# On Optimal Overcomplete Subband Expansions for Multiple Description Coding

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## ABSTRACT

We study the use of overcomplete (also known as oversampled) subband expansions for multiple description coding. The system proposed uses an overcomplete subband expansion (a subband expansion in which the number of subbands is larger than the downsampling factor of each subband) followed by independent coding (quantization and coding) of each of the subbands. The entire coding of each of the subbands is entirely contained within a single description. Using analysis in the polyphase domain, we study finding frequency responses of optimal filterbanks and the rate allocation across the subbands to minimize the end-to-end reconstruction error subject to a transmission rate constraint when a distribution over channel states is known. A general analysis of the problem is given and results are shown for a $3 \times 2$ expansion of a first-order autoregressive process.

**Keywords:** Multiple Description Coding, Overcomplete Subband Expansions, Oversampled Subband Expansions, Overcomplete Linear Expansions, Frame Expansions, Joint Source-Channel Coding, Erasure Channel Coding

## 1. INTRODUCTION

Multiple description (MD) source coding is the problem of encoding a source $\{x[n]\}_{n=0}^{L-1}$ into $P$ separate binary descriptions at rates $R_0, \ldots, R_{P-1}$ bits such that any subset $\mathcal{S}$ of the descriptions may be received and together decoded to an expected distortion $D_{\mathcal{S}}$ commensurate with the total bit rate of $\mathcal{S}$. Early papers on multiple description coding are information theoretic and consider the problem of determining for $P = 2$ the set of rates and expected distortions $\{(R_0, R_1, D_0, D_1, D_{0,1})\}$ that are asymptotically achievable.[1–5] More recent papers consider the problem of designing practical multiple description quantizers, and their use over erasure channels.

The papers on practical MD quantization have so far taken three distinctly different approaches. In the first approach, pioneered by Vaishampayan, MD scalar, vector, tree-structured vector, or trellis quantizers are designed to produce the $P$ descriptions using a generalized Lloyd-like clustering algorithm that minimizes the Lagrangian of the rates and expected distortions $R_0, R_1, D_0, D_1, D_{0,1}$.[6–8] In the second approach, pioneered by Wang, Orchard, and Reibman, MD quantizers are constructed by separately describing (i.e., quantizing and coding) the $N$ coefficients of an $N \times N$ block linear transform that has been designed to introduce a controlled amount of correlation between the transform coefficients.[9–13] In the third approach, pioneered by Goyal, Kovačević, and Vetterli, MD quantizers are constructed by separately describing the $N$ coefficients of an overcomplete $N \times K$ tight frame expansion.[14,12] The correlating transforms approach and the overcomplete tight frame expansions are both extensions of standard linear transform approaches to ordinary source coding. In particular the correlating transforms approach provides redundancy across descriptions via a transform that is suboptimal for ordinary source coding and the overcomplete linear expansion provides redundancy because of the fact that the $N > K$ projections of a $K$ dimensional vector onto $N$ vectors can never be independent.

The traditional extensions of linear transform coders in[9–14] for multiple description coding were only made on block based transform coders. However, many ordinary source coders using linear transforms employ subband expansions rather than block based expansions for better compression efficiency. Examples include EZW[15] and SPIHT[16] for image compression and MP3 for audio compression. Therefore, recent work by Yang et al.[17,18] has extended the use of block based correlating transforms to correlating subband expansions. The only constraint is that the coding of each subband is entirely contained within a single description. The papers by Yang et al. find the frequency responses of the filterbanks in order to minimize the end-to-end reconstruction error given a redundancy constraint (i.e. the bitrate above the minimum required to code a source at a certain fidelity). This system has been

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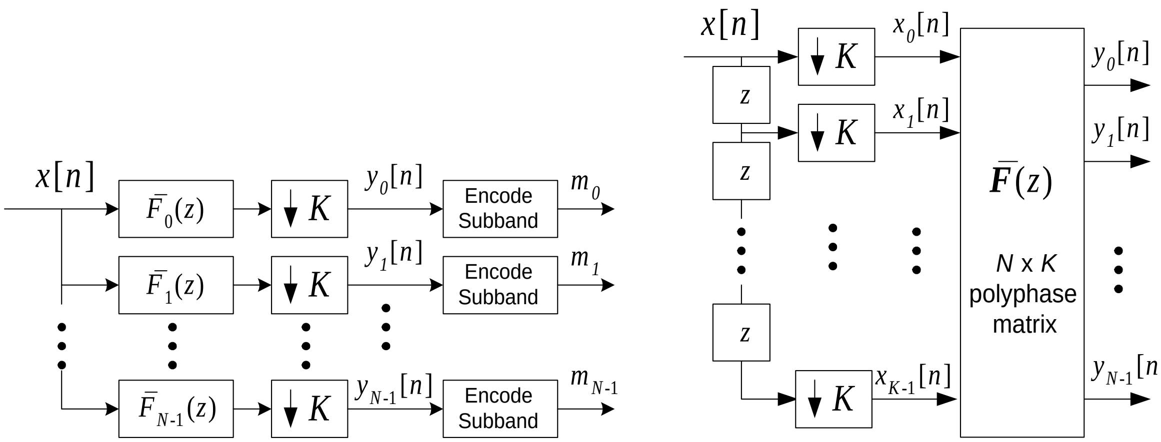


(a) Traditional subband setup

(b) Polyphase representation

**Figure 1.** Overcomplete subband expansion encoder.

further studied by Dragotti et al.[19] to try to find the achievable region of rates and distortions for such a system. In this paper, we present an extension of block based frame expansions to overcomplete subband expansions for multiple description coding. The analysis presented by Yang et al. for optimization of a correlating subband expansion is analogous to the optimization of the block based case[10,13,11] by performing the analysis in the polyphase domain. Similarly the optimization presented here is an extension from the block based case[20] by performing the analysis in the polyphase domain.

A general analysis for optimizing an overcomplete subband expansion for multiple description coding over an erasure channel when the input is a Gaussian $K$-block stationary process is presented. Wide-sense $K$-block stationary processes are those in which the first and second moments are invariant to shifts of $K$. Ordinary wide-sense stationary (WSS) processes are a subset of block stationary processes. A general analysis of the problem is shown and specific solutions are shown for a Gaussian first-order autoregressive process with a redundancy of 3/2.

In the optimization, analyzing infinite duration processes is difficult as convergence issues come up when taking Fourier transforms. Thus, in this discussion, it is assumed that a sequence is formed by taking a finite duration segment of length $L$ from a real valued zero-mean WSS process, for asymptotically large $L$. Such an analysis is not too restrictive since all real-world signals are finite duration. Plus the analysis holds for arbitrarily large segments, as long as they remain finite. Since the process is WSS, the analysis remains the same regardless of which $L$ samples are taken, so without loss of generality we assume a source of the form $x = \{x[n] : n = 0, \ldots, L-1\}$.

## 2. SYSTEM SETUP

In MD coding using overcomplete subband expansions, a length $L$ input signal $\boldsymbol{x} \in \mathbb{R}^L$ is represented by a vector $\boldsymbol{y} = \boldsymbol{F}\boldsymbol{x} \in \mathbb{R}^{NL/K}$, $N > K$, $N/K \in \mathbb{Z}$. $\boldsymbol{F}$ is a so-called frame operator, whose $NL/K$ rows $\{\boldsymbol{\varphi}\}_{i=1}^{NL/K}$ span $\mathbb{R}^L$. The coefficients of $\boldsymbol{y}$ are encoded and transmitted in $P$ descriptions. The decoder receives descriptions of only $P_{r,s} \le P$ descriptions after potential erasures, and reconstructs the signal $\hat{\boldsymbol{x}}$ from the received descriptions. The matrix $\boldsymbol{F}$ is constrained to be a subband expansion and the encodings of each subband of $\boldsymbol{y}$ are constrained to be individual descriptions. Constraining $\boldsymbol{F}$ to be a subband expansion reduces the degrees of freedom in the transform to $NL$ from $NL^2/K$.

Such a system is shown in Figure 1. A source of length $L$, $x = \{x[n] : n = 0, \ldots, L-1\}$ is first passed through an overcomplete subband expansion (also known as an oversampled subband expansion) resulting in $N$ subbands, each of length $L/K$. Assume $L$ is a multiple of $K$ so that $L/K$ is an integer. Let $y_i = \{y_i[n] : n = 0, \ldots, L/K-1\}$, $i = 0, \ldots, N-1$ be the $N$ subbands and let $\boldsymbol{y} = [y_0 \quad \ldots \quad y_{N-1}]^T$ be a vector of the subbands with a discrete Fourier transform vector (DFT) $\boldsymbol{Y} = \{\boldsymbol{Y}[m] : m = 0, \ldots, L/K-1\}$ whose components are the Fourier coefficients of the subbands. Each subband $y_i$ is independently coded using any reasonable method subject to a total rate constraint of $B_i$ bits, where the total rate constraint for coding of the source is given by

$$\frac{1}{L}\sum_{i=0}^{N-1} B_i \leq B. \tag{1}$$

The coding is typically lossy and thus there is a quantization noise associated with coding of the subbands. The encoding portion of quantization specifies an interval in which the values lie, rather than the actual value. Let $\tilde{y}$ be the geometric midpoint of these intervals and let $q$ be a sequence of quantization noises defined as $q = y - \tilde{y}$. Let $\boldsymbol{Q} = \{\boldsymbol{Q}[m] : m = 0, \ldots, L/K - 1\}$ be an $N$ dimensional vector of Fourier transforms of the quantization noises on each subband.

The setup for performing the transform shown in Figure 1(a) is the "traditional" method for performing subband decompositions. However for purposes of analysis, it is easier to use the equivalent polyphase representation as is shown in Figure 1(b), where $\bar{\boldsymbol{F}}(z)$ is an $N \times K$ matrix. Details of polyphase representations can be found in.[21] The relationship between $\bar{F}_i(z)$ in Figure 1(a) and the components of the matrix $\bar{\boldsymbol{F}}(z)$ in Figure 1(b) is given by $\bar{F}_i(z) = \sum_{k=0}^{K-1} z^k(\bar{\boldsymbol{F}}(z^K))_{ik}$. The $x_i = \{x_i[n] : n = 0, \ldots, L/K - 1\}$, $i = 0, \ldots, K-1$ are defined to be the $K$ polyphase components of the signal given by $x_i[n] = x[Kn + i]$, with a vector of polyphase components given by $\boldsymbol{x} = [x_0 \quad \ldots \quad x_{K-1}]$ with Fourier transform $\boldsymbol{X} = \{\boldsymbol{X}[m] : m = 0, \ldots, L/K - 1\}$. Then, if $\boldsymbol{F}$ is defined to be a sequence of matrices with $\boldsymbol{F} = \{\boldsymbol{F}[m] : m = 0, \ldots, L/K - 1\}$ with $(\boldsymbol{F}[m])_{ik} = (\bar{\boldsymbol{F}}(e^{j2\pi \frac{m}{L/K}}))_{ik}$, where it is assumed that any $K$ rows of $\boldsymbol{F}[m]$ form an invertible matrix for all $m$, then the following relationships can be obtained

$$\begin{aligned} \boldsymbol{Y}[m] &= \boldsymbol{F}[m]\boldsymbol{X}[m] & (2)\\ \boldsymbol{X}[m] &= \boldsymbol{F}^{+}[m]\boldsymbol{X}[m]; & (3) \end{aligned}$$

$\boldsymbol{F}^{+}$ is the pseudo-inverse of $\boldsymbol{F}$ given by $\boldsymbol{F}^{+} = (\boldsymbol{F}^H\boldsymbol{F})^{-1}\boldsymbol{F}^H$ (only true since $\boldsymbol{F}$ has full rank), where $\boldsymbol{F}^H$ is the complex-conjugate transpose of $\boldsymbol{F}$.

The decoder receives a subset of these $N$ subbands as is shown in Figure 2 depending on the channel state and reconstructs an estimate of the source, $\hat{x}$. The subset of subbands received is dependent on the state of the channel $S$. In a $P$ description system, the number of possible channel states is $2^P$ since this is the number of subsets of descriptions that can be received by the decoder (each description can be received or not). The channel state $S$ is given by the decimal representation of a $P$-bit vector $b_{P-1} \ldots b_1 b_0$, where $b_i = 1$ if the $i$th description is received and 0 else. For example, for $P = 3$, $S = 5 = (101)_2$ implies that descriptions 2 and 0 are received and description 1 is not. We write $D_s$ to be the distortion when the channel is in state $s$, $s = 0, 1, \ldots, 2^P - 1$ and $R_i$ to be the rate of the $i$th description, $i = 0, 1, \ldots, P-1$. Also, we define $\mathcal{I}_s$ to be the set of indices of the received descriptions in state $s$, so that $\mathcal{I}_5 = \{0, 2\}$. The cardinality of this set, $N_{r,s} = |\mathcal{I}_s|$, gives the number of received descriptions in state $s$. The distortion is taken to be the mean-square error and can be calculated by conditioning on the channel state $s$ as

$$\begin{aligned} D &= \frac{1}{L}\sum_{n=0}^{L-1} E[(x[n] - \hat{x}[n])^2] \\ &= \sum_{s=0}^{2^P-1} p_s \underbrace{\frac{1}{L}\sum_{n=0}^{L-1} E[(x[n] - \hat{x}[n])^2 | S = s]}_{D_s}, \end{aligned} \tag{4}$$

where $p_s$ is the probability of being in state $s$. Using Parseval's theorem on the vector of polyphase components of the source, $D_s$ can be written

$$D_s = \frac{1}{L}\sum_{m=0}^{L/K-1} \underbrace{E[\|\boldsymbol{X}[m] - \hat{\boldsymbol{X}}[m]\|^2 | S = s]}_{\mathcal{E}_s[m]}.$$

$\mathcal{E}_s$ is obviously a function of the channel state $s$, the source distribution of $x$, the coding method used to code the subbands, and the decoder. If $e$ is defined to be the reconstruction error as $e = x - \hat{x}$, with a vector of polyphase components $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ with Fourier transform $\boldsymbol{E}$, then we can simply write $\mathcal{E}_s[m] = E[\boldsymbol{E}^H[m]\boldsymbol{E}[m] | S = s]$.

Before obtaining a value for $\mathcal{E}_s$, we make the following assumptions.

1. $x$ is taken by windowing a length $L$ segment from a $K$-block wide-sense stationary (WSS) Gaussian process, $w$, with a diagonal power spectral density matrix $\boldsymbol{S_{ww}}(f)$. If $w$ is a $K$-block stationary process with full power spectral density matrix (not necessarily diagonal), then we can always find a principle component filterbank so that the output has the desired characteristics. For large $L$, we define $\boldsymbol{S_{xx}}$ as the power spectrum matrix for the vector of polyphase components as the covariance matrix of the Fourier coefficients of $\boldsymbol{x}$. Thus

$$\boldsymbol{S_{xx}}[m] = E[\boldsymbol{X}[m]\boldsymbol{X}^H[m]] \approx \boldsymbol{S_{ww}}\left(\frac{m}{L/K}\right).$$

2. Regardless of the coding method used to code the subbands, we assume that the power spectrum of the quantization noise of the subbands is given by

$$E[\boldsymbol{Q}[m]\boldsymbol{Q}^H[m]] = \begin{bmatrix} E[Q_0^2[m]] & 0 & \cdots & 0 \\ 0 & E[Q_1^2[m]] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E[Q_{N-1}^2[m]] \end{bmatrix}, \tag{5}$$

with

$$E[Q_i^2[m]] \approx cE[|Y_i[m]|^2]2^{-2R_i[m]}, \tag{6}$$

where $R_i[m]$ is a sequence with

$$\sum_{m=0}^{L/K-1} R_i[m] = B_i, \tag{7}$$

$E[|Y_i[m]|^2]$ is the variance of the $m$th Fourier coefficient of the $i$th subband, and $c$ is a constant depending on how the subbands are coded. $E[|Y_i[m]|^2]$ is given by

$$\begin{aligned} E[|Y_i[m]|^2] &= \Big(\boldsymbol{F}[m]E[\boldsymbol{X}[m]\boldsymbol{X}^H[m]]\boldsymbol{F}^H[m]\Big)_{ii} \\ &= \sum_{k=0}^{K-1} \|F_{ik}[m]\|^2 E[|X_k[m]|^2]. \end{aligned} \tag{8}$$

Equation (5) is a high-rate approximation. If the input is a $K$-block stationary process as in the first assumption, then each subband is a WSS process. Thus, a reasonable method of encoding the subbands is to use scalar quantization on the coefficients of a real DFT with rate $R_i[m]$ used to code the $m$th coefficient of the $i$th subband. This is because the Fourier coefficients of a segment from a Gaussian WSS process are approximately independent. If the Fourier coefficients are optimally entropy coded after scalar quantization, $c = \pi e/6$. The assumption of a diagonal matrix is only approximately true since the matrix would only be diagonal if all $N$ subbands were independent. However, even when they are not, as is the case in overcomplete subband expansions, this assumption is approximately correct at high rate as was shown by Bennett.[22] In fact, practically, we will constrain the optimization to cover only those transforms for which this is true; at high rate this will be all subband expansions in which no two rows of $\boldsymbol{F}[m]$ are identical at any $m$.

3. The decoder is assumed to use linear reconstruction techniques as presented in Section 2.1. Although this is suboptimal in cases when the number of received subbands is larger than $K$, such an assumption makes the analysis of distortion tractable.

### 2.1. Decoder

The decoder receives quantized versions of some of the subbands as is shown in Figure 2, i.e. it knows the intervals in which the received subband coefficients lie. We partition $\boldsymbol{Y}$ into received and not received subbands. Let $\boldsymbol{Y}_{r,s}$ denote the $N_{r,s}$ dimensional vector of the Fourier transform of the received subbands when the channel is in state $s$ and let $\boldsymbol{Y}_{nr,s}$ denote the $N_{nr,s} = N - N_{r,s}$ dimensional vector of erased subbands in state $s$, so that

$$\boldsymbol{\Pi}_s\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_{r,s} \\ \boldsymbol{Y}_{nr,s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{r,s} \\ \boldsymbol{F}_{nr,s} \end{bmatrix} \boldsymbol{X} = \boldsymbol{\Pi}_s\boldsymbol{F}\boldsymbol{X},$$

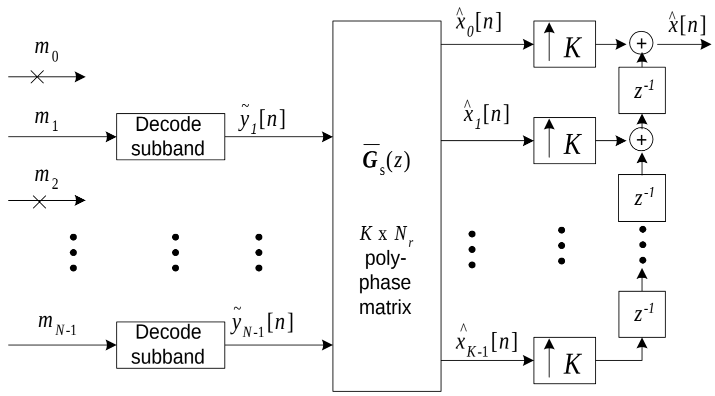


**Figure 2.** Overcomplete subband expansion decoder.

where $\boldsymbol{\Pi}_s$ is an $N \times N$ permutation matrix, $\boldsymbol{F}_{r,s}$ is an $N_{r,s} \times K$ matrix, and $\boldsymbol{F}_{nr,s}$ is an $N_{nr,s} \times K$ matrix. Since $\mathcal{I}_s$ is an $N_{r,s}$ dimensional set of indices of received components, the $j$th subband in $\boldsymbol{Y}_{r,s}$ is the $\mathcal{I}_s(j)$th subband in $\boldsymbol{Y}$.

Let $\boldsymbol{v}$ be a vector formed by concatenating all $L/K$ Fourier coefficients of the $N$ subbands as as $\boldsymbol{v} = [\boldsymbol{Y}[0] \ldots \boldsymbol{Y}[L/K-1]]^T$ with the subvector corresponding to the received subbands denoted as $\boldsymbol{v}_{r,s}$ in state $s$. The decoder knows that the coefficients in the received subbands lie within a certain interval so that

$$\boldsymbol{v}_{r,s} \in \mathcal{C}.$$

Let $\tilde{\boldsymbol{v}}_{r,s}$ be the midpoint of the region $\mathcal{C}$. Thus if the decoder is to minimize $E[(x[n] - \hat{x}[n])^2]$ at each $n$, then the decoder is given by $\hat{x}_{opt}[n] = E[x[n] | \boldsymbol{v}_{r,s} \in \mathcal{C}]$. Equivalently since the Fourier transform is a linear orthonormal transform, the optimal decoder can find the Fourier transform coefficients of the vector of polyphase components as $\hat{\boldsymbol{X}}_{opt}[m] = E[\boldsymbol{X}[m] | \boldsymbol{v}_{r,s} \in \mathcal{C}]$. Furthermore since the Fourier coefficients of a Gaussian WSS process are assumed to be independent, the optimal decoder is given by

$$\hat{\boldsymbol{X}}_{opt}[m] = E[\boldsymbol{X}[m] | \boldsymbol{Y}_{r,s}[m] \in \mathcal{C}[m]]. \tag{9}$$

In cases in which $N_{r,s} \geq K$, we can write

$$\begin{aligned} \hat{\boldsymbol{X}}_{opt}[m] &= \boldsymbol{F}^{+}_{r,s}[m] E[\boldsymbol{F}_{r,s}[m] \boldsymbol{X}[m] | \boldsymbol{Y}_{r,s}[m] \in \mathcal{C}[m]] \\ &= \boldsymbol{F}^{+}_{r,s}[m] E[\boldsymbol{Y}_{r,s}[m] | \boldsymbol{Y}_{r,s}[m] \in \mathcal{C}[m]], \end{aligned} \tag{10}$$

since $\boldsymbol{F}^{+}_{r,s}[m] \boldsymbol{F}_{r,s}[m] = \boldsymbol{I}_{N_{r,s}}$. However for cases in which $N_{r,s} > K$, the geometric midpoint of the region is not a good estimate for the expected value of the received transform coefficients[14,23] resulting in *inconsistent* estimates. That is $E[\boldsymbol{Y}_{r,s} | \boldsymbol{Y}_{r,s} \in \mathcal{C}] \neq \tilde{\boldsymbol{Y}}_{r,s}$. But in order to make the distortion analysis tractable, we use the geometric midpoint to calculate the reconstruction as

$$\hat{\boldsymbol{X}}[m] = \boldsymbol{F}^{+}_{r,s}[m] \tilde{\boldsymbol{Y}}_{r,s}[m]. \tag{11}$$

Since the channel states in which $N_{r,s} \geq K$ have significantly lower distortions than cases in which $N_{r,s} < K$, hopefully such an assumption will not affect the overall performance averaged over all states. Furthermore, hopefully a system that minimizes the distortion for such a decoder will also perform reasonably for one that obtains optimal reconstructions. In cases in which $N_{r,s} < K$, (10) does not hold since $\boldsymbol{F}^{+}_{r,s} \boldsymbol{F}_{r,s} \neq \boldsymbol{I}_{N_{r,s}}$. However the optimal decoder can be obtained by partitioning the source into a component in the subspace corresponding to the received subbands and another one in the subspace orthogonal to this one. That is,

$$\boldsymbol{X}[m] = \boldsymbol{F}^{+}_{r,s}[m] \boldsymbol{Y}_{r,s}[m] + (\boldsymbol{F}^{\perp}_{r,s}[m])^H \boldsymbol{Y}_{r,s}[m], \tag{12}$$

where $\boldsymbol{F}^{\perp}_{r,s}$ is a $(K - N_{r,s}) \times K$ dimensional matrix whose rows form an orthonormal basis for the subspace orthogonal to the span of the rows of $\boldsymbol{F}_{r,s}$. By definition of $\boldsymbol{F}^{\perp}_{r,s}$, it is easy to see that

$$\boldsymbol{F}^{\perp}_{r,s} (\boldsymbol{F}^{\perp}_{r,s})^H = \boldsymbol{I}_{N_{nr,s}} \qquad \boldsymbol{F}^{\perp}_{r,s} \boldsymbol{F}^{H}_{r,s} = \boldsymbol{0}_{N_{nr,s} \times N_{r,s}} \qquad \boldsymbol{F}^{\perp}_{r,s} \boldsymbol{F}^{+}_{r,s} = \boldsymbol{0}_{N_{nr,s} \times N_{r,s}}. \tag{13}$$

Using (9) and (12), the optimal decoder is given by

$$\hat{\boldsymbol{X}}[m] = \boldsymbol{F}^{+}_{r,s}[m]\tilde{\boldsymbol{Y}}_{r,s}[m] + (\boldsymbol{F}^{\perp}_{r,s}[m])^H E[\tilde{\boldsymbol{Y}}^{\perp}_{r,s}[m]|\boldsymbol{Y}_{r,s}[m] = \tilde{\boldsymbol{Y}}_{r,s}[m]]. \tag{14}$$

If the process is Gaussian, then the expected value in the second portion of this expression can be easily calculated to be

$$E[\boldsymbol{Y}^{\perp}_{r,s}|\boldsymbol{Y}_{r,s} = \tilde{\boldsymbol{Y}}_{r,s}] = \boldsymbol{F}^{\perp}_{r,s}\boldsymbol{S}_{\boldsymbol{xx}}\boldsymbol{F}^{H}_{r,s}(\boldsymbol{F}_{r,s}\boldsymbol{S}_{\boldsymbol{xx}}\boldsymbol{F}^{H}_{r,s})^{-1}\tilde{\boldsymbol{Y}}_{r,s}.$$

## 3. SYSTEM OPTIMIZATION

To minimize the overall distortion subject to a rate constraint, we need to derive expressions for $\mathcal{E}_s[m]$ in each state. Again, there are two cases to consider: $N_{r,s} \geq K$ and $N_{r,s} < K$ since the assumption of how the decoder operates is different in each of these cases.

When $N_{r,s} \geq K$ the decoder has information to localize the input source to a finite volume cell. Since $\boldsymbol{X}[m] = \boldsymbol{F}^{+}_{r,s}[m]\boldsymbol{Y}_{r,s}[m]$,

$$\boldsymbol{E}[m] = \boldsymbol{F}^{+}_{r,s}[m]\boldsymbol{Q}_{r,s}[m]. \tag{15}$$

Thus

$$\begin{aligned} \mathcal{E}_s[m] &= E[\boldsymbol{E}^H[m]\boldsymbol{E}[m]] && (16)\\ &= \operatorname{tr}\left(\boldsymbol{F}^{+}_{r,s}[m]E[\boldsymbol{Q}_{r,s}[m]\boldsymbol{Q}^{H}_{r,s}[m]](\boldsymbol{F}^{+}_{r,s}[m])^H\right). && (17)\end{aligned}$$

Since $\boldsymbol{Q}[m]\boldsymbol{Q}^H[m]$ is assumed to be diagonal for all $m$,

$$\mathcal{E}_s[m] = c\sum_{i=0}^{N_{r,s}-1}\left((\boldsymbol{F}^{+}_{r,s}[m])^H\boldsymbol{F}^{+}_{r,s}[m]\right)_{ii} E[|Y_{\mathcal{I}_s(i)}[m]|^2]2^{-2R_{\mathcal{I}_s(i)}[m]}. \tag{18}$$

When $N_{r,s} < K$, then there is not enough information to localize the source to a finite volume cell. Using (12) and (14), the error spectra can be written

$$\boldsymbol{E}[m] = \boldsymbol{F}^{+}_{r,s}[m]\boldsymbol{Q}_{r,s}[m] + (\boldsymbol{F}^{\perp}_{r,s}[m])^H\Big(\boldsymbol{Y}^{\perp}_{r,s}[m] - E[\boldsymbol{Y}^{\perp}_{r,s}[m]|\boldsymbol{Y}_{r,s}[m] = \tilde{\boldsymbol{Y}}_{r,s}[m]]\Big). \tag{19}$$

Thus

$$\begin{aligned} \mathcal{E}_s[m] &= E[\boldsymbol{E}^H[m]\boldsymbol{E}[m]] && (20)\\ &= \operatorname{tr}\left(\boldsymbol{F}^{+}_{r,s}[m]E[\boldsymbol{Q}_{r,s}[m]\boldsymbol{Q}^{H}_{r,s}[m]](\boldsymbol{F}^{+}_{r,s}[m])^H\right) + E[\|\boldsymbol{Y}^{\perp}_{r,s}[m] - E[\boldsymbol{Y}^{\perp}_{r,s}[m]|\boldsymbol{Y}_{r,s}[m] = \tilde{\boldsymbol{Y}}_{r,s}]\|^2], && (21)\end{aligned}$$

which has been obtained using (19) and simplified using (13). The first portion of the distortion is due to the quantization in the received subbands and the second component is due to the out of subspace components. Again if the process is Gaussian, the second portion can be calculated using

$$\begin{aligned} E[\|\boldsymbol{Y}^{\perp}_{r,s}[m] - E[\boldsymbol{Y}^{\perp}_{r,s}[m]|\boldsymbol{Y}_{r,s}[m]]\|^2] = \operatorname{tr}\Big[&\boldsymbol{F}^{\perp}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}[m](\boldsymbol{F}^{\perp}_{r,s}[m])^H - \\ &\boldsymbol{F}^{\perp}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}\boldsymbol{F}^{H}_{r,s}[m]\Big(\boldsymbol{F}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}[m]\boldsymbol{F}^{H}_{r,s}[m]\Big)^{-1}\boldsymbol{F}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}[m](\boldsymbol{F}^{\perp}_{r,s}[m])^H\Big].\end{aligned}$$

Again, since $E[\boldsymbol{Q}[m]\boldsymbol{Q}^H[m]]$ is assumed to be diagonal,

$$\begin{aligned} \mathcal{E}_s[m] = &\left(c\sum_{i=0}^{N_{r,s}-1}\left((\boldsymbol{F}^{+}_{r,s}[m])^H\boldsymbol{F}^{+}_{r,s}[m]\right)_{ii} E[|Y_{\mathcal{I}_s(i)}[m]|^2]2^{-2R_{\mathcal{I}_s(i)}[m]}\right) + \\ &\operatorname{tr}\Big[\boldsymbol{F}^{\perp}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}[m](\boldsymbol{F}^{\perp}_{r,s}[m])^H - \\ &\boldsymbol{F}^{\perp}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}\boldsymbol{F}^{H}_{r,s}[m]\Big(\boldsymbol{F}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}[m]\boldsymbol{F}^{H}_{r,s}[m]\Big)^{-1}\boldsymbol{F}_{r,s}[m]\boldsymbol{S}_{\boldsymbol{xx}}[m](\boldsymbol{F}^{\perp}_{r,s}[m])^H\Big]. \end{aligned} \tag{22}$$

Using (18) and (22) in (4) gives the distortion that is to be minimized over the subband expansion polyphase matrix, $\boldsymbol{F}[m]$, and the rate allocation, $R_i[m]$, for each of the Fourier coefficients in the subbands $i = 0, \ldots, N-1$, $m = 0, \ldots, L/K - 1$. From (18) and (22), the distortion can be rearranged and written

$$D = \underbrace{\frac{1}{L}\sum_{m=0}^{L/K-1} \mathcal{E}_b[m]}_{D_b} + \underbrace{\frac{1}{L}\sum_{m=0}^{L/K-1} \mathcal{E}_{nb}[m]}_{D_{nb}} \tag{23}$$

where $D_b$ is the portion that can be minimized by both the transform and rate allocation and $D_{nb}$ is the portion that can only be minimized by choice of transform. In particular $\mathcal{E}_b[m]$ is

$$\mathcal{E}_b[m] = \sum_{i=0}^{N-1} \alpha_i[m] S_{y_i y_i}[m] 2^{-2R_i[m]}, \tag{24}$$

where

$$\alpha_i[m] = c \sum_{s \in \mathcal{S}(i)} p_s \Big( (\boldsymbol{F}^{+}_{r,s}[m])^H \boldsymbol{F}^{+}_{r,s}[m] \Big)_{\mathcal{I}_s^{-1}(i)\mathcal{I}_s^{-1}(i)}, \tag{25}$$

where $\mathcal{S}(i)$ is the set of states in which subband $i$ is received and $\mathcal{I}_s^{-1}(i)$ gives the position of the $i$th subband of $\boldsymbol{Y}$ in $\boldsymbol{Y}_{r,s}$ when the channel is in state $s$. The term $\mathcal{E}_{nb}[m]$ can be written

$$\begin{aligned} \mathcal{E}_{nb}[m] \quad = \quad & \sum_{s \in \mathcal{S}} \mathrm{tr} \Big[ \boldsymbol{F}^{\perp}_{r,s}[m] \boldsymbol{S}_{\boldsymbol{xx}}[m] (\boldsymbol{F}^{\perp}_{r,s}[m])^H - \\ & \boldsymbol{F}^{\perp}_{r,s}[m] \boldsymbol{S}_{\boldsymbol{xx}}[m] \boldsymbol{F}^{H}_{r,s}[m] \Big( \boldsymbol{F}_{r,s}[m] \boldsymbol{S}_{\boldsymbol{xx}}[m] \boldsymbol{F}^{H}_{r,s}[m] \Big)^{-1} \boldsymbol{F}_{r,s}[m] \boldsymbol{S}_{\boldsymbol{xx}}[m] (\boldsymbol{F}^{\perp}_{r,s}[m])^H \Big], \end{aligned}$$

where $\mathcal{S}$ is the set of states in which $N_{r,s} < K$.

### 3.1. Optimal Rate Allocation

First consider the problem of optimal rate allocation. The problem of optimal bit allocation is to minimize

$$D_b = \frac{1}{L} \sum_{m=0}^{L/K-1} \sum_{i=0}^{N-1} \alpha_i[m] S_{y_i y_i}[m] 2^{-2R_i[m]},$$

subject to the rate constraint. Using (1) and (7), the overall rate constraint is

$$\frac{1}{L} \sum_{i=0}^{N-1} \sum_{m=0}^{L/K-1} R_i[m] \leq B.$$

The constrained problem is turned into an unconstrained problem using a Lagrangian formulation and minimizing,

$$J = \sum_{m=0}^{L/K-1} \sum_{i=0}^{N-1} \Big[ \alpha_i[m] S_{y_i y_i}[m] 2^{-2R_i[m]} + \lambda R_i[m] \Big], \tag{26}$$

where $\lambda$ is chosen to meet the rate constraint. Differentiating with respect to $R_j[l]$ and setting the derivatives $\partial J / \partial R_j[l] = 0$ for $j = 0, \ldots, N-1$, $l = 0, \ldots, L/K - 1$ and solving for the rate to meet the rate constraint gives

$$R_i[m] = \frac{BK}{N} + \frac{1}{2} \log_2 \left[ \frac{\alpha_i[m] S_{y_i y_i}[m]}{\left( \prod_{l=0}^{L/K-1} \prod_{j=0}^{N-1} \alpha_j[l] S_{y_j y_j}[l] \right)^{K/LN}} \right]. \tag{27}$$

This gives an optimal $D_{b,opt}$ of

$$D_{b,opt} = \frac{N}{K} \left( \prod_{l=0}^{L/K-1} \prod_{j=0}^{N-1} \alpha_j[l] S_{y_j y_j}[l] \right)^{K/LN} 2^{-2BK/N} \tag{28}$$

This solution is only valid provided all $R_i[m] \geq 0$, which will be the case at high rates. The rate constraint for coding of the individual subbands can easily be obtained by using (7).

### 3.2. Optimal Filterbank

Transform optimization involves finding the filterbanks to minimize $D_{b,opt} + D_{nb}$. Although an actual filterbank will most likely have constraints of filter length and causality that will impose constraints on the allowed frequency responses of the filter, we will assume no such constraints in this optimization. Furthermore, there might be additional constraints for using fast decoding algorithms such as constraining the filterbank to be a concatenation of two orthonormally related filterbanks.[23,24] These are also not considered in this optimization.

The constraints imposed on the filterbanks are those that do not hinder its performance. Namely, we assume that the polyphase matrix of frequency responses, $\boldsymbol{F}[m]$, is assumed to have $\sum_{k=0}^{K-1}(\boldsymbol{F}[m])_{ik}^2 = 1$, and $(\boldsymbol{F}[m])_{ik}$ real, for $i = 0, \ldots, N-1$, $m = 0, \ldots, L/K-1$. The assumption of the matrix having unit norm at all $m$ is not restrictive since arbitrary scaling will only change the stepsizes in relation to the variance. The bit allocation, $R_i[m]$, would not change. Although not rigorously proven, the assumption of having real coefficients in the matrix also seems intuitively correct since all terms in the distortion are only a function of the magnitude of the values in the matrix $\boldsymbol{F}$ and its pseudo-inverse in various channel states. Furthermore, we will restrict the optimization over transforms for which (5) is approximately true by limiting the minimum angular separation between the rows of $\boldsymbol{F}$ at each $m$.[24]

The actual minimization of $D_{b,opt} + D_{nb}$ is difficult and there seems to be no closed form solution for it. Therefore, we propose using a descent algorithm by varying a single parameter at a time in round robin fashion. All other parameters remain fixed and the parameter is varied over a certain range to find the value that gives the minimum distortion. In this case, the $NL$ parameters are the values for $(\boldsymbol{F}[m])_{ij}$, $i = 0, \ldots, N-1$, $j = 0, \ldots, K-1$, $m = 0, \ldots, L/K-1$. Although one could iterate directly over all these parameters, this is computationally intensive since the value for $F_{ij}[m]$ affects the overall distortion at all $m$, since it affects $\mathcal{E}_{b,opt}$ at all $m$. This is because $\mathcal{E}_{b,opt}[m]$ is a constant given by

$$\mathcal{E}_{b,opt}[m] = N 2^{-2BK/N} \prod_{l=0}^{L/K-1} \prod_{j=0}^{N-1} \left(\alpha_j[l] S_{y_j y_j}[l]\right)^{K/LN}, \quad m = 0, \ldots, L/K-1.$$

For optimization purposes, it would be easier to optimize the transform at each $m$ independently. For this purpose, we propose a two step iteration.

To jointly optimize the rate allocation and the transform, we define the rate allocation across frequencies as

$$\bar{R}[m] = \sum_{i=0}^{L/K-1} R_i[m]. \tag{29}$$

Then, the following two iteration steps are used.

1. Given $\boldsymbol{F}[m]$, find the optimal rate allocation, $R_i[m]$ using (27). From $R_i[m]$ calculate $\bar{R}[m]$ using (29).
2. Given a rate allocation across frequencies, $\bar{R}[m]$, find the optimal rate allocation across the subbands and optimal transform.

The first step is straightforward. For the second step, to find the optimal transform and new rate allocation across the subbands, we minimize $D_{b,opt} + D_{nb}$ subject to (29). Note that the way $\bar{R}$ is calculated, it always meets the overall rate constraint so long as (29) is met. To minimize the overall distortion subject to the rate constraint, a Lagrangian is formed at each $m$ as

$$J = \sum_{i=0}^{N-1} \alpha_i[m] S_{y_i y_i}[m] 2^{-2R_i[m]} + \lambda R_i[m],$$

where $\lambda$ is chosen to meet (29) at each $m$. Again, setting partial derivatives equal to 0 and solving for the rate gives

$$R_i[m] = \frac{\bar{R}[m]}{N} + \frac{1}{2}\log_2 \frac{\alpha_i[m] S_{y_i y_i}[m]}{\prod_{j=0}^{N-1} \left[\alpha_j[m] S_{y_j y_j}[m]\right]^{1/N}}.$$

Using this rate allocation gives a distortion of

$$D_{b,opt} = \frac{1}{L} \sum_{k=0}^{L/K-1} N \left( \prod_{j=0}^{N-1} \alpha_j[m] S_{y_j y_j}[m] \right)^{1/N} 2^{-2\bar{R}[m]/N}. \tag{30}$$

Thus the optimal $\mathcal{E}_b[m]$ is given by

$$\mathcal{E}_{b,opt}[m] = N \left( \prod_{j=0}^{N-1} \alpha_j[m] S_{y_j y_j}[m] \right)^{1/N} 2^{-2\bar{R}[m]/N}. \tag{31}$$

Now, since the value of $\alpha_i[m]$ and $S_{y_i y_i}[m]$ only depend on the value of the transform at that $m$, given $\bar{R}[m]$, the value of $\mathcal{E}_{b,opt}[m]$ only depends on the value of the transform at $m$. Thus, to minimize $D$, we can independently minimize the value of $\mathcal{E}_{b,opt}[m] + \mathcal{E}_{nb}[m]$ for all $m$ over the choice of transforms. The problem of minimizing $\mathcal{E}_{b,opt}[m] + \mathcal{E}_{nb}[m]$ at each $m$ can be done by using a decent algorithm at each $m$. However, now the convergence is much quicker as there are only $NK$ parameters to find at each $m$ independently of other $m$.

## 4. CODING OF FIRST ORDER AUTOREGRESSIVE PROCESS

Suppose the input sequence, $x$, is taken from a WSS process, $w$ with autocorrelation function , $R_{ww}$ and power spectral density $S_{ww}$. Then $w$ is definitely a block-WSS process with a period of $K$. The vector of polyphase components, $\boldsymbol{w}$ gives a vector sequence with an autocorrelation matrix $\boldsymbol{R_{ww}}$, where

$$\begin{aligned} (\boldsymbol{R_{ww}})_{ik}[n] &= E[w_i[n_0+n]w_k^*[n_0]] \\ &= R_{ww}[Kn+i-k]. \end{aligned} \tag{32}$$

And, the power spectral density matrix is given by $\boldsymbol{S_{ww}}$, where

$$\begin{aligned} (\boldsymbol{S_{ww}})_{ik}(f) &= \sum_{n=-\infty}^{\infty} R_{ww}[Kn+i-k]e^{-j2\pi f n} \\ &= \frac{e^{j2\pi f(i-k)/K}}{K} \sum_{m=0}^{K-1} S_{ww}\left(\frac{f+m}{K}\right) e^{j2\pi m(k-i)/K}. \end{aligned} \tag{33}$$

In order to find the optimal overcomplete subband expansion for such a process, first, we have to find a principle component filter bank and then find the optimal overcomplete subband expansion for the output of the PCFB.

In this section, we consider the optimization of a first order autoregressive Gaussian process with $K = 2$ and $N = 3$. A first order autoregressive process with correlation coefficient $r$ and variance $\sigma^2$ has an autocorrelation function given by

$$R_{ww}[n] = \sigma^2 r^{|n|}.$$

and a power spectral density

$$S_{ww}(f) = \frac{\sigma^2(1-r^2)}{|1-re^{-j2\pi f}|^2}.$$

Using (32), with $K = 2$ gives

$$R_{w_0w_0}[n] = R_{w_1w_1}[n] = R_{ww}[2n] = \sigma^2 r^{|2n|}$$

$$R_{w_0w_1}[n] = R_{w_1w_0}[-n] = R_{ww}[2n-1] = \sigma^2 r^{|2n-1|}.$$

Using (33), we get

$$S_{w_0w_0}(f) = S_{w_1w_1}(f) = \frac{1}{2}\left[S_{ww}\left(\frac{f}{2}\right) + S_{ww}\left(\frac{f+1}{2}\right)\right] = \frac{\sigma^2(1-r^4)}{|1-r^2e^{-j2\pi f}|^2},$$

$$S_{w_0w_1}(f) = S_{w_1w_0}^*(f) = \frac{e^{-j\pi f}}{2}\left[S_{ww}\left(\frac{f}{2}\right) + e^{j\pi} S_{ww}\left(\frac{f+1}{2}\right)\right] = \frac{\sigma^2 r(1-r^2)(1+e^{-j2\pi f})}{|1-r^2e^{-j2\pi f}|^2}.$$

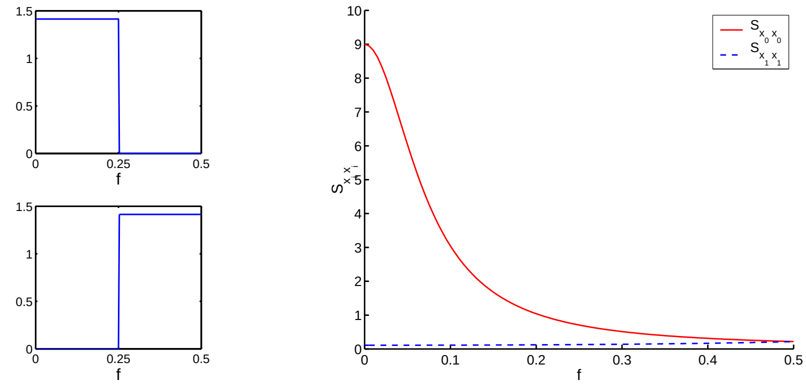


**Figure 3.** Principle component filter bank (left) and power spectral density of polyphase components after principle component filter bank (right).

Since $\boldsymbol{S_{ww}}$ is not diagonal we have to find a principle component filter bank $\bar{\boldsymbol{U}}(z)$ such that the output is a block-WSS random process with diagonal power spectral density. The eigenvalues of the power spectral density matrix are found to be $\lambda(f) = S_{w_0w_0}(f) \pm |S_{w_0w_1}(f)|$, with eigenvectors $\boldsymbol{v}(f) = \pm S^*_{w_0w_1}(f)/|S_{w_0w_1}(f)|$, with $S_{w_0w_1}(f)/|S_{w_0w_1}(f)| = e^{j\pi f}$ for $f \in [-1/2, 1/2)$ and $-e^{j\pi f}$ for $f \in [1/2, 3/2)$ with a period of 2. The filters in the PCFB for a first-order autoregressive process turn out to be brickwall lowpass and highpass filters as is shown in Figure 4.1. A $3 \times 2$ subband expansion combines the lowpass and highpass components into 3 subbands.

If a finite segment of the autoregressive process is put through the principle component filter bank, the output of the principle component filter bank will be a sequence $x$ whose polyphase components will have Fourier transforms $\boldsymbol{X}$ with

$$E[\boldsymbol{X}[m]\boldsymbol{X}[m]^H] = \boldsymbol{S_{xx}}[m] \approx \begin{bmatrix} \lambda_0\left(\frac{m}{L}\right) & 0 \\ 0 & \lambda_1\left(\frac{m}{L}\right) \end{bmatrix}. \tag{34}$$

If the optimal filter bank for coding of $x$ over the lossy channel is found to be $\boldsymbol{F}$, then the optimal filter bank for the finite segment of the autoregressive process will be given by $\boldsymbol{FU}$.

## 4.1. Results

We consider the coding of a length $L = 800$ segment from a first-order autoregressive process with $r = 0.8$ and $\sigma_2 = 1$, with $K = 2$, $N = 3$, $P = 3$, and $B = 6$ bits. The channel is assumed to lose subbands independently with a loss probability of $\epsilon$. With this channel model the distribution over channel states is given by, $p_0 = \epsilon^3$, $p_1 = p_2 = p_4 = \epsilon^2(1-\epsilon)$, $p_3 = p_5 = p_6 = \epsilon(1-\epsilon)^2$, and $p_7 = (1-\epsilon)^3$. The plots of $S_{x_0x_0}$ and $S_{x_1x_1}$ after the principle component filter bank are shown in Figure 4.1.

The polyphase components of the optimal filter bank, $\boldsymbol{F}$, are shown for $\epsilon = 0.05$ and $\epsilon = 0.95$ in Figure 4. At low loss rates, $\boldsymbol{F}[m]$ is approximately a real harmonic frame at all $m$. A row permutation of a $3 \times 2$ real harmonic frame is given by

$$\boldsymbol{H}_{3\times 2} = \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \cos(0) & \sin(0) \\ \cos(\pi/3) & \sin(-\pi/3) \end{bmatrix}. \tag{35}$$

At high loss rates, $\boldsymbol{F}[m]$ is repeating the subband with the higher variance. However at high frequencies, from Figure 4.1, it can be seen that $\boldsymbol{S_{xx}}$ becomes white at high $m$ and thus even at high loss rates, $\boldsymbol{F}[m]$ is a real harmonic frame at high frequencies.

The filter banks combining the principle component filter bank with the optimal overcomplete filter bank as $\boldsymbol{FU}$ and then obtaining the filterbanks for the traditional setup of Figure 1(a) are shown in Figure 5 for $\epsilon = 0.1, 0.3, 0.7, 0.9$. At low loss rates, the combined filter bank consists of filters that are brickwall lowpass and highpass filters followed by a real harmonic frame. As the loss rate increases the filter bank becomes one in which the lowpass filter is repeated.

Empirical tests show the performance of the system to be within 4% of the theoretical performance. We compare the empirical performance of the optimal filter bank with that of a principle component filter bank followed by a real

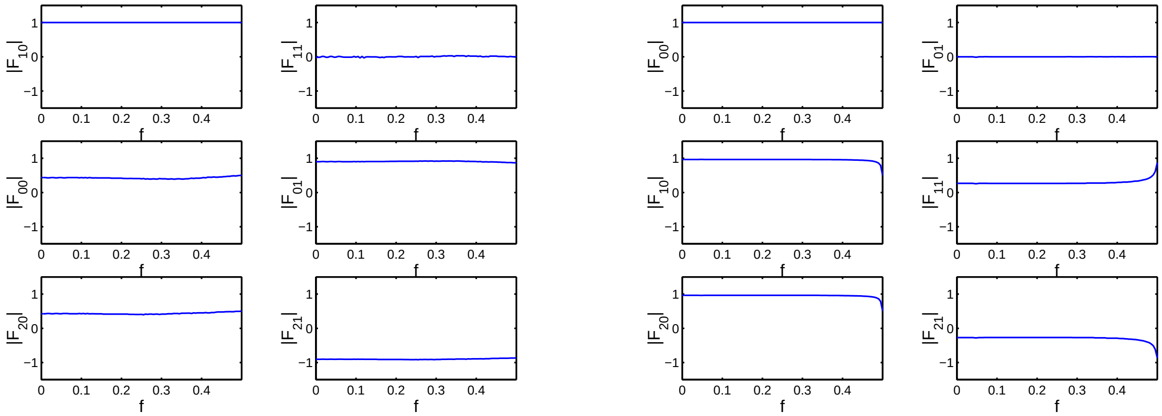


**Figure 4.** Polyphase components of optimal filter banks for $\epsilon = 0.05$ (left) and $\epsilon = 0.95$(right).

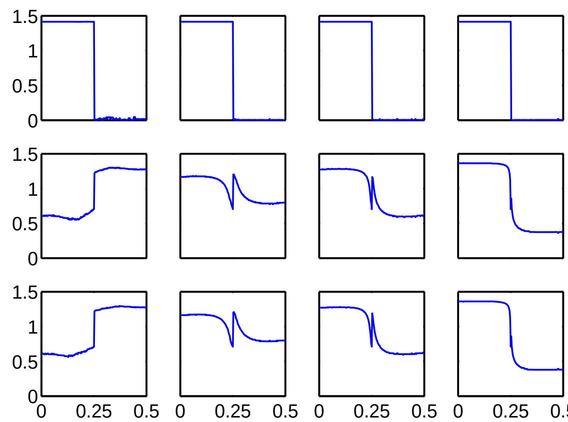

**Figure 5.** Optimal combined filter banks for $\epsilon = 0.1, 0.3, 0.7$ and $0.9$ (left to right).

harmonic frame, i.e. $\boldsymbol{F}[m] = \boldsymbol{H}_{3\times 2}$ for all $m$. Such an overcomplete expansion is easy to implement since it is not a function of $m$ and thus it can be implemented by simply taking coefficients from all the subbands at a particular spatial location and applying the overcomplete expansion. The results as a function of number of descriptions received is shown in Figure 6. The overall empirical distortion as a function of loss rate is also shown in Figure 6 along with the relative decrease in distortion by using an optimized frame.

## 5. CONCLUSION

The use of overcomplete (oversampled) subband expansions for multiple description coding was studied. Ideas from optimization of block based frame expansions were applied to optimize the rate allocation across the subbands and the subband expansion in the polyphase domain. Results for optimal overcomplete subband expansions for a first-order autoregressive Gaussian process were also provided. The performance of an optimized subband expansion was compared to an unoptimized one and empirical results were obtained to verify the performance. However, for practical use the subband expansions must have simpler implementations by constraining them to be a concatenation of orthonormally related critically sampled filter banks with finite-impulse response (FIR) forward and inverse implementations. This is left as future work.

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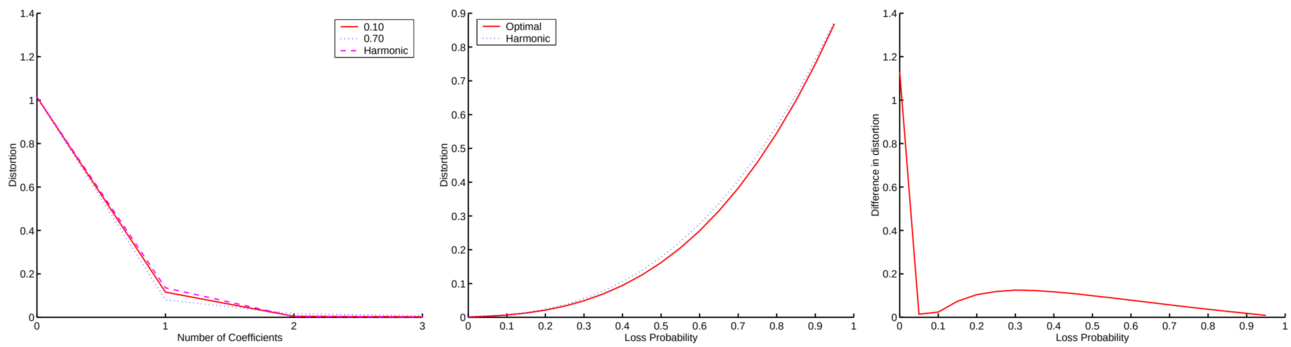


(a) Distortion as function of number of descriptions received.

(b) Distortion as function of loss rate.

(c) $(D_{harmonic} - D_{optimal})/D_{optimal}$.

**Figure 6.** Distortion results.

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