thus $v(\gamma) = \frac{c\cdot n\cdot t_n}{(t_n-t_{n-1})(t_n-t_0)}$. For AND technology $\frac{t_n}{t_n-t_{n-1}} = \frac{(1-\gamma)^n}{(1-\gamma)^n-\gamma\cdot(1-\gamma)^{n-1}} = \frac{1-\gamma}{1-2\gamma}$. For $0<\gamma<\frac{1}{2}$ this is a monotonic function of γ. Additionally, $\frac{1}{t_n-t_0} = \frac{1}{(1-\gamma)^n-\gamma^n}$ is also a monotonic function of γ, for $0<\gamma<\frac{1}{2}$. ☐

B.3 The Price Of Unaccountability

For an anonymous technology that exhibits both UPC and Over-Payment, we can analyze the price of unaccountability. An AND technology is an example of such a technology.

Lemma 2 *For any anonymous technology (t,c) that exhibits both UPC and Over-Payment and $t_0>0$, the POU is*

$$POU = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$$

Proof. By Theorem 7 and Theorem B.1 the technology has a single transition in both the strategic and non-strategic cases. Let v_a be value in which the phase transition occur in the agency (strategic) case, and let v_n be value in which the phase transition occur in the non-agency (non-strategic) case. The phase transition value is the value in which the principal is indifferent between contracting with 0 agents and contracting with n agents. Thus v_n solves the equation $v_n\cdot t_n - c\cdot n = v_n\cdot t_0$, so $v_n = \frac{c\cdot n}{t_n-t_0}$. Additionally, v_a solves the equation $t_n\cdot(v_a - \frac{c\cdot n}{t_n-t_{n-1}}) = v_a\cdot t_0$, so $v_a = \frac{c\cdot n}{t_n-t_0}\cdot\frac{t_n}{t_n-t_{n-1}} = v_a\cdot\frac{t_n}{t_n-t_{n-1}}$.

As we assumed that $t_0>0$ then $t_{n-1}>0$, thus $\frac{t_n}{t_n-t_{n-1}}>1$, and therefore $v_a>v_n$, that is, the phase transition in the strategic case occur in a larger value than in the non-strategic case. Up to v_n, in both cases no agent is contracted, thus the social welfare is the same and the POU is 1 for such values. At any value larger than v_a in both cases all n agents are contracted, thus the social welfare is the same and the POU is 1 for such values. The only difference occur in $[v_n,v_a]$ range, in which all agents are contracted in the non-strategic case, while non is contracted in the strategic case. In that range, the social welfare ratio for a given value v is $\frac{v\cdot t_n - c\cdot n}{v\cdot t_0} = \frac{t_n}{t_0} - \frac{1}{v}\cdot\frac{c\cdot n}{t_0}$, which is a monotonically increasing function of the value v. Thus the ratio is maximized at the highest value of the range, which is v_a.

Thus, the POU is $POU = \frac{v_a\cdot t_n - c\cdot n}{v_a\cdot t_0} = \frac{t_n}{t_0} - \frac{c\cdot n}{v_a\cdot t_0}$. As $\frac{c\cdot n}{v_a\cdot t_0} = \frac{c\cdot n}{\frac{c\cdot n}{t_n-t_0}\cdot\frac{t_n}{t_n-t_{n-1}}\cdot t_0} = \left(\frac{t_n}{t_0}-1\right)\cdot\left(1-\frac{t_{n-1}}{t_n}\right)$, we derive that

$$POU = \frac{t_n}{t_0} - \frac{c\cdot n}{v_a\cdot t_0} = \frac{t_n}{t_0} - \left(\frac{t_n}{t_0}-1\right)\cdot\left(1-\frac{t_{n-1}}{t_n}\right) = \frac{t_{n-1}}{t_n}\cdot\frac{t_n}{t_0} + 1 - \frac{t_{n-1}}{t_n} = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$$

☐

Lemma 13. *A technology function that exhibits IRS also exhibits UPC.*

Proof. Assume in contradiction that the technology exhibits IRS and not UPC. Then there is a $k\in\{1,\dots,n\}$ s.t. $\frac{k}{n}\le\frac{t_k-t_0}{t_n-t_0}$. As the technology exhibits IRS, it holds that for any $i\in\{2,\dots,n\}$, $t_i-t_{i-1} > t_{i-1}-t_{i-2}$, thus for any $i\in\{1,\dots k-1\}$ it holds that $t_k-t_{k-1} > t_i-t_{i-1}$. Therefore by summation

$$(k-1)(t_k-t_{k-1}) = \sum_{i=1}^{k-1}(t_k-t_{k-1}) > \sum_{i=1}^{k-1}(t_i-t_{i-1}) = t_{k-1}-t_0$$

Equivalently $t_k-t_{k-1} > \frac{t_k-t_0}{k}$. As for k we assumed that $\frac{k}{n}\le\frac{t_k-t_0}{t_n-t_0}$, it implies that $t_k-t_{k-1} > \frac{t_n-t_0}{n}$. As for any $i\in\{(k+1),\dots,n\}$, by IRS $t_k-t_{k-1} < t_i-t_{i-1}$,

$$(n-k)(t_k-t_{k-1}) = \sum_{i=k+1}^{n}(t_k-t_{k-1}) < \sum_{i=k+1}^{n}(t_i-t_{i-1}) = t_n-t_k$$

which implies that $t_n-t_k > \frac{n-k}{n}(t_n-t_0)$. With the assumption that $\frac{k}{n}(t_n-t_0)\le t_k-t_0$ we observe that $t_n-t_0 = (t_n-t_k)+(t_k-t_0) > t_n-t_0$, a contradiction. ☐

Lemma 14. *AND technology exhibits both UPC and Over-Payment, thus has a single phase transition in both the strategic and non-strategic cases.*

Proof. First we observe that AND exhibits IRS thus exhibits UPC (Lemma 13). Next we show that it exhibits Over-Payment. As for any k, $\frac{t_k}{t_k-t_{k-1}} = \frac{\gamma}{1-\gamma}$, it implies that $\frac{Q_k}{Q_n} = \frac{\left(\frac{c\cdot k\cdot t_k}{t_k-t_{k-1}}\right)}{\left(\frac{c\cdot n\cdot t_n}{t_n-t_{n-1}}\right)} = \frac{k}{n}$. Thus, Over-Payment is equivalent to UPC for AND technology. ☐

Corollary 3. *The price of unaccountability for AND technology for any c and $\gamma>0$ is obtained for the phase transition value of the strategic case, and is*

$$POU = \left(\frac{1}{\gamma}-1\right)^{n-1} + 1 - \frac{\gamma}{1-\gamma}$$

Proof. By the above Lemma, AND technology exhibits both UPC and Over-Payment. Now, the proof is a direct result of Lemma 2 and the fact that for AND technology with $\gamma > 0$, $t_{n-1} > 0$ for any n and $\frac{t_{n-1}}{t_0} = \frac{(1-\gamma)^{n-1}\cdot\gamma}{\gamma^n} = \left(\frac{1}{\gamma} - 1\right)^{n-1}$ and $\frac{t_{n-1}}{t_n} = \frac{(1-\gamma)^{n-1}\cdot\gamma}{(1-\gamma)^n} = \frac{\gamma}{1-\gamma}$.

□

Lemma 15. *For an anonymous OR technology with two agents and parameter γ, the POU is obtained at the transition point of the agency case from 0 to 1 agent, and is*

$$\forall\gamma \leq 0.282.., POU = \frac{3\gamma^2 - 4\gamma + 2}{1 - \gamma + \gamma^2}$$

$$\forall\gamma \geq 0.282.., POU = \frac{(-1+\gamma)(-1+10\gamma-16\gamma^2+9\gamma^3)}{(1-\gamma+\gamma^2)\gamma(-2+\gamma)}$$

Proof. Let v_{ij}^A and v_{ij}^{NS} denote the transition point from investing in i to j agents in the agency and non-strategic cases, respectively. Since in the OR technologies all the transitions exist (see Theorem 2), the transition points with two agents are v_{01}^{NS}, v_{12}^{NS}, v_{01}^A, and v_{12}^A.

It is easy to verify that $v_{01}^{NS} < v_{12}^{NS} < v_{12}^A$, and that $v_{01}^{NS} < v_{01}^A < v_{12}^A$. Thus, there are two possible orders on the transition points:

1. $v_{01}^{NS} < v_{12}^{NS} < v_{01}^A < v_{12}^A$
2. $v_{01}^{NS} < v_{01}^A < v_{12}^{NS} < v_{12}^A$

The order of the transitions depends on the relation between v_{01}^A and v_{12}^{NS}.

$$v_{12}^{NS} > v_{01}^A$$

$$\Updownarrow$$

$$-\frac{-1+5\gamma-6\gamma^2+3\gamma^3}{(-1+2\gamma)^2(-1+\gamma)^2\gamma} > 0$$

$$\Updownarrow$$

$$\gamma < 0.282..$$

We denote the critical γ by $\gamma_c = 0.282...$ Solving for the transition points, we get:

$$v_{01}^A = \frac{-\gamma+1+\gamma^2}{(-1+\gamma)^2(-1+2\gamma)^2}$$

$$v_{12}^A = -\frac{2-3\gamma-\gamma^2+\gamma^3}{(-1+\gamma)(-1+2\gamma)^2\gamma^2}$$

$$v_{01}^{NS} = \frac{1}{(-1+\gamma)(-1+2\gamma)}$$

$$v_{12}^{NS} = -\frac{1}{\gamma(-1+2\gamma)}$$

Suppose k_1 agents are contracted in the non-strategic case and k_2 agents are contracted in the agency case. The ratio between the social welfare in the two cases is $\frac{w_{k_1}}{w_{k_2}} = \frac{t_{k_1}v - k_1 c}{t_{k_2}v - k_2 c}$. By Lemma 1, the POU is obtained at a transition point of either the agency or the non-agency case. In addition, $\frac{w_0}{w_1}$, $\frac{w_1}{w_2}$, and $\frac{w_0}{w_2}$ are all monotonically increasing in v. Therefore, for any a, b, the maximal ratio for $v \in [a, b]$ is obtained at b.

In case (1), where $\gamma \geq \gamma_c$, there are three possible transition points, r_1, r_2, r_3 as follows:

$$\begin{aligned} r_1 &= \frac{w_0}{w_2}(v = v_{01}^A) \\ &= \frac{(-1+\gamma)(-1+10\gamma-16\gamma^2+9\gamma^3)}{(1-\gamma+\gamma^2)\gamma(-2+\gamma)} \\ r_2 &= \frac{w_0}{w_1}(v = v_{12}^{NS}) = \frac{1-2\gamma+3\gamma^2}{\gamma(2-\gamma)} \\ r_3 &= \frac{w_1}{w_2}(v = v_{12}^A) \\ &= \frac{(-1+\gamma)(-2+\gamma+6\gamma^2-8\gamma^3+7\gamma^4)}{2-5\gamma+3\gamma^2+4\gamma^3-10\gamma^4+5\gamma^5} \end{aligned}$$

and we get:

$$POU = \max\{r_1, r_2, r_3\} = \frac{1-2\gamma+3\gamma^2}{\gamma(2-\gamma)}$$

In case (2), where $\gamma \leq \gamma_c$, there are only two transition points, and we get:

$$\begin{aligned} POU &= \max\{\frac{w_0}{w_1}(v = v_{01}^A), \frac{w_1}{w_2}(v = v_{12}^A)\} \\ &= \max\{\frac{2-4\gamma+3\gamma^2}{1-\gamma+\gamma^2}, \\ &\quad \frac{(-1+\gamma)(-2+\gamma+6\gamma^2-8\gamma^3+7\gamma^4)}{2-5\gamma+3\gamma^2+4\gamma^3-10\gamma^4+5\gamma^5}\} \\ &= \frac{2-4\gamma+3\gamma^2}{1-\gamma+\gamma^2} \end{aligned}$$

□

C Non Anonymous Technologies

C.1 Orbit Characterization

Let g and h be two Boolean functions on disjoint inputs with any cost vectors, and let $f = h \bigwedge g$ (i.e., take their networks in series). An optimal contract for f for some v, denoted by S, is composed of some agents from the h-part and some from the g-part, call them T and R

respectively.

Lemma 5 *Let S be an optimal contract for $f = g \bigwedge h$ on v. Then, T is an optimal contract for h on $v \cdot t_g(R)$, and R is an optimal contract for g on $v \cdot t_h(T)$.*

Proof. We will abuse the notation and use the functions to denote the technology as well ($f(S)$ will denote the probability of success with the function f and the contract S).

The utility of the principal with value v from S when using technology f is

$$U(S,v) = f(S)\left(v - \sum_{i \in S} \frac{c_i}{\Delta_i^f(S \setminus i)}\right)$$

$$= h(T) \cdot g(R) \cdot \left(v - \sum_{i \in T} \frac{c_i}{\Delta_i^f(S \setminus i)} + \sum_{i \in R} \frac{c_i}{\Delta_i^f(S \setminus i)}\right)$$

For any $i \in T$, $\Delta_i^f(S \setminus i) = h(1, T \setminus i) \cdot g(R) - h(0, T \setminus i) \cdot g(R) = g(R) \cdot \Delta_i^h(T \setminus i)$. Similarly, for any $i \in R$, $\Delta_i^f(S \setminus i) = h(T) \cdot \Delta_i^g(R \setminus i)$.

We derive that

$$U(S,v) = h(T) \cdot \left(g(R) \cdot v - \sum_{i \in T} \frac{g(R) \cdot c_i}{g(R) \cdot \Delta_i^h(T \setminus i)}\right) + h(T) \cdot g(R) \cdot \sum_{i \in R} \frac{c_i}{h(T) \cdot \Delta_i^g(R \setminus i)} =$$

$$h(T)\left(g(R) \cdot v - \sum_{i \in T} \frac{c_i}{\Delta_i^h(T \setminus i)}\right) + g(R) \cdot \sum_{i \in R} \frac{c_i}{\Delta_i^g(R \setminus i)}$$

the first term is maximized exactly at a set T that is optimal for h on the value $g(R) \cdot v$, while the second term is independent of T and h. We conclude that S is an optimal contract for f on v if and only if T is an optimal contract for h on $v \cdot t_g(R)$. The proof that R is an optimal contract for g on $v \cdot t_h(T)$ is similar and is omitted. □

Lemma 16. *The real function $v \to t_h(T)$, where T is the $h-part$ of an optimal contract for f on v, is monotone non-decreasing (and similarly for the function $v \to t_g(R)$).*

Proof. Let $S_1 = T_1 \cup R_1$ be the optimal contract for f on v_1, and let $S_2 = T_2 \cup R_2$ be the optimal contract for f on $v_2 < v_1$. Assume in contradiction that $h(T_1) < h(T_2)$. By Lemma 5, R_1 is optimal for g on $v \cdot h(T_1)$, and R_2 is optimal for g on $v \cdot h(T_2)$. Since $h(T_1) < h(T_2)$, R_2 is optimal for g on a larger value then R_2, thus by Lemma 3, $g(R_2) \geq g(R_1)$. As $f = g \cdot h$ we conclude that $f(S_2) = h(T_2) \cdot g(R_2) > h(T_1) \cdot g(R_1) = f(S_1)$, contracting Lemma 3 that shows that the function $v \to f(S)$ is monotone non-decreasing, thus $f(S_1) \geq f(S_2)$. □

Observation 6. *The k-orbit of any technology with symmetric cost c in the non-strategic case is of size at most one. Thus, the orbit of any such technology (with non anonymous success function and symmetric cost c) in the non-strategic case is of size at most $n + 1$.*

Proof. At most one set of size k, one with the maximal probability, can be on the orbit. □

By the above observation, in the non-strategic case any k-orbit is very small (of size at most 1). We show that in the strategic case, this is far from being the case.

Theorem 4 *Every admissible collection of k size sets can be obtained as the $k-orbit$ of some t.*

Proof. Let $\mathcal{S}$ be some admissible collection of k size sets. For each set $S \in \mathcal{S}$ in the collection we pick $\epsilon_S \in (0.17, 0.2]$, such that for any two admissible sets $S_i \neq S_j$, $\epsilon_{S_i} \neq \epsilon_{S_j}$, and for some $S \in \mathcal{S}$, $\epsilon_S = 0.2$. We also pick a smaller $\epsilon > 0$.

Next, we define the technology t:

- For any set T such that $T = S \in \mathcal{S}$ let $t(S) = 1/2 - \epsilon_S$.
- For any set $S \in \mathcal{S}$ and every $i \in S$, $t(S \setminus i) = 1/2 - 2\epsilon_S$.
- Let $\mathcal{Z}$ be the family of all sets for which the above defined t for ($\mathcal{Z} = \mathcal{S} \cup \bigcup_{S \in \mathcal{S}, i \in S}\{S \setminus i\}$). For any set T that is not in $\mathcal{Z}$, we define: $t(T) = max_{Z: T \subset Z, Z \in \mathcal{Z}}(t(Z) + (|T| - |Z|) \cdot \epsilon)$ (if there is not $Z \in \mathcal{Z}$ such that $T \subset Z$, then $t(T) = \epsilon \cdot |T|$).

Note that t is well defined, as $\mathcal{S}$ is an admissible set, thus for each set T, $t(T)$ was only defined once (that is, for any two sets $S, S' \in \mathcal{S}$ and any two agents i, j, $S \setminus i \neq S' \setminus j$).

We show that each admissible set S is the optimal at the value $v_S = \frac{c \cdot k}{2\epsilon_S{}^2}$. We first show that it is better than contract with a different $S' \in \mathcal{S}$, and than we show that it is better than any contract for other sets (in particular $k - 1$ size sets).

The utility of the principal from a contract with a set $S \in \mathcal{S}$ is $u(S, v) = t(S) \cdot (v - \sum_{i \in S} \frac{c}{t(S) - t(S \setminus i)}) = (\frac{1}{2} - \epsilon_S)(v - \frac{c \cdot k}{\epsilon_S}) = \frac{v}{2} + k \cdot c - (\frac{c \cdot k}{2\epsilon_S} + v \cdot \epsilon_S)$. The utility is maximized when $(\frac{c \cdot k}{2\epsilon_S} + v \cdot \epsilon_S)$ is minimized, which happens when $\frac{c \cdot k}{2\epsilon_S} = v \cdot \epsilon_S$, or when $v = \frac{c \cdot k}{2\epsilon_S{}^2}$. We denote by v_S the value for which ϵ_S maximizes the utility of the principal ($v_S = \frac{c \cdot k}{2\epsilon_S{}^2}$). Note that at v_S, any set $S' \neq S$ has a lower utility for the principal.

Additionally, note that for any admissible set S, and any set T of size $k - 1$, $t(S) > t(T)$. This is true as $t(S) = \frac{1}{2} - \epsilon_S > 0.3$, while even if $T = S' \setminus i$ for some S', i, then $t(T) = \frac{1}{2} - 2\epsilon_{S'} < 0.16$. This implies that in order to show that any $S \in \mathcal{S}$ has higher utility than any T of size $k-1$ at the value v_S, it is sufficient to show that this hold for the smallest v_S, which is achieved for the largest ϵ_S, which is 0.2 by our construction. We denote the largest ϵ_S by $\bar{\epsilon} = 0.2 = 1/5$, its corresponding set by $\bar{S}$, and its corresponding value by $\bar{v} = \frac{k \cdot c}{2 \cdot \bar{\epsilon}^2}$.

The utility of the principal from a set $T = S \setminus i$ for some admissible set S and agent i, at the value $\bar{v}$ is $u(T, \bar{v}) = (\frac{1}{2} - 2\epsilon_S) \cdot (\bar{v} - \frac{(k-1)\cdot c}{(\frac{1}{2} - 2\epsilon_S) - (k-2)\epsilon})$. As we can take ϵ to be as small as we like, we can neglect the $(k-2)\epsilon$ term. Thus, $u(T, \bar{v}) = \frac{\bar{v}}{2} - 2\epsilon_S \cdot \bar{v} - (k-1) \cdot c$.

We need to verify that $u(T, \bar{v}) = \frac{\bar{v}}{2} - 2\epsilon_S \cdot \bar{v} - (k-1) \cdot c < \frac{\bar{v}}{2} + k \cdot c - \frac{c \cdot k}{\bar{\epsilon}} = u(\bar{S}, \bar{v})$. As $\bar{v} = \frac{k \cdot c}{2 \cdot \bar{\epsilon}^2}$, this is equivalent to $c(1 - 2k + \frac{k}{\bar{\epsilon}}) < 2\epsilon_S \cdot \frac{k \cdot c}{2 \cdot \bar{\epsilon}^2}$. Equivalently, $\epsilon_S > \bar{\epsilon} - \bar{\epsilon}^2 \cdot (2 - \frac{1}{k})$. As $k = 1$ maximizes the right side, we need to verify that $\epsilon_S > \bar{\epsilon} - \bar{\epsilon}^2 = 1/5 - 1/25 = 4/25 = 0.16$, and this holds as $\epsilon_S > 0.17$.

Finally, note that for any other contract, there is at least one agent that is paid $\frac{1}{\epsilon}$. As we can choose ϵ arbitrarily small, we can make sure that at any $v_S = \frac{c \cdot k}{2 \epsilon_S{}^2}$ (recall that $\epsilon_S > 0.17$) the payment to any one agent is at least as the value v_S. This implies that for any v_S, the optimal contract is not any set other that $S \in \mathcal{S}$ or $T = S \setminus i$ for some $S \in \mathcal{S}$ and agent $i \in S$.

We conclude that at any v_S the optimal contract is the admissible set $S \in \mathcal{S}$. □

Lemma 4 *For any $n \geq k$, there exists an admissible collection of k-size sets of size $\Omega(\frac{1}{n} \cdot \binom{n}{k})$.*

Proof. Take an error correcting code that corrects a single error. Since the distance between any two code words is at least 3, the set of code words is an admissible collection. Now, it is known that there exist such codes with $\Omega(2^n/n)$ code words. The only thing left is to show that an appropriate fraction of these code words have weight k. This can be easily achieved by renaming at random for each coordinate 0 and 1. I.e. choose a random n-bit vector r and Xor each codeword (bitwise) with r. Now each single codeword is uniformly mapped to the whole cube, and thus its probability of having weight k (after the Xor-ing) is exactly $\binom{n}{k}/2^n$. Thus the expected number of weight k words (in the code after Xor-ing) is the product of this probability and the number of code words, which gives $\Omega(\binom{n}{k}/n)$, and for some r this expectation is achieved or exceeded. □

Corollary 1 *There exists a technology (t, c) with orbit of size with orbit of size $\Omega(\frac{2^n}{n\sqrt{n}})$.*

Proof. By Stirling approximation $\binom{k}{2k} = \frac{2^{2k}}{\sqrt{\pi k}}(1 + O(1/k))$, thus for n=2k we derive that there exists an orbit of size $\Omega\left(\binom{n}{n/2} \cdot \frac{1}{n}\right) = \Omega\left(\frac{2^n}{\sqrt{\pi \cdot n/2}} \cdot \frac{1}{n}\right) = \Omega(\frac{2^n}{n\sqrt{n}})$ □

Lemma 6 *Let g and h be two Boolean functions on disjoint inputs and let $f = g \bigwedge h$ (i.e., take their networks in series). Suppose x and y are the respective orbit sizes of g and h; then, the orbit size of f is less or equal to $x + y - 1$.*

Proof. By Lemma 5 an optimal contract for f is constructed from optimal contracts for h and g. By Lemma 16 the orbit of h consists of sets $T_1, T_2, \ldots, T_y$ with increasing success probabilities (because of consistent tie breaking). Similarly, the orbit of g consists of sets $R_1, R_2, \ldots, R_x$ with increasing success probabilities.

The orbit of f consists of contracts of the form $T_i \cup R_j$. If we order the orbit of f by increasing success probabilities: $S_1, S_2, \ldots S_z$, where $S_l = T_{i(l)} \cup R_{j(l)}$, then By Lemma 16 both $i(l)$ and $j(l)$ are monotonically non decreasing, and at least one of them must increase when we move from l to $l+1$. As for any l , $x \geq i(l) \geq 1$ and $y \geq j(l) \geq 1$, the orbit size of f is of size at most $x + y - 1$. □

Corollary 2 *Assume that $\{(g_j, c_j)\}_{j=1}^m$ is a set of anonymous technologies on disjoint inputs, each with identical agent cost (all agents of technology g_j has the same cost c_j). Than the orbit of $f = \bigwedge_{j=1}^m g_j$ is of size at most $(\sum_{j=1}^m n_j) - 1$, where n_j is the number of agents in technology g_j (the orbit is linear in the number of agents).*

Proof. The size of the orbit of the technology (g_j, c_j) is at most $n_j + 1$ (as it is anonymous with identical costs, see Observation 6). By induction we get that he size of f is at most $\sum_{j=1}^m (n_j + 1) - (m-1) = (\sum_{j=1}^m n_j) - 1$. □

C.2 AOO and OOA Technologies

Theorem 3 *Let h be an anonymous OR technology, and let $f = \bigwedge_{j=1}^{n_c} h$ be the AOO technology that is obtained by a conjunction of n_c of these OR-components on disjoint inputs. Then for any value v, an optimal contract contracts with the same number of agents in each OR-component. Thus, the orbit of f is of size at most $n_l + 1$, where n_l is the number of agents in h.*

Proof. We prove that for any *AOO* technology with n_c OR-components, each with n_l symmetric agents, any optimal contract has equal number of agents contracted in each OR-component (for any v, c, γ, and any n_c, n_l).

We prove the claim by induction on n_c. The base case of $n_c = 2$ is proven first. Let h denote the OR technology for a single component (by symmetry h is the same for all components).

Claim 2. *For any AOO technology with two OR-components, each with n_l symmetric agents, any optimal contract has the same number of agents contracted in each OR-component.*

Proof. Assume that for some v the optimal contract has k_1 agents in the first OR-component, and k_2 in the second OR-component. Assume in contradiction (wlog) that $k_1 > k_2$, this $h(k_1) > h(k_2)$. By Lemma 5, k_1 is optimal for h on $v \cdot h(k_2)$, and k_2 is optimal for h on $v \cdot h(k_1) > v \cdot h(k_2)$. This contradicts Observation 3 which shows that if k_2 is optimal for a larger value than k_1, than $k_2 \geq k_1$. □

By induction, assume that for any number of OR-components that is smaller than n_c ($n_c > 2$), the optimal contract has the same number of agents in each component. We show that in the optimal contract has the same number of agents in each component if there are n_c components. Assume that in the optimal contract has k_1 agents on the first OR-component. Let g be the conjunction of the rest $n_c - 1$ components. By Lemma 5, the contract on g is an optimal contract at the value $v \cdot h(k_1)$, thus by our induction hypothesis has the same number of agents k_2 contracted at each OR-component. To conclude the proof we need to show that $k_1 = k_2$.

Let h_2 be the conjunction of the first two OR-components. Again by Lemma 5 the contract on h_2 is an optimal contract for some value, and by the induction hypothesis has the same number of agents contracted in each of the two components, k_3. Since in the first component k_1 agents are contracted then $k_1 = k_3$. Since in the second component k_2 agents are contracted then $k_2 = k_3$. Thus $k_1 = k_2$, and we conclude the proof. □

Corollary 4. *The orbit of AND technology with any n, any $\{\gamma_i\}_{i \in N}$ and any cost vector c is of size at most $n + 1$ and can be calculated in polynomial time.*

Proof. We can look at each agent as a technology (an "OR" technology over a single agent), and the AND technology is just and "AND" over these n technologies. Each of these n technologies has two optimal contracts (0 and 1), and clearly the transition point between these contracts can be calculated in constant time. We apply Lemma 6 to conclude by induction that the orbit of the AND technology is of size at most $2n - (n-1) = n+1$. Any AND technology is a special case of the family of $AND - of - OR$ technologies of Lemma 7, thus we can calculate the orbit in polynomial time. □

Theorem 10. *In the OOA technology with two parallel paths of length two, for any values of c and γ, there exist values $v_1 < v_2$ in the optimal contract such that:*

- *for any $v \leq v_1$, no agent is contracted.*
- *for any $v \in [v_1, v_2]$, two agents on the same path are contracted.*
- *for any $v \geq v_2$, all four agents are contracted.*

Proof. Let $t(k_1, k_2)$ be the probability of success when k_1 agents are contracted on the first path and k_2 on the second path. Similarly, let $p(k_1, k_2)$ and $u(k_1, k_2)$ be the respective total payments and principal's utility under these profiles. Since u_{k_1,k_2} is a linear function of v with a slope of t_{k_1,k_2}, the functions $u(k_1, k_2)$ and $u(k_1', k_2')$ cannot intersect in more than a single point.

We will show that the profiles $(k_1 = 0, k_2 = 1)$, $(k_1 = 1, k_2 = 1)$, and $(k_1 = 2, k_2 = 1)$ cannot be optimal:

1. For any γ, $t(2,0) > t(1,1)$ and $p(2,0) < p(1,1)$. Therefore, $u(1,1) < u(2,0)$.
2. Let v_1 denote the intersection point of $u(0,0)$ and $u(0,2)$. It is easy to verify that:

$$v_1 = \frac{2(-1 + 2\gamma - 2\gamma^3 - \gamma^2 + \gamma^4)}{(\gamma^2 - 1)(\gamma - 1)(2\gamma - 1)(-2\gamma + 2\gamma^3 + 1 - \gamma^2)}$$

Calculating the gap between $u(0,0)$ and $u(0,1)$ at v_1, we get:

$$(u(0,0) - u(0,1))_{v=v_1} = \frac{-\gamma^3 + 3\gamma^2 - 2\gamma + 1}{(\gamma - 1)^2} > 0$$

$$\Downarrow$$

$$\forall v \leq v_1, \quad u(0,0) > u(0,1)$$

and:

$$\forall v \geq v_1, \quad u(0,2) > u(0,1)$$

$$\Downarrow$$

$$u(0,1) < \max\{u(0,0), u(0,2)\}$$

Let v_2 denote the intersection point of $u(2,0)$ and $u(2,2)$. It is easy to verify that:

$$v_2 = \frac{2(-2\gamma - 5\gamma^2 + 4\gamma^3 + 3\gamma^4 - 4\gamma^5 + \gamma^6 + 2)}{(\gamma^2 - 1)(\gamma - 1)(2\gamma - 1)\gamma^2(\gamma - 2)(2 + 2\gamma^2 - 5\gamma)}$$

Calculating the gap between $u(2,2)$ and $u(2,1)$ at v_2, we get:

$$(u(2,2) - u(2,1))_{v=v_2} = \frac{\gamma^8 - 4\gamma^7 + 4\gamma^6 - 3\gamma^4 - \gamma^3 + \gamma^2 + \gamma - 1}{\gamma^2(2-\gamma)(1 - \gamma + \gamma^2)(-1 + \gamma^2)}$$

$$\Downarrow$$

$$\forall v \leq v_2, u(2,0) > u(2,1) \text{ and: }$$
$$\forall v \geq v_2, u(2,2) > u(2,1)$$

$$\Downarrow$$

$$u(2,1) < \max\{u(0,2), u(2,2)\}$$

.

From the above claims, we get that for $v \leq v_1$, the profile $(0,0)$ is optimal, for $v_1 \leq v \leq v_2$, the profile $(2,0)$ is optimal, and for $v \geq v_2$, the profile $(2,2)$ is optimal. To complete the proof, we have to show that $v_1 < v_2$. Indeed,
$v_i - v_2 = \frac{2(\gamma^6-4\gamma^5+3\gamma^4+4\gamma^3-7\gamma^2+2\gamma+2)}{(2+2\gamma^2-5\gamma)(2-\gamma)\gamma^2(-1+\gamma^2)^2(-1+\gamma)} < 0$. □

D Algorithmic Aspects

Proposition 1 *Given as input the full description of a technology (the values $t_0, \ldots, t_n$ and the identical cost c for an anonymous technology, or the value $t(S)$ for all the 2^n possible subsets $S \subseteq N$ of the players, and a vector of costs $\vec{c}$ for non-anonymous technologies), the following can all be computed in polynomial time:*

- *The orbit of the technology in both the agency and the non-strategic cases.*
- *An optimal contract for any given value v, for both the agency and the non-strategic cases.*
- *The price of unaccountability $POU(t)$.*

Proof. We prove the claims for the non-anonymous case, the proof for the anonymous case is similar.

We first show how to construct the orbit of the technology (the same procedure apply in both cases). To construct the orbit we find all transition points and the sets that are on the orbit. The empty contract is always optimal for $v = 0$. Assume that we have calculated the optimal contracts and the transition points up to some transition point v for which S is an optimal contract with the higher success probability. We show how to calculate the next transition point and the next optimal contract.

By Lemma 3 the next contract on the orbit (for higher values) has a higher success probability (there are no two sets with the same success probability on the orbit). We calculate the next optimal contract by the following procedure. We go over all sets T such that $t(T) > t(S)$, and calculate the value for which the principal is indifferent between contracting with T and contracting with S. The minimal indifference value is the next transition point and the contract that has the minimal indifference value is the next optimal contract. Linearity of the utility in the value and monotonicity of the success probability of the optimal contracts ensure that the above works. Clearly the above calculation is polynomial in the input size.

Once we have the orbit, it is clear that an optimal contract for any given value v can be calculated. We find the largest transition point that is not larger than the value v, and the optimal contract at v is the set with the higher success probability at this transition point.

Finally, as we can calculate the orbit of the technology in both the agency and the non-strategic cases in polynomial time, we can find the price of unaccountability in polynomial time. By Lemma 1 the price of unaccountability $POU(t)$ is obtained at some transition point, so we only need to go over all transition points, and find the one with the maximal social welfare ratio. □

A more interesting question is whether if given the function t as a black box, we can compute the optimal contract in time that is polynomial in n. We can show that, in general this is not the case:

Theorem 5 *Given as input a black box for a success function t (when the costs are identical), and a value v, the number of queries that is needed, in the worst case, to find the optimal contract is exponential in n.*

Proof. Consider the construction of an orbit for an admissible collection of k size sets of Theorem 4. We can view the constructed t as encoding all, exponentially many, values of epsilon used in the construction. Any algorithm that finds the optimal contract, can be directly used to answer queries of the form "is a given ϵ inside the set encoded by t?" by simply finding the optimal contract for the v whose optimal contract is given by that ϵ_S (the expression for v is given in the construction). A lower bound for this problem is quite trivial as an adversary will just keep providing values for the queried locations of t that reflect arbitrary other ϵ_S's, keeping the desired ϵ_S for the last position. □

Theorem 6 *The Optimal Contract Problem for Read Once Networks is #P-complete (under Turing reductions).*

Proof. We will show an algorithm for this problem can be used to compute $t(E)$. This will be done as follows: first define a new graph G' which is obtained by "And"ing G with a new player x, with γ_x very close to $\frac{1}{2}$. By choosing γ_x close enough to $\frac{1}{2}$, we can make sure that player x will enter the optimal contract only for very large values of v, after all other agents are contracted (if we can find the optimal contract for any value, it is easy to find a value for which in the original network the optimal contract is E, by keep doubling the value and asking for the optimal contract. Once we find such a value, we choose γ_x s.t. $\frac{c}{1-2\gamma_x}$ is larger than that value). Let us denote $\beta_x = 1 - 2\gamma_x$.

The critical value of v where player x enters the optimal contract of G', can be found using binary search over the algorithm that supposedly finds the optimal contract for any network and any value. Note that at this critical value v, the principal is indifferent between the set E and $E \cup \{x\}$. Now when we write the expression for this indifference, in terms of $t(E)$ and $\Delta_i^t(E)$, we observe the following.

$$t(E) \cdot \gamma_x \cdot \left(v - \sum_{i \in E} \frac{c}{\gamma_x \cdot \Delta_i^t(E \setminus i)} \right) =$$

$$t(E)(1-\gamma_x) \left(v - \sum_{i \in E} \frac{c}{(1-\gamma_x) \cdot \Delta_i^t(E \setminus i)} - \frac{c}{t(E) \cdot \beta_x} \right)$$

$$\Updownarrow$$

$$t(E) \cdot \gamma_x \cdot v - t(E) \cdot \sum_{i \in E} \frac{c}{\Delta_i^t(E \setminus i)} =$$

$$t(E) \cdot (1-\gamma_x) \cdot v - t(E) \cdot \sum_{i \in E} \frac{c}{\Delta_i^t(E \setminus i)} - \frac{(1-\gamma_x) \cdot c}{\beta_x}$$

$$\Updownarrow$$

$$t(E) \cdot \gamma_x \cdot v = t(E) \cdot (1-\gamma_x) \cdot v - \frac{(1-\gamma_x) \cdot c}{\beta_x}$$

$$\Updownarrow$$

$$t(E) = \frac{(1-\gamma_x) \cdot c}{(\beta_x)^2 \cdot v}$$

thus, if we can find the optimal contract we are also able to compute the value of $t(E)$. □

Lemma 7 *Given a Read Once $AND - of - OR$ network such that each OR-component is an anonymous technology (has the same γ and c for all agents), the optimal contract problem can be solved in polynomial time.*

Proof. Let us assume that there are n_l OR-components and that the c OR-component (for $c \in \{1, \ldots, n_l\}$) has n_c agents. We will calculate all the transition points and the optimal contract at each interval between any two consecutive points. We begin by calculating the transition points and the optimal contract at each interval for each OR-component separately, this can be done in polynomial time by Proposition 1.

Next we start to combine each OR-component with all previous ones, starting from combining the second component with the first. Let h_k be the AND technology of the first k OR-components. Assume by inductions that its orbit was calculated in polynomial time and by Lemma 6 its size $r(k)$ is at most $\sum_{c=1}^{k} n_c - (k-1)$. Let X be the set of optimal contracts for h_k. We sort the contracts by increasing success probabilities, thus $X = (X_1, X_2, \ldots, X_{r(k)})$ where $h_k(X_i) < h_k(X_{i+1})$. Let o_{k+1} denote the (OR) technology of the $k+1$ OR-component, let $Y = (Y_1, Y_2, \ldots, Y_y)$ be the set of optimal contracts for o_{k+1} sorted by increasing probabilities. Note that $y \leq n_{k+1} + 1$.

Using Lemma 5 we show how to combine the $k+1$ OR-component with h_k. By the Lemma, the only candidates for optimal contracts of $h_{k+1} = h_k \bigwedge o_{k+1}$, are $(X_i, Y_j) \in (X, Y)$. We denote the indexes of the $l - th$ optimal contract for $h(k+1)$ by $(i(l), j(l))$. The first optimal contract for h_{k+1} is $(X_{i(1)}, Y_{j(1)}) = (X_1, Y_1)$, which is the $(0, 0)$ contract. Assume that we calculated the optimal contracts for h_{k+1} up to the l-th contract. We calculate the $l+1$-th optimal contract by the following procedure. We go over all i, j pairs such that $i \geq i(l)$ and $j \geq j(l)$, and calculate the value for which the principal is indifferent between the contract $(X_i(l), Y_j(l))$ and the contract (X_i, Y_j). The contract that has the minimal indifference value is the next optimal contract, and we continue to find the next optimal contract by the same procedure.

Clearly, finding each additional optimal contract can be done in polynomial time, as the number of optimal contracts in h_k grows at most linearly in the number of agents in h_k. □