The two-eigenvalue problem and density of Jones representation of braid groups

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Introduction

In 1983 V. Jones discovered a new family of representations ρ of the braid groups. They emerged from the study of operator algebras (type Π_1 factors) and unlike earlier braid representations had no naive homological interpretation. Almost immediately he found that the trace or "Markov" property of ρ allowed new link invariants to be defined and this ushered in the era of quantum topology. There has been an explosion of link and 3-manifold invariants with beautiful inter-relations, asymptotic formulae, and enchanting connections to mathematical physics: Chern-Simons theory and 2-dimensional statistical mechanics. While many sought to bend Jones' theory toward classical topological objectives, we have found that the relation between the Jones polynomial and physics allows potentially realistic models of quantum computation to be created [FKW][FLW][FKLW][F]. Unitarity, a hidden locality, and density of the Jones representation are central to computational applications. With this application in mind, we have returned to some of Jones' earliest questions about these representations and the distributions of his invariants. A few concise answers are stated here in the introduction. Question 9 of Jones in [J2] asked for the closed images of the irreducible components of his representation. We answer Jones' question, and also identified the closed images for the general $SU(N)$ case completely.

A salient feature of Jones representation is the *two-eigenvalue property:*

the image of each braid generator has only two distinct eigenvalues $\{-1, q\}$. This is obvious from the quadratic Hecke relation $(\sigma_i + 1)(\sigma_i - q) = 0$. This two-eigenvalue property plays a key role in the following theorem:

Theorem 0.1. *Fix an integer $r \geq 5, r \neq 6, 10, n \geq 3$ or $r = 10, n \geq 5$. Let*

$$\rho_n^{(2,r)} = \oplus_{\lambda \in \wedge_n^{(2,r)}} \rho_\lambda^{(2,r)} : B_n \to \prod_{\lambda \in \wedge_n^{(2,r)}} U(\lambda)$$

be the unitary Jones representation of the n-strand braid group B_n. Then the closed image $\overline{\rho_n^{(2,r)}(B_n)}$ contains $\prod_{\lambda \in \wedge_n^{(2,r)}} SU(\lambda)$.

Our original motivation for studying Jones representation is for quantum computation. The special case $r = 5$ has already been used to show that the $SU(2)$ Witten-Chern-Simons modular functor at the fifth root of unity is universal for quantum computation [FLW]. Combining that paper with the above result, we conclude that the $SU(2)$ Witten-Chern-Simons modular functor at an r-th root of unity is universal for quantum computation if $r \neq 3, 4, 6$.

Jones was also concerned with the range of values his invariants assumed and their statistical properties. For this we must understand the topology and measure theory of the image Γ of ρ, since the Jones polynomial is obtained by tracing them.

There are three levels of detail in the discussion of a finitely generated group (or semi-group) Γ approximating a Lie group G. First is density and the rate at which density is achieved. From [Ki][So] [NC], we extract:

Theorem 5.6. *Let X be a set closed under inverse in a compact semisimple Lie group G (with Killing metrics) such that the group closure $\langle X \rangle$ is dense in G. Let X_l be the words of length $\leq l$ in X, then X_l is an ϵ-net in G for $l = \mathcal{O}(log^2(\frac{1}{\epsilon}))$, i.e., for all $g \in G$, $dist(g, X_l) < \epsilon$.*

Conjecturally the theorem should still hold for $l = \mathcal{O}(log(\frac{1}{\epsilon}))$ and there are some number theoretically special generating sets of $SU(2)$ [GJS] for which such an estimate for l can in fact be obtained. Such results now translate into topological statements:

Corollary 5.7. *Given a "conceivable" value v for the evaluation of Jones polynomial of $\hat{b}$ at a root of unity, i.e., one that lies in the computed support of the limiting distribution for $b \in B_n$, the n-string braids, to approximate*

v by v', $||v - v'|| < \epsilon$, it is sufficient to consider braids $b'_l \in B_n$ of length $l = \mathcal{O}(log^2(\frac{1}{\epsilon}))$ with Jones evaluations $b'_l = v'$, $||v - v'|| < \epsilon$.

The second level is uniformity in measure: if $\Gamma =< \gamma_1, \cdots, \gamma_m >$, i.e., Γ is generated as a semi-group by $\gamma_1, \cdots, \gamma_m$, let W_l be the set of unreduced words of length=l and μ_l be the equally weighted atomic measure on W_l (mass m^{-l} on each word in W_l), it is known that density implies uniformity in measure [Bh], $\mu_l \to \mathrm{Haar}(G)$ in the weak-* topology (i.e., when integrated against continuous functions.) Third is the rate of convergence of measures, which is also addressed in [Bh].

Returning to the Jones polynomial evaluations which are weighted traces of dense representations, we can determine the statistics. Recall n is the number of strands, and l is the length of a braid. One may consider the double limit when l, and later n are taken to infinity. In this case, if r is a fixed integer $r \geq 5$, $r \neq 6$, the distribution of evaluations at $e^{\frac{\pm 2\pi i}{r}}$ of the Jones polynomial of a "random" link with n strands tends to a fixed Gaussian. The variance of this Gaussian depends on r and grows like r^3 as $r \to \infty$.

Our density result follows from the solution of a general two-eigenvalue problem: Let G be a compact Lie group, and V a faithful, irreducible, unitary representation of G. The pair (G, V) is said to have the *k-eigenvalue property* if there exists a conjugacy class $[g]$ of G such that

(1) the class $[g]$ generates G topologically;

(2) any element $g \in [g]$ acts on V with exactly k different eigenvalues such that for each $2 \leq r \leq k$, no set of r eigenvalues forms a coset of the multiplicative group $\{1, \omega, \omega^2, \cdots, \omega^{r-1}\}$, where ω is a primitive r-th root of unity.

The k-eigenvalue problem is to classify all such pairs (G, V). Note that G is not assumed to be connected. The problem naturally divides into two cases according to whether G is or is not finite modulo its center. The solution to the first case is essentially known to the experts and we content ourselves with a statement at the end of section 1. The solution to the case that $G/Z(G)$ has positive dimension is:

Theorem 1.1 *Suppose (G, V) is a pair with the two-eigenvalue property. Let G_1 be the universal covering of the derived group $[G_0, G_0]$ of the identity component G_0 of G. If G is of positive dimension modulo its center, then V is an irreducible G_1-module, with highest weight ϖ, and (G_1, ϖ) is one of the following:*

(1) $(SU(l+1), \varpi_i)$ for some $l \geq 1$, and $1 \leq i \leq l$.

(2) $(Spin(2l+1), \varpi_l)$ *for some* $l \geq 2$.

(3) $(Sp(2l), \varpi_1)$ *for some* $l \geq 3$.

(4) $(Spin(2l), \varpi_i)$ *for some* $l \geq 4$ *and* $i = 1, l-1, l$,

where ϖ_i *denotes the* i*-th fundamental representation.*

There is a fairly close analogy between this theorem and J. Serre's classification [Se] of inertial monodromy types for Hodge-Tate modules with only two different weights. Not only are the problems formally similar, the solution is identical. However, it does not seem that either result implies the other. In the Hodge-Tate case, one looks for a cocharacter taking two distinct values on the set of weights of an irreducible representation of a semisimple group; in our case, one looks for a rational cocharacter taking two different values (mod $\mathbb{Z}$) which are not congruent (mod $\frac{1}{2}\mathbb{Z}$). Our technique here works for the 3-eigenvalue problem.

1 The two-eigenvalue problem

Let G be a compact Lie group, and V a faithful, irreducible, unitary representation of G. The pair (G, V) is said to have the *two-eigenvalue property* if there exists a conjugacy class $[g]$ of G such that

(1) the class $[g]$ generates G topologically;

(2) any element $g \in [g]$ acts on V with exactly two different eigenvalues whose ratio is not ± 1.

Note that G is not assumed to be connected. The problem naturally divides into two cases according to whether G is or is not finite modulo its center. The solution to the first case is essentially known to the experts and we content ourselves with a statement at the end of this section. The rest of the section is devoted to the case that $G/Z(G)$ has positive dimension.

Theorem 1.1. *Suppose* (G, V) *is a pair with the two-eigenvalue property. Let* G_1 *be the universal covering of the derived group* $[G_0, G_0]$ *of the identity component* G_0 *of* G*. If* G *is of positive dimension modulo its center, then* V *is an irreducible* G_1*-module, with highest weight* ϖ*, and* (G_1, ϖ) *is one of the following:*

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We begin with a lemma from linear algebra.

Lemma 1.2. *Suppose W is a vector space with a direct sum decomposition $W = \oplus_{i=1}^{n} W_i$, and U is an operator on W such that $U : W_i \to W_{i+1}$ $(1 \leq i \leq n)$ cyclically. Then any eigenvalue of U multiplied by any nth root of unity is again an eigenvalue of U.*

Proof: Choose a basis of W consisting of bases of $W_i, i = 1, 2, \cdots, n$. If k is not a multiple of n, then tr $U^k = 0$ because all diagonal entries of U^k are 0 with respect to the above basis. Let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of U with multiplicity. For each integer $m > 0$, consider tr $U^m = \sum {\lambda_i}^m$. Let ω be an nth root of unity. Then $\sum (\omega\lambda_i)^m = \sum \omega^m {\lambda_i}^m = \omega^m \sum {\lambda_i}^m$. We claim this sum is equal to tr $U^m = \sum {\lambda_i}^m$. Indeed, when m is not a multiple of n, they are both 0, when m is a multiple of n, $\omega^m = 1$. Recall that the symmetric polynomials $\{\sum x_i^m\}$ uniquely determine all the symmetric polynomials of x_i. It follows that $\prod_i(\lambda - \omega\lambda_i) = \prod_i(\lambda - \lambda_i)$. Therefore, the set of the eigenvalues of T is invariant under multiplication by any nth root of unity. $\square$

In the two-eigenvalue problem, the generating conjugacy class cannot lie in the identity component G_0 unless G is connected. However, the following lemma allows us to reduce to the connected case:

Lemma 1.3. *Given a compact Lie group G, and an irreducible representation of G. If an element g has two eigenvalues under ρ whose ratio is $\neq \pm 1$, then g is a product of an element in G_0 with an element in $Z_G(G_0)$, the centralizer of G_0 in G.*

Proof: The action of Ad_g defines an automorphism of G_0. By [St] Theorem 7.5, there exists a maximal torus T of G_0 such that Ad_g fixes T as a set.

Recall any automorphism of G_0 fixing T pointwise is an inner automorphism by an element in T.

To show that Ad_g fixes T pointwise, consider all the characters $\{\chi\}$ of ρ, and the weight space decomposition $V = \oplus_{\chi \in \chi^*(T)} V_\chi$. As Ad_g fixes T as a set, $\rho(g)$ permutes the weight spaces V_χ according to the permutation of characters by Ad_g. Suppose the longest permutation cycle of weight spaces by Ad_g has length=l. If $l \geq 3$, then by Lemma 1.2, $\rho(g)$ have at least l distinct eigenvalues, contrary to hypothesis. If $l = 2$, then by Lemma 1.2, the two possible eigenvalues of $\rho(g)$ has ratio -1. Therefore, $l = 1$, i.e., $\rho(g)$ fixes every weight space V_χ. It follows that Ad_g fixes the maximal torus T of G_0 pointwise. The lemma follows. □

Theorem 1.4. *Let (G, V) be a pair with the two eigenvalue property. If G is of positive dimension modulo its center, then the derived group $[G_0, G_0]$ of G_0 is a simple Lie group, and $G = G_0 Z(G)$.*

Proof: Let $[g]$ satisfy the two-eigenvalue property. As the conjugates of g (topologically) generate G/G_0, if the restriction of V to G_0 had more than one isotypic component, g would permute these components nontrivially, contrary to Lemma 1.2. Thus, the restriction of V to G_0 is the tensor product of an irreducible representation V_0 and a trivial representation V^0. By Lemma 1.3, $g = g_0 z$, where $g_0 \in G_0$ and z centralizes G_0. By Schur's Lemma, $\rho(z) = 1 \otimes B$, while $\rho(g_0) = A \otimes 1$. The two-eigenvalue property implies that either A or B is scalar. Since $[g]$ generates a dense subgroup of G, the same is true of $[g_0]$ and G_0. As V is a faithful representation, A cannot be scalar, so B must be. Thus, (G_0, V_0) satisfies the two-eigenvalue property with generating class $[g_0]$. Moreover, V^0 must be one-dimensional since otherwise V would be a reducible representation of G.

Let G_1 denote the universal cover of $[G_0, G_0]$. Let $g_1 \in G_1$ denote an element whose image in $[G_0, G_0]$ lies in the coset $g_0 Z(G_0)$. The pull-back V_1 of V_0 to G_1 is again irreducible, and the image of g_1 has two eigenvalues with the same ratio as the original image of g_0. Moreover, $[g_1]$ generates a dense subgroup of G_1 since no proper closed subgroup of G_1 can generate G_0 modulo $Z(G_0)$. It follows that (G_1, V_1) satisfies the two-eigenvalue property.

If G_1 were not simple, it would factor as $G_2 \times G_3$, and V_1 would factor as an external tensor product of representations V_2 and V_3. Writing $\rho(g_1) = A \otimes B$, we see that A or B must be a scalar. Thus $[g_1]$ cannot generate a dense subgroup of the product. We conclude that G_1, and therefore $[G_0, G_0]$, must be simple. □

Theorem 1.5. *Let G be a connected, simply connected compact simple Lie group and V an irreducible representation of G satisfying the two-eigenvalue property. Let ϖ denote the highest weight of V. Then (G, ϖ) is one of the following:*

(1) $(SU(r+1), \varpi_i)$ *for some* $r \geq 1$ *and* $1 \leq i \leq r$.

(2) $(Spin(2r+1), \varpi_r)$ *for some* $r \geq 2$.

(3) $(Sp(2r), \varpi_1)$ *for some* $r \geq 3$.

(4) $(Spin(2r), \varpi_i)$ *for some* $r \geq 4$ *and* $i \in \{1, r-1, r\}$.

In other words G is classical and V is minuscule.

Proof: Fix a maximal torus T of G. As the conjugates of T cover G, there exists $g \in T$ satisfying the two-eigenvalue property. There is a natural identification of T with the quotient $W/X_*(T)$, where $W = X_*(T) \otimes \mathbf{R}$ is the universal covering space of T, and where we identify $\mathbf{R}/\mathbf{Z}$ with the set of complex numbers of norm 1. Let $\tilde{g}$ denote an element of W mapping to g. The two-eigenvalue condition means that the values $\chi(\tilde{g})$, as χ ranges over the characters of V, lie in exactly two cosets of $\mathbf{Z}$ which do not differ by a half-integer.

Let α denote the highest short root of G and $\varpi, \varpi-\alpha, \ldots, \varpi-k\alpha$ a string of weights of V. If $k \geq 2$, then $\alpha(\tilde{g})$ must be an integer. As the set of weights is invariant under the Weyl group, all short roots of G lie in the Weyl-orbit of α, and as the short roots span the root lattice, this would imply that all $\chi(\tilde{g})$ lie in a single coset, contrary to hypothesis. It follows that $k = 1$, or equivalently,

$$\sum_{i=1}^{r} a_i b_i \cdot \frac{\alpha_i^2}{\alpha^2} = 1,$$

where

$$\varpi = a_1\varpi_1 + \cdots + a_r\varpi_r,\ \ \alpha = b_1\alpha_1 + \cdots + b_r\alpha_r.$$

Indeed, in the notation of [Hu],

$$1 = \langle \varpi, \alpha \rangle = 2\frac{\varpi \cdot \alpha}{\alpha^2} = 2\sum_{i,j} a_i b_j \frac{\varpi_i \cdot \alpha_j}{\alpha^2} = \sum_{i,j} a_i b_j \langle \varpi_i, \alpha_j \rangle \frac{\alpha_j^2}{\alpha^2} = \sum_i a_i b_i \frac{\alpha_i^2}{\alpha^2}.$$

Note that $\frac{\alpha_i^2}{\alpha^2} \in \{1, 2, 3\}$. Since all the coefficients b_i in the representation of the longest short root as a linear combination of simple roots are ≥ 1, this implies that ϖ is a fundamental weight ϖ_i for some i such that $a_i = b_i = 1$, and α_i is a short root. In addition to the cases listed above, we have the cases (E_6, ϖ_1), (E_6, ϖ_6), and (E_7, ϖ_7). We claim that none of these exceptional cases correspond to actual solutions of the two-eigenvalue problem.

For E_6, the two representations in question are dual to one another, so we consider only the one corresponding to the highest weight ϖ_1. By [MP], the restriction of this representation to $H = SU(3) \times SU(3) \times SU(3)$ is

$$\sigma \otimes \sigma^* \otimes 1 \oplus 1 \otimes \sigma \otimes \sigma^* \oplus \sigma^* \otimes 1 \otimes \sigma,$$

where σ denotes the standard representation of $SU(3)$. Since H can be chosen to contain T, we may write $g = (g_1, g_2, g_3) \in H$. The two-eigenvalue property guarantees that one of the $\sigma(g_i)$ has two eigenvalues and the other two are scalars. Without loss of generality, we assume $\sigma(g_1)$ has eigenvalues α (with multiplicity 2) and α^{-2}, while the scalars for g_2 and g_3 are β and γ. The set of eigenvalues is

$$\{\alpha\beta^{-1}, \alpha^{-2}\beta^{-1}, \beta\gamma^{-1}, \gamma\alpha^{-1}, \gamma\alpha^2\}.$$

Since two pairs of eigenvalues have ratio α^3, either $\alpha\beta^{-1} = \gamma\alpha^2$ or $\alpha^3 = 1$. In the first case, $\alpha\beta\gamma = 1$, and since $\beta^3 = \gamma^3 = 1$, this implies $\alpha^3 = 1$. We conclude that the eigenvalues are α/β, β/γ, and γ/α, all cube roots of unity. Since they multiply to 1, all are the same or all are different, contrary to hypothesis.

For E_7, we restrict to $SU(2) \times SU(4) \times SU(4)$ and obtain

$$1 \otimes \sigma \otimes \sigma \oplus 1 \otimes \sigma^* \otimes \sigma^* \oplus \tau \otimes 1 \otimes S^2\sigma \oplus \tau \otimes S^2\sigma \otimes 1,$$

where σ and τ are the standard representations of $SU(4)$ and $SU(2)$ respectively. Writing $g = (g_1, g_2, g_3)$, we conclude that $\sigma(g_2)$ and $\sigma(g_3)$ are scalars β and γ, while $\tau(g_1)$ has eigenvalues $\alpha^{\pm 1}$. Thus, the set of eigenvalues is

$$\{\beta\gamma, \beta^{-1}\gamma^{-1}, \alpha\gamma^2, \alpha^{-1}\gamma^2, \alpha\beta^2, \alpha^{-1}\beta^2\}.$$

Note that $\gamma^2 = \beta^2 = \pm 1$ since β and γ determine unimodular scalar 4×4 matrices. If $\alpha^2 = 1$, then all the eigenvalues are the same up to sign, contrary to hypothesis. If not the squares of eigenvalues are 1, α^2, and α^{-2}, so $\alpha^2 = -1$. But this implies that two eigenvalues have ratio -1, contrary to hypothesis. $\square$

Now we state the solution to the two-eigenvalue problem for finite groups. Our list is based on [Za] and depends on the classification of finite simple groups. The cases $m \geq 5$ are classical [Bl].

Theorem 1.6. *Suppose $(G, V, [g])$ has the two-eigenvalue property, and $G/Z(G)$ is finite. Then $g^m \in Z(G)$ for some $m \in \{3, 4, 5\}$, and $G = H \cdot Z(G)$ for some group H with an element $h \in H$ such that $h^{-1}g \in Z(G)$. Furthermore, one of the following holds:*

(a) $m = 5$, $H \cong SL(2, 5)$ and $\dim V = 2$;

(b) $m = 4$, G contains a normal subgroup E such that $E/Z(E)$ is of exponent 2 and of order 2^{2k}, $\dim V = 2^k$, $V|_E$ is irreducible and $H/E \in \{Sp(2k, 2), U(k, 2), O^-(2k, 2)$ with $k > 2, S_{2k+1}, S_{2k+2}\}$;

(c) $m = 3$ and one of the following holds:
(1) $H \cong Sp(2n, 3), n > 1$ and $\dim V = \frac{(3^n-(-1)^n)}{2}$;
(2) $H \cong PSp(2n, 3), n > 1$ and $\dim V = \frac{(3^n+(-1)^n)}{2}$;
(3) $H \cong SU(n, 2)$ and n is a multiple of 3, or $H \cong U(n, 2)$, $V|_H$ is a Weil representation of H and $\dim V = \frac{(2^n+2(-1)^n)}{3}$ or $\frac{(2^n-(-1)^n)}{3}$;
(4) $H \cong \tilde{A}_n$, the two-fold central extension of the alternating group A_n, and $\dim V = 2^{\frac{n-3}{2}}$ for n odd, and $\dim V = 2^{\frac{n-2}{2}}$ for n even;
(5) G contains a normal subgroup E such that $E/Z(E)$ is of exponent 2 and of order 2^{2k}, $\dim V = 2^k$, $V|_E$ is irreducible and $H/E \in \{Sp(2k, 2), U(k, 2), O^+(2k, 2), O^-(2k, 2)$ with $k > 2, A_{2k+1}, A_{2k+2}\}$;
(6) G contains a normal extraspecial subgroup E of order 3^{2k}, $\dim V = 3^k$, and $V|_E$ is irreducible, and $H/E \cong Sp(2k, 3)$;
(7) $H \cong PSp(4, 3)$, and $\dim V = 6$;
(8) $H/Z(H) \cong PSU(4, 3)$, $|Z(G)| = 6$, and $\dim V = 6$;
(9) $H/Z(H) \cong J_2, |Z(G)| = 2$, and $\dim V = 6$;
(10) $H/Z(H) \cong Sp(6, 2)$, $|Z(G)| = 2$, and $\dim V = 8$;
(11) $H/Z(H) \cong O^+(8, 2), |Z(G)| = 2$, and $\dim V = 8$;
(12) $H/Z(H) \cong G_2(4), |Z(G)| = 2$, and $\dim V = 12$;
(13) $H/Z(H) \cong Suz, |Z(G)| = 6$, and $\dim V = 12$.
(14) $H \cong Co_1$, and $\dim V = 24$;

2 Hecke algebra representations of braid groups

The n-strand braid group B_n has the well-known presentation:

$$B_n = \{\sigma_1, \cdots, \sigma_{n-1} | \quad \begin{array}{r} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1\}. \end{array}$$

Hecke algebra representations of the braid groups in the root of unity case are indexed by two parameters: a compact Lie group and an integer $l \geq 1$, called the *level* of the theory. The cases of Jones and Wenzl representations correspond to the special unitary groups $SU(k), k \geq 2$. For each pair of integers (k, r) with $r \geq k+1$, there is a unitary representation of the braid groups with level $l = r - k$. Jones representations correspond to $SU(2)$, and the general $SU(k)$ theory gives rise to the HOMFLY polynomial.

We describe the Jones-Wenzl representation explicitly, following [We]. Let $q = e^{\pm\frac{2\pi i}{r}}$, and $[m]$ be the quantum integer $\frac{q^{\frac{m}{2}}-q^{-\frac{m}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$. The constant $[2] = q^{\frac{1}{2}} + q^{-\frac{1}{2}} = 2cos\frac{\pi}{r}$ is ubiquitous in quantum topology. The Hecke algebra $H_n(q)$ of type A is the (finite dimensional) complex algebra generated by $e_1, \ldots, e_{n-1}$ such that

1. $e_i^2 = e_i$,
2. $e_i e_{i+1} e_i - [2]^{-2} e_i = e_{i+1} e_i e_{i+1} - [2]^{-2} e_{i+1}$,
3. $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.

A representation π of $H_n(q)$ on a Hilbert space is called a $\mathbb{C}^*$ *representation* if each $\pi(e_i)$ is self-adjoint.

Lemma 2.1. *Each $\mathbb{C}^*$ representation of the Hecke algebra $H_n(q)$ gives rise to a unitary representation of the braid group B_n by the formula:*

$$\rho(\sigma_i) = q - (1+q)\pi(e_i). \qquad (1)$$

Proof: The defining relations 1—3 of $H_n(q)$ imply that the elements $\rho(\sigma_i)$ satisfy the braid relations. Writing e_i for $\pi(e_i)$, since $\rho^*(\sigma_i) = \bar{q} - (1+\bar{q})e_i^*$,

$$\rho(\sigma_i)\rho^*(\sigma_i) = q\bar{q} + (1+q)(1+\bar{q})e_i e_i^* - \bar{q}(1+q)e_i - q(1+\bar{q})e_i^* = 1.$$

Cancellation of the last three terms follows from the facts $e_i^* = e_i$ and $e_i^2 = e_i$. □

Jones-Wenzl $\mathbb{C}^*$ representation of $H_n(q)$ are reducible; their irreducible constituents, referred to as *sectors*, are indexed by Young diagrams. A Young diagram with n boxes is the diagram of a partition of the integer n:

$$\lambda = [\lambda_1, \ldots, \lambda_k], \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0, \sum_{i=1}^{k} \lambda_i = n.$$

Note that λ is allowed to have empty rows. Given a Young diagram λ with n boxes, a standard tableau of shape λ is an assignment of integers $\{1, 2, \cdots, n\}$ into the boxes so that the entries of each row and column are increasing.

Definition 1. *Suppose t is a standard tableau with n boxes, and m_1 and m_2 are two entries in t. Suppose m_i appears in row r_i and column c_i of t.*

(1) Set $d_{t,m_1,m_2} = (c_1 - c_2) - (r_1 - r_2)$.

(2) Set $\alpha_{t,i} = \frac{[d_{t,i,i+1}+1]}{[2][d_{t,i,i+1}]}$ if $[d_{t,i,i+1}] \neq 0$, and $\beta_{t,i} = \sqrt{\alpha_{t,i}(1-\alpha_{t,i})}$.

(3) A Young diagram $\lambda = [\lambda_1, \cdots \lambda_k], \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ is (k,r)-admissible if $\lambda_1 - \lambda_k \leq r - k$.

(4) Suppose t is a standard tableau of shape λ with n boxes, let $t^{(i)}(1 \leq i \leq n)$ be the standard tableaux obtained from t by deleting boxes with entries $n, n-1, \cdots, n-i+1$. A standard tableau t is (k,r)-admissible if the shape of each tableau $t^{(i)}$ is a (k,r)-admissible Young diagram.

The irreducible sectors of the Jones-Wenzl representations of the Hecke algebras $H_n(q)$ (and hence of the braid groups B_n) are indexed by the the pair (k,r) and a (k,r)-admissible Young diagram λ with n boxes. A $\mathbb{C}^*$ representation $\pi_\lambda^{(k,r)}$ of the Hecke algebra $H_n(q)$ can be constructed as follows: let $V_\lambda^{(k,r)}$ be the complex vector space with basis $\{\vec{v}_t\}$, where t ranges over (k,r)-admissible standard tableaux of shape λ. Let $s_i(t)$ be the tableau obtained from t by interchanging the entries i and $i+1$. If $s_i(t)$ is also (k,r)-admissible, then we define

$$\pi_\lambda^{(k,r)}(e_i)(\vec{v}_t) = \alpha_{t,i}\vec{v}_t + \beta_{t,i}\vec{v}_{s_i(t)}. \tag{2}$$

If $s_i(t)$ is not (k,r)-admissible, set $\beta_{t,i} = 0$ in formula (2). In this case, $\alpha_{t,i}$ is either 0 or 1. It follows that $\pi_\lambda^{(k,r)}(e_i)$ (with respect to the basis $\{\vec{v}_t\}$) is a matrix consisting of only 2×2 blocks

$$\begin{pmatrix} \alpha_{t,i} & \beta_{t,i} \\ \beta_{t,i} & 1-\alpha_{t,i} \end{pmatrix} \tag{3}$$

and 1×1 blocks 0 or 1. The identity $\alpha_{t,i} = \alpha_{t,i}^2 + \beta_{t,i}^2$ implies that (3) is a projector. So all eigenvalues of e_i are either 0 or 1. We write $\rho_\lambda^{(k,r)}$ for the restriction of $\pi_\lambda^{(k,r)}$ to B_n. When n and r are fixed, they may be suppressed.

Definition 2. *Given a pair of integers (k,r) with $r \geq k+1$. Let $\Lambda_n^{(k,r)}$ be the set of all (k,r)-admissible Young diagrams with n boxes.* The Jones-Wenzl representation *of the braid group* B_n *is:*

$$\rho_n^{(k,r)} = \oplus_{\lambda \in \Lambda_n^{(k,r)}} \rho_\lambda^{(k,r)} : B_n \to \prod_{\lambda \in \Lambda_n^{(k,r)}} U(\lambda).$$

Here we write $U(\lambda)$ *for the unitary group of the Hilbert space* $V_\lambda^{(k,r)}$ *with the orthonormal basis* $\{\vec{v}_t\}$*.*

Definition 3. *A* (k,r)*-admissible diagram is of* trivial type *if* λ *is a row or column or if* $k = r-1$*. A* (k,r)*-admissible diagram is a* hook *if the second row has exactly one box. A hook with exactly two rows is a* Burau hook*, and the corresponding sector is a* Burau representation*.*

We note that ρ_λ is one-dimensional if and only if λ is of trivial type.

Theorem 2.2. *Let* h *be a* (k,r)*-admissible hook with* $(b+1)$ *rows and* $(a+1)$ *columns.*

(1) If $a+b < r-1$*, then* $\rho_h^{(k,r)}$ *is equivalent up to tensoring by a character to the* b*th exterior power of the Burau representation associated to the hook with* $(a+b)$ *columns.*

(2) If $a+b = r-1$*, then* $\rho_h^{(k,r)}$ *is equivalent up to tensoring by a character to the* $(b-1)$*th exterior power of the Burau representation associated to the hook with* $(a+b-1)$ *columns.*

Proof: For the first part, we explicitly identify a basis of V_h with that of the b-th exterior power of the Burau representation ρ_β associated to the hook β with $(a+b)$ columns. The basis of V_β can be indexed conveniently by the entry i of the box in the second row. The set

$$\{v_{i_2} \wedge v_{i_3} \wedge \cdots \wedge v_{i_{b+1}} \mid 2 \leq i_2 < \cdots < i_{b+1} \leq a+b+1\}$$

spans $\wedge^b V_\beta$. We identify each element of this basis with the basis element of V_h given by the standard tableau whose first column entries are $1, i_2, \cdots, i_{b+1}$, which we denote $v_{1,i_2,\cdots,i_{b+1}}$. Now we just compare the action of the braid generator σ_k on corresponding basis elements: $v_{1,i_2,\cdots,i_{b+1}}$ and $v_{i_2} \wedge v_{i_3} \wedge \cdots \wedge v_{i_{b+1}}$. For the Burau representation, we have $\rho_\beta(\sigma_k)(v_i) = \quad qv_i \quad \text{if} \quad i \neq k, k+1$. We drop ρ from the notation now. First we compare two special cases:

$$\sigma_k(v_{1,i_2,\cdots,i_{b+1}}) = \begin{cases} q & \text{if } k \text{ and } k+1 \text{ do not appear in } i_2, \ldots, i_{b+1} \\ -1 & \text{if } k \text{ and } k+1 \text{ both appear in } i_2, \ldots, i_{b+1} \end{cases}$$

$$\sigma_k(v_{i_2} \wedge \cdots \wedge v_{i_{b+1}}) = \begin{cases} q^b & \text{if } k \text{ and } k+1 \text{ do not appear in } i_2, \ldots, i_{b+1} \\ -q^{b-1} & \text{if } k \text{ and } k+1 \text{ both appear in } i_2, \ldots, i_{b+1} \end{cases}$$

There are two remaining cases: k appears in $\{i_2, \cdots, i_{b+1}\}$ but $k+1$ not, or $k+1$ appears in $\{i_2, \cdots, i_{b+1}\}$ but k not. Note for both cases, the hook distance between k and $k+1$ in the two hooks h and β is the same $\mp k$. Therefore, the action of σ_k on the respective 2-dimensional subspace is the same. Since there are $(b-1)$ basis elements $v_i, i \neq k$ in $\{i_2, \cdots, i_{b+1}\}$, we have a factor of q^{b-1} when comparing to the action of σ_k on $v_{i_2} \wedge \cdots \wedge v_{i_{b+1}}$.

The second part is proved similarly. The admissibility condition for standard Young tableaux reduces the rank by 1. $\square$

In general, Jones-Wenzl sectors $\rho_\lambda^{(k,r)}$ have the following properties:

Theorem 2.3. *Let λ be an admissible Young diagram which is not of trivial type.*

(1) For each i, the image $\rho_\lambda^{(k,r)}(\sigma_i)$ has exactly two distinct eigenvalues, -1 and q.

*(2) (**Bratteli** diagram) Given a (k,r)-admissible Young diagram λ with n boxes, then the restriction of $\rho_\lambda^{(k,r)}$ from B_n to B_{n-1} is the direct sum of the irreducible representations associated to all (k,r)-admissible Young diagrams λ' of size $n-1$ obtained from λ by removing a single corner box.*

(3) If $r \geq 5$ and $r \notin \{6, 10\}, n \geq 3$, or $r = 10, n \geq 5$, then the image group of $\rho_\lambda^{(k,r)}(B_n)$ is infinite modulo its center.

All three statements are in [J2]. The first is obvious from the construction given above. One can easily deduce (3) from (1) and (2) given Theorem 1.6.

3 Duality of Jones-Wenzl representations

The Hecke algebra $H_n(q)$ has an automorphism which intertwines the Jones-Wenzl representations of $H_n(q)$ associated to a pair of Young diagrams. This duality was first discovered by F. Goodman and H. Wenzl [GW] and by A. Kuniba and T. Nakanishi [KN]. It is called *rank-level* duality in conformal field theory. This duality accounts for the appearance of the symplectic and orthogonal groups as closed images of certain Jones-Wenzl representations.

Let $\mathbb{N}$ denote the set of natural numbers (including 0).

Definition 4. *Fix an integer $r > 0$. An* r-tile *is a $k \times (r-k)$ matrix $T = (t_{ij})_{k \times r-k}$ satisfying the following conditions:*

(1) $t_{ij} \in \mathbb{N}$,

(2) the entries in each row and column are non-increasing,

(3) the difference of any two entries in a single row or column is ≤ 1.

The relation between r-tiles and (k,r)-admissible Young diagrams is given by the following constructions.

The r-tile T_λ of a Young diagram λ: Suppose $\lambda = [\lambda_1, \cdots, \lambda_k]$ is a Young diagram with k rows and $r \geq k+1$. Let $l = r-k$, and let T_λ be the $k \times l$ matrix with

$$t_{ij} = \left\lfloor \frac{\lambda_i + l - j}{l} \right\rfloor.$$

The Young diagram λ_T of an r-tile T: the (k,r)-admissible Young diagram λ_T is a Young diagram with at most k rows whose ith row has $\sum_{j=1}^{l} t_{ij}$ boxes.

Definition 5. *(1) Given a (k,r)-admissible Young diagram λ, the* r-conjugate of *λ, denoted λ_r^*, is the Young diagram associated with the transpose tile of T_λ.*

(2) A Young diagram is r-symmetric *if T_λ is a symmetric matrix after discarding all 0-rows and 0-columns.*

(3) Given a Young tableau t of shape λ, the r-conjugate *t^* is the tableau of shape λ_r^* such that the shape of $t^{(i)}$ is r-conjugate to the shape of $t^{*(i)}$ for all i.*

We have the following duality:

Theorem 3.1. *For any (k,r)-admissible Young diagram λ, $\rho_{\lambda_r^*}$ is equivalent to $\chi \otimes \rho_\lambda^*$, where ρ_λ^* is the contragredient representation of ρ_λ and $\chi : B_n \to U(1)$ denotes the character with $\chi(\sigma_i) = -q$.*

Proof: We describe this duality explicitly in terms of bases. From the definition of the representations ρ_λ and $\rho_{\lambda_r^*}$, the basis elements of the representation spaces V_λ and $V_{\lambda_r^*}$ are in 1-1 correspondence by r-conjugation of Young tableaux: $t \leftrightarrow t^*$. We define the *duality transformation* J as the linear map $J : V_\lambda \to V_{\lambda_r^*}$ with $J(\vec{v}_t) = \pm \vec{v}_{t^*}$, where the sign $\pm$ is determined as follows. Let t_0 be the standard vertical tableau of shape λ. This is the tableau in which numbers 1 through n are filled in one column at a time, working left to right, and it is not necessarily admissible. Each standard tableau t of shape λ determines a permutation of $\{1, 2, \cdots, n\}$ by comparison to t_0. The sign $\pm$ is the sign of this permutation.

We show that $\rho_{\lambda_r^*} = \chi \otimes \rho_\lambda^*$ for each braid generator σ_i. Given a standard tableau t, there are two cases depending on whether or not $s_i(t)$ is standard. If $s_i(t)$ is not standard, then the proof is straightforward. If $s_i(t)$ is standard, then

$$\rho_\lambda(\sigma_i) = \begin{pmatrix} q-(1+q)\alpha_{t,i} & -\beta_{t,i} \\ -\beta_{t,i} & q-(1+q)(1-\alpha_{t,i}) \end{pmatrix}.$$

Note that $d_{t^*,i,i+1} = -d_{t,i,i+1}$, therefore $\alpha_{t^*,i} = 1 - \alpha_{t,i}$. Since $\det(\rho_\lambda(\sigma_i)) = -q$, we have

$$\rho_\lambda(\sigma_i) = (-q) \cdot \frac{1}{\det(\rho_\lambda(\sigma_i))} \cdot \rho_\lambda(\sigma_i) = \chi \cdot \rho_\lambda^{-1}(\sigma_i) = \chi \otimes \rho_\lambda^*(\sigma_i).$$

$\square$

Corollary 3.2. *(1) If λ is r-symmetric, then $\dim V_\lambda$ is even.*

(2) If λ is r-symmetric, then ρ_λ is self-dual up to the character χ. More precisely, suppose $T = (t_{ij})$ is the r-tile of λ, then if $\sum_{i>j} t_{ij}$ is odd, ρ_λ is symplectic up to χ, and if $\sum_{i>j} t_{ij}$ is even, ρ_λ is orthogonal up to χ.

Proof: Let us examine more carefully the matrix J representing the above duality. First note that r-conjugation is an involution on the basis elements of V_λ without any fixed points as long as λ has ≥ 2 boxes. This implies (1). If the sign of t is the same as that of t^*, then J is either $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Therefore, J defines an orthagonal pairing. If the signs of t and t^* are different, then J is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so J defines a symplectic pairing. As $\rho_\lambda \cdot J^{-1} = \chi \otimes \rho_\lambda^*$, up to the character χ, ρ is either a symplectic or an orthogonal matrix with respect to either the symplectic form or inner product given by J^{-1}. Checking signs gives (2). $\square$

The converse of (2) is also true for $r > 4$. This is a slight refinement of a result of [GW], and we follow the proof given there.

Theorem 3.3. *Let $r > 4$ and $1 < k_1, k_2 < r - 1$.*

(1) Let $\lambda_1 \in \Lambda_n^{(k_1,r)}$ and $\lambda_2 \in \Lambda_n^{(k_2,r)}$. If λ_i are not of trivial type, then ρ_{λ_1} is equivalent to the tensor product of ρ_{λ_2} with a character of B_n if and only if $\lambda_1 = \lambda_2$.

(2) Let $\lambda_1 \in \Lambda_n^{(k_1,r)}$ and $\lambda_2 \in \Lambda_n^{(k_2,r)}$. If λ_i are not of trivial type, then ρ_{λ_1} is equivalent to the tensor product of $\rho^*_{\lambda_2}$ with a character of B_n if and only if $\lambda_1 = (\lambda_2)^*_r$.

Proof. For any pair of distinct diagrams λ_1 and λ_2, the sets of diagrams of the form $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ cannot coincide. In other words, there exists an admissible subdiagram μ of one of the two, obtained by removing a single box, which cannot be so obtained from the other. Unless one or both is the Burau hook $[n-1,1]$ or its conjugate, μ is not of trivial type. If ρ_{λ_1} and ρ_{λ_2} are equivalent up to tensoring by a character, the same is true of their restrictions to B_{n-1}. We may therefore proceed by induction, the base case being that in which either λ_1 or λ_2 is $[n-1,1]$ and the other is $[2,1,\dots,1]$. These are not equivalent for $n \geq 4$ by Theorem 2.2.

Part (2) is an immediate consequence of (1) and Theorem 3.1. $\square$

4 Closed images of Jones-Wenzl sectors

In this section, we compute the universal cover G_1 of the identity component G_0 of the closure of $\rho_\lambda(B_n)$ for each ρ_λ with infinite image. We also give the ambient representation V of G_0 (specified as a representation of G_1.) Since $\overline{\rho_\lambda(B_n)}$ is the product of G_0 and a group of scalar matrices, this is enough information to determine the actual closure of the image of the sector.

Theorem 4.1. *Fix integers r, n such that $r \geq 5$, $r \neq 6$, and $n \geq 3$. Let k be an integer less than $r-1$ and let $\lambda \in \Lambda_n^{(k,r)}$. We assume that λ is not of trivial type, and if $r = 10$, we assume that λ is neither $[2,1]$ nor $[2,2]$. Let G_1 denote the universal cover of the identity component of the closure of $\rho_\lambda(B_n)$ and V, of dimension N, denote the representation space of ρ_λ regarded as a G_1-module. Then*

(1) if λ is neither r-symmetric nor a hook, then (G_1, V) is equivalent to $(SU(N), V_{\varpi_1})$.

(2) if λ is a hook with $a+1$ columns and $b+1$ rows, then (G_1, V) is equivalent to $(SU(a+b), V_{\varpi_b})$.

(3) if λ is not a hook but is r-symmetric, $T_\lambda = (t_{ij})$ is the r-tile of λ, and $\Sigma = \sum_{i>j} t_{ij}$, then

if Σ is even, then (G_1, V) is equivalent to $(Spin(N), V_{\varpi_1})$;
if Σ is odd, then (G_1, V) is equivalent to $(Sp(N), V_{\varpi_1})$

The rest of the section is devoted to the proof of this theorem. We remark that the excluded cases, $r \in \{3, 4, 6\}$, $r = 10$ and $\lambda \in \{[2,1], [2,2]\}$, or λ of trivial type, are precisely the cases in which the image was already known to be finite [J2][BW][GJ].

We have already seen that $\rho_\lambda(\sigma_i)$ has two distinct eigenvalues whose ratio $-q$ is not -1. Since the braid generators are all conjugate to one another, the conjugacy class of $\rho_\lambda(\sigma_i)$ topologically generates the closure of $\rho_\lambda(B_n)$. Thus, G_1 is simple, V is irreducible with highest weight ϖ, and (G_1, ϖ) appears on the list given in Theorem 1.1.

Definition 6. *A pair (G_1, V) consisting of a simply connected simple Lie group and an irreducible representation is* standard *if G_1 is isomorphic to $SU(N)$, $Sp(N)$, or $Spin(N)$, and* $\dim V = N$.

Our main goal is to show that the pairs (G_1, V) arising from diagrams which are not hooks are standard. We rule out the other possibilities offered by Theorem 1.1 by means of two pieces of information: $\dim V$, and the closure of B_{n-1} in G_0, as computed by means of the Bratteli diagram. In order to start the induction argument, we need to compile results in a number of special cases. We begin with hooks.

Proposition 4.2. *Theorem 4.1 holds for all hooks λ.*

Proof: By Theorem 2.2, it suffices to consider the case of Burau hooks $\lambda = [m, 1]$. We use induction on m. For $m = 2$ (resp. $m = 3$), we can appeal to Theorem 1.6 or to classical results characterizing all finite subgroups of $GL(2)$ (resp. $GL(3)$) [Ft] to show that $G_0 = G_1 = SU(2)$ (resp. SU(3)) except when $m = 2$ and $r = 10$. For general $m < r$, $\dim \rho_{[m,1]} = m$, and by the induction hypothesis, $G_0 \supset SU(m-1)$, so $G_0 = G_1 = SU(m)$. $\square$

We now consider diagrams λ with ≤ 7 boxes which are neither hooks nor of trivial type. For $n = 4$, $\lambda = [2, 2]$, and $\dim \rho_\lambda = 2$, so $G_1 = SU(2)$, except when $r = 10$, in which case G_1 is trivial. For $n = 5$, there are two possible diagrams, and

$$\dim \rho_{[3,2]} = \dim \rho_{[2,2,1]} = 5,$$

and by Theorem 1.1, $G_1 = SU(5)$ in each case. This is enough information for the induction argument when $r = 5$, so we now restrict attention to

$r \geq 7$. For $n = 6$, the diagrams $[4,2]$, $[3,3]$, $[3,2,1]$, $[2,2,2]$, and $[2,2,1,1]$ give sectors of dimensions 9, 5, 16, 5, and 9 respectively. Thus, (G_1, V) is obviously standard for each case except the symmetric diagram $[3,2,1]$, which contains the admissible subdiagram $[2,2,1]$. In this case, therefore, G_0 contains $SU(5)$. It follows that here again, the pair is standard. For $n = 7$, we have $[5,2]$, $[4,3]$, $[4,2,1]$, and $[3,2,2]$ together with their conjugates; the dimensions are 14, 14, 35, and 21 respectively, so Theorem 1.1 implies all are standard. For $n \geq 8$, $\lambda \in \{[4,4],[2,2,2,2]\}$ gives $\dim \rho_\lambda = 14$ and (G_1, V) standard, and otherwise, $\dim \rho_\lambda > 15$.

We can already prove the main theorem in the case that $r = 5$. Indeed, every λ with three rows is 5-conjugate to one with two, so we consider only diagrams of the form $[l,m]$, $0 \leq l - m \leq 3$. By a Bratteli diagram computation,

$$\dim \rho_{[l,m]} = \begin{cases} F_{2m-1} & \text{if } l = m, \\ F_{2m+1} & \text{if } l = m+1, \\ F_{2m+2} & \text{if } m+2 \leq l \leq m+3, \end{cases}$$

where F_k denotes the kth Fibonacci number. If $\dim V = F_{k+1}$ and $G_0 \supset SU(F_k)$, then $G_0 = G_1 = SU(F_{k+1})$, so the theorem follows by induction on k.

The general proof of the theorem follows this strategy but is technically more difficult. We assume henceforth that $r \geq 7$.

Lemma 4.3. *The pair $(Spin(8), 8)$ never appears among pairs $(G_1, \dim V)$. The pairs $(SU(5), 10)$, and $(SU(6), 15)$ occur only when λ is a hook.*

Proof: We know already that as λ ranges over diagrams which are not hooks, $\dim \rho_\lambda$ is never 8, 10, or 15. When λ is a hook, G_1 is always a special unitary group. □

Lemma 4.4. *Let $\Lambda \subset \bigcup_k \Lambda_n^{(k,r)}$ denote a set of diagrams. Suppose that for each $\lambda \in \Lambda$, the corresponding pair (G_1, V) is standard. Let ρ_Λ denote the direct sum of the representations ρ_λ, $\lambda \in \Lambda$. Then*

$$\operatorname{rank}(\overline{\rho_\Lambda(B_n)}^0) \geq \frac{\dim \rho_\Lambda}{3}. \tag{4}$$

Proof: Let Λ' denote a maximal subset of Λ containing no two r-conjugate diagrams. Let H_λ denote the quotient of $\overline{\rho_\lambda(B_n)}$ by its center. This is always a simple group, either $PSU(N)$, $PSO(N)$, or $PSp(N)$. The closure of the direct sum $\rho_\lambda \oplus \rho_\mu$ maps to $H_\lambda \times H_\mu$, and its image maps onto each factor. By Goursat's Lemma, either the image is the graph of an isomorphism between H_μ and H_λ, or it is the whole product. Up to isomorphism, $PSU(N)$ has exactly two non-trivial N-dimensional projective representations, and they are dual to one another. By Theorem 3.3, if $\lambda, \mu \in \Lambda'$, there cannot be an isomorphism $H_\lambda \to H_\mu$ commuting with the maps from B_n, in the $PSU(N)$ case. There is only one isomorphism class of non-trivial projective N-dimensional representations of $PSp(N)$, and the same is true for $PSO(N)$ when $N \geq 6$ and $N \neq 8$. Thus, again there cannot be an isomorphism $H_\lambda \to H_\mu$ commuting with the maps from B_n. By Goursat's lemma, we conclude that the closure of $\rho_{\Lambda'}(B_n)$ maps onto $\prod_{\lambda \in \Lambda'} H_\lambda$. The same is true a fortiori of the closure of $\rho_\Lambda(B_n)$. If λ is not r-symmetric, then H_λ has rank $N-1 \geq 2$, and the sum of the dimensions of ρ_λ and $\rho_{\lambda^*_r}$ is $2N \leq 3(N-1)$. Otherwise the rank of ρ_λ is $N/2$ and the contribution of λ to $\dim \rho_\Lambda$ is N. Thus, $\dim \rho_\Lambda$ is at most 3 times the rank of $\rho_\Lambda(B_n)$. □

We note that among pairs (G, V) satisfying Theorem 1.1, the only non-standard ones satisfying

$$\operatorname{rank} G \leq \frac{\dim V}{3}$$

are $Spin(7)$ with its spin representation and $SU(4)$ and $SU(5)$ with their fundamental representations of dimensions 6 and 10 respectively. By Lemma 4.3, these cases are ruled out for pairs arising from $\rho_\lambda(B_n)$. We cannot proceed immediately by induction, however, since the base cases, which are the hooks, do not in general satisfy the inequality (4). To remedy this, we need to analyze partitions λ from which hooks can be obtained by removing a single box. We therefore define

$$h_{a,b} = [a+1, \underbrace{1, 1, \ldots, 1}_{b}], \;\; \lambda_{a,b} = [a+1, 2, \underbrace{1, \ldots, 1}_{b-1}].$$

Note that the admissibility of $\lambda_{a,b}$ implies the admissibility of $h_{a,b}$ except in the case $a = r-2, b = 1$.

Proposition 4.5. *If $a + b \geq 5$ and $h_{a,b}$ is admissible, then*

$$\dim \rho_{\lambda_{a,b}} \geq (14/5) \dim \rho_{h_{a,b}}.$$

Proof: Either $a + b < r - 1$ and $h_{a,b}$ has two admissible subdiagrams with $a + b$ boxes, $h_{a-1,b}$ and $h_{a,b-1}$, or $a + b = r - 1$ and there is only one:

$h_{a-1,b}$. In the first case and if $b > 1$, $\lambda_{a,b}$ has three admissible subdiagrams with $a + b + 1$ boxes, $h_{a,b}$, $\lambda_{a-1,b}$, and $\lambda_{a,b-1}$; in the second or if $b = 1$, only the first two are admissible. We proceed by induction, the proposition being true in the case $a + b = 5$ and sharp when $(a, b) = (4, 1)$. Suppose that n is given and the proposition is true when $a + b = n - 1$. Now take $a + b = n$. In the first case, if $b > 1$,

$$\begin{aligned}\dim \rho_{\lambda_{a,b}} &= \dim \rho_{\lambda_{a-1,b}} + \dim \rho_{\lambda_{a,b-1}} + \dim \rho_{h_{a,b}} \\ &\geq (14/5)(\dim \rho_{h_{a-1,b}} + \dim \rho_{h_{a,b-1}}) + \dim \rho_{h_{a,b}} = (19/5) \dim \rho_{h_{a,b}},\end{aligned}$$

while if $b = 1$, than $a \geq 4$, so

$$\dim \rho_{\lambda_{a,1}} = \frac{a^2 + 3a}{2} \geq \frac{14(a + 1)}{5} = \frac{14}{5} \dim \rho_{h_{a,1}}.$$

In the second case,

$$\begin{aligned}\dim \rho_{\lambda_{a,b}} &= \dim \rho_{\lambda_{a-1,b}} + \dim \rho_{h_{a,b}} \geq (14/5) \dim \rho_{h_{a-1,b}} + \dim \rho_{h_{a,b}} \\ &= (19/5) \dim \rho_{h_{a,b}}.\end{aligned}$$

□

Proposition 4.6. *For any $a, b \geq 1$, $\lambda_{a,b}$ satisfies Theorem 4.1.*

Proof: By the case analysis following Proposition 4.2, we may take $a+b = n \geq 6$, and we may assume the proposition is true when $a + b < n$. The induction hypothesis gives $\operatorname{rank} G_1 \geq 13$. Applying Lemma 4.4 to $\lambda_{a-1,b}$ and (assuming $b > 1$ and $a + b < r - 1$) $\lambda_{a,b-1}$, the induction hypothesis together with Lemma 4 implies that the rank of G_1 is at least $3/14$ times the dimension of the representation. Among the possible pairs (G_1, V) in Theorem 1.1, only the standard ones satisfy both conditions. By Lemma 3.2, G_1 is unitary, spin, or orthogonal, depending on which of the conditions in Theorem 4.1 $\lambda_{a,b}$ satisfies. The proposition follows by induction on n. □

We can now prove Theorem 4.1.

Proof: We use induction on n. We may assume that λ is not a hook and that for every admissible tableau with shape λ, neither is $\lambda^{(1)}$. Let Λ denote the set of admissible diagrams of the form $\lambda^{(1)}$ for some admissible tableau. By inequality (4),

$$\operatorname{rank} \overline{\rho_\lambda(B_n)} \geq \operatorname{rank} \overline{\rho_\Lambda(B_{n-1})} \geq \frac{\dim \rho_\lambda}{3}.$$

By Lemma 4.3, this inequality together with the fact that λ is not a hook implies that the pair (G_1, V) arising from ρ_λ is standard. The theorem follows by induction. □

For completeness, we point out the closed images of the remaining cases using Theorem 1.6. They have all been identified earlier in [J2][BW][GJ]. As we mentioned earlier, they are all finite groups. The images for $SU(2), r = 4$ are given by Theorem 1.6, (b) [J2]; $SU(2), r = 6$ by Theorem 1.6, (c) cases $(1), (2), (6)$ [BW]; $SU(2), r = 10$ and $n = 3, 4$ by Theorem 1.6, (a) [J2]; The images for $SU(3), r = 6$ are identified first by D. Goldschmidt and V. Jones (see [GJ]), the images are given by Theorem 1.6, (c) cases $(3), (5)$. The images for $SU(4), r = 6$ are the same as those for $SU(2), r = 6$ by rank-level duality.

5 Distribution of evaluations of Jones polynomials

In this section, we fix an integer $r \geq 3, r \neq 3, 4, 6$, and $q = e^{\pm \frac{2\pi}{r}}$. Given a braid $\sigma \in B_n$, let $\hat{\sigma}$ be the usual closure of σ. Then the Jones polynomial of the link $\hat{\sigma}$ at q is:

$$J(\hat{\sigma}, q) = (-1)^{n-1+e(\sigma)} \cdot q^{-\frac{3e(\sigma)}{2}} \cdot \sum_{\lambda=[\lambda_1,\lambda_2]\in\Lambda_n^{(2,r)}} \frac{[\lambda_1 - \lambda_2 + 1]}{[2]} \cdot Tr(\rho_\lambda^{(2,r)}(\sigma)),$$

where $e(\sigma)$ is the sum of all exponents of standard braid generators appearing in σ. In the following, we denote $\frac{[\lambda_1-\lambda_2+1]}{[2]}$ by w_λ.

The sum of exponents $e(\sigma)$ defines a homomorphism from B_n to $\mathbb{Z}$. Let ρ denote the direct sum of the representations ρ_λ as λ ranges over $\Lambda_n^{(2,r)}$. Let $G = \overline{\rho(B_n)} \times \mathbb{Z}_{2r}$. There is a natural map $\rho' : B_n \to G$ defined by $\rho'(\sigma) = (\rho(\sigma), r(n - 1 + e(\sigma)) - 3e(\sigma) \pmod{2r})$. Let

$$T_n : \Big(\prod_{\lambda\in\Lambda_n^{(2,r)}} U(\lambda)\Big) \times \mathbb{Z}_{2r} \to \mathbb{C}$$

be defined by

$$T_n((u_\lambda), m) = q^{\frac{m}{2}} \sum_{\lambda \in \Lambda_n^{(2,r)}} w_\lambda Tr(u_\lambda).$$

The definitions are designed so that

$$J(\hat{\sigma}, q) = T_n(\rho'(\sigma)).$$

Let $G' \subset G$ denote the closure of $\rho'(B_n)$.

Lemma 5.1. *If $n \geq 5$, then*

$$(G')_0 = \prod_{\lambda \in \Lambda_n^{(2,r)}} SU(\lambda),$$

and $G' = (G')_0 Z(G')$.

Proof: As $n > 4$, a diagram with two rows cannot be symmetric, nor can two distinct diagrams with two rows be conjugate to one another. The computation of $(G')_0$ now follows immediately from the proof of Lemma 4.4. As G' is a subgroup of

$$\Big(\prod_{\lambda \in \Lambda_n^{(2,r)}} \overline{\rho_\lambda(B_n)} \Big) \times \mathbb{Z}_{2r}$$

and has the same identity component, it suffices to prove that the latter group is the product of its identity component and its center. This is immediate from Theorem 1.4. ☐

Lemma 5.2. *Let $\mu_{n,k}$ denote probability measure on $\mathbb{C}$ given by values of $J(\hat{\sigma}, q)$, if σ is chosen randomly and uniformly from (non-reduced) words of length k in the braid generators $\sigma^{\pm}, \ldots, \sigma_{n-1}^{\pm} \in B_n$. The weak-* limit of $\mu_{n,k}$ as $k \to \infty$ is the push-forward of Haar measure on G', $T_{n*} dg'$.*

Proof: Let ν denote the probability measure on G' given by the average of δ-functions centered at $\rho'(\sigma_1)^{\pm}, \ldots, \rho'(\sigma_{n-1})^{\pm}$. By [Bh], since $\rho'(B_n)$ is dense in G', the weak-* limit of the k-fold convolution ν^{*k} is Haar measure dg'. Thus the weak-* limit of $T_{n*}(\nu^{*k})$ is $T_{n*} dg'$. ☐

The only significance of the choice of the set $\{\sigma_i^{\pm}\}$ is that it generates B_n; any other semigroup generators would do as well. Much more sophisticated results in ergodic theory can be applied to prove convergence of measure on more refined ensembles of braids. For example, the Stein-Nevo theorem [SN] allows the study of reduced words in the free group. If μ_r and μ_{r+1} are measures uniformly supported on reduced words in $\gamma_1, \cdots \gamma_m$ and their inverses, then $\frac{1}{2}(\mu_r + \mu_{r+1})$ will also converge weakly to Haar(G'). One may also ask about using the braid group—not the free group—to count braids and whether a similar uniformity is obtained. We do not know at present.

Lemma 5.3. *If $n \geq r - 2$, then*

$$\sum_{\lambda \in \Lambda_n^{(2,r)}} w_\lambda^2 = \frac{r}{\sin^2 \frac{2\pi}{r}}.$$

Proof: There are four cases, depending on the parity of n and r. If both are even, the sum in question is

$$[2]^{-2} \sum_{k=0}^{r/2-1} [2k+1]^2 = (q - q^{-1})^{-2} \sum_{k=0}^{r/2-1} \left(q^{2k+1} + q^{-1-2k} - 2\right) = \frac{r}{\sin^2 \frac{2\pi}{r}}.$$

If r is even and n is odd, the sum is

$$[2]^{-2} \sum_{k=0}^{r/2-2} [2k+2]^2 = (q - q^{-1})^{-2} \sum_{k=0}^{r/2-2} \left(q^{2k+2} + q^{-2-2k} - 2\right) = \frac{r}{\sin^2 \frac{2\pi}{r}}.$$

If r is odd and n is even, the sum is

$$[2]^{-2} \sum_{k=0}^{r/2-3/2} [2k+2]^2 = (q - q^{-1})^{-2} \sum_{k=0}^{r/2-3/2} \left(q^{2k+2} + q^{-2-2k} - 2\right) = \frac{r}{\sin^2 \frac{2\pi}{r}}.$$

Finally, if both are odd,

$$[2]^{-2} \sum_{k=0}^{r/2-3/2} [2k+1]^2 = (q - q^{-1})^{-2} \sum_{k=0}^{r/2-3/2} \left(q^{2k+1} + q^{-1-2k} - 2\right) = \frac{r}{\sin^2 \frac{2\pi}{r}}.$$

$\square$

The fact that $\sum_\lambda w_\lambda^2$ does not depend on the parity of n has the interesting consequence that the distribution of values of J on braids of n strands tends to a limit as n goes to ∞:

Theorem 5.4. *The weak-* limit of the sequence of measures $T_{n*}dg'$ is the Gaussian distribution $\frac{1}{2\pi\sigma_r}e^{-\frac{z\bar{z}}{\sigma_r}}dzd\bar{z}$, where $\sigma_r = \frac{r}{\sin^2 2\pi/r}$.*

Proof: By Lemma 5.1, we can write $G' = (H \times A)/H \cap A$, where H is a product of special unitary groups and A is finite and abelian. Every representation of G' can be regarded as a representation of $H \times A$ and every irreducible representation as an exterior tensor product of an irreducible representation of H and an irreducible character of A. In particular, the restriction of T_n to G' can be regarded as a function on $H \times A$: namely a w_λ-weighted sum of traces of representations $\sigma_\lambda \boxtimes \tau_\lambda$, where σ_λ is the composition of the standard representation with the projection onto the factor $SU(\lambda)$ of H.

Let $N = \inf_{\lambda\in\Lambda_n^{(2,r)}} \dim \rho_\lambda$. If a_λ, b_λ are non-negative integers with

$$\sum_{\lambda\in\Lambda_n^{(2,r)}} (a_\lambda + b_\lambda) < N,$$

then

$$\bigotimes_{\lambda\in\Lambda_n^{(2,r)}} (\sigma_\lambda \boxtimes \tau_\lambda)^{\otimes a_\lambda} \otimes ((\sigma_\lambda \boxtimes \tau_\lambda)^{*\otimes b_\lambda}$$

is isotypic on $Z(H)$ and non-trivial unless $a_\lambda = b_\lambda$ for all λ. In this case, the representation is trivial on A, so the dimension of the space of invariants is

$$\dim \left(\bigotimes_{\lambda\in\Lambda_n^{(2,r)}} \sigma_\lambda^{\otimes a_\lambda} \otimes \sigma_\lambda^{*\otimes a_\lambda} \right)^H = \prod_{\lambda\in\Lambda_n^{(2,r)}} a_\lambda!$$

by the invariant theory of $SU(\lambda)$ [Wl].

Let $\{X_\lambda\}$ denote a set of independent Gaussian random variables with distribution $\frac{1}{2\pi}e^{-z\bar{z}}dzd\bar{z}$ indexed by $\lambda \in \Lambda_n^{(2,r)}$. The expectation is

$$E(X_\lambda^a \bar{X}_\lambda^b) = \begin{cases} a! & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Since all X_λ, $\lambda \in \Lambda_n^{(2,r)}$, are independent, if

$$X = \sum_{\lambda\in\Lambda_n^{(2,r)}} w_\lambda X_\lambda,$$

then

$$E(X^a \bar{X}^b) = \int_{G'} T_n(g')^a \overline{T_n(g')}^b dg' = \int_{\mathbb{C}} z^a \bar{z}^b T_{n*} dg'$$

whenever $a + b < N$. As N goes to ∞ with n, by [Fe], this implies that each moment of $T_{n*}dg'$ equals the corresponding moment of the measure $\frac{1}{2\pi\sigma_r} e^{-\frac{z\bar{z}}{\sigma_r}} dz d\bar{z}$ of X when n is sufficiently large. This implies weak convergence by [Fe] VIII.6 and XV.5. (Actually, the results in [Fe] are stated only for distributions on $\mathbb{R}$, but the method works for $\mathbb{R}^n$.) ☐

We conclude that if r is a fixed integer $r \geq 5$, $r \neq 6$, then in the limit as $n \to \infty$, the distribution of values at $e^{\frac{\pm 2\pi i}{r}}$ of the Jones polynomial of a "random" link with n strands tends to a fixed Gaussian. The variance of this Gaussian depends on r and grows like r^3 as $r \to \infty$.

Theorem 5.5. *For each n and k, let $\mu_{n,k}^{\mathrm{knot}}$ denote the distribution of values of $J(\hat{\sigma}, e^{2\pi i/r})$, where σ ranges over those non-reduced words of length k in B_n for which $\hat{\sigma}$ is a knot. If $r = 5$ or $r \geq 7$, then in the weak-* topology,*

$$\lim_{n\to\infty} \lim_{k\to\infty} \mu_{n,k}^{\mathrm{knot}} = \frac{1}{2\pi\sigma_r} e^{-\frac{z\bar{z}}{\sigma_r} dz d\bar{z}}, \ \sigma_r = \frac{r}{\sin^2 \frac{2\pi}{r}}.$$

Proof: A braid σ gives rise to a knot $\hat{\sigma}$ if and only if the image of σ under the standard quotient map $B_n \to S_n$ is an n-cycle. For each $n \geq 5$ we consider the homomorphism $\phi : B_n \to G' \times S_n$ obtained from ρ' and the standard quotient map $B_n \to S_n$. By Goursat's lemma, the closure of the image is either all of $G' \times S_n$ or an index-2 subgroup. Applying [Bh] to the topological generators $\phi(\sigma_i^{\pm 1})$ of this subgroup, we see that in the large k limit, if we condition on a fixed element of S_n, the resulting distribution on G' approaches one of three possible limits: Haar measure dg' on G', twice the restriction of dg' to an index-2 subgroup $G'_{\mathrm{even}} \subset G'$, or twice the restriction of dg' to the non-trivial coset $G'_{\mathrm{odd}} = G' \setminus G'_{\mathrm{even}}$. (Note that the factor of 2 is needed in the last two cases to give a probability measure.) The argument of Lemma 5.1 goes through unchanged when G' is replaced by G'_{even}, so the integral of $z^a \bar{z}^b$ with respect to $T_{n*} dg'_{\mathrm{even}}$ coincides with the integral with respect to $T_{n*}dg'$ when $a + b < N$. By additivity in measure, the decomposition

$$dg' = dg'|_{G'_{\mathrm{even}}} + dg'|_{G'_{\mathrm{odd}}} = \frac{1}{2} dg'_{\mathrm{even}} + \frac{1}{2} dg'_{\mathrm{odd}}$$

gives

$$\int z^a \bar{z}^b T_{n*} dg'_{\text{odd}} = 2\int z^a \bar{z}^b T_{n*} dg' - \int z^a \bar{z}^b T_{n*} dg'_{\text{even}} = \int z^a \bar{z}^b T_{n*} dg'$$

for $a + b < N$. The theorem now follows from [Fe]. $\square$

Remark: In [DLL], the evaluations of Jones polynomials at several roots of unity are plotted for prime knots, or prime alternating knots up to 13 crossings. While density still holds for these cases, we do not know if there exist any limiting distributions for these ensembles of knots (note that our filtration in Theorem 5.5 and their filtration for the plotting are different.)

Another interesting direction is to study subgroups of the braid groups. By [Sta], a braid b belonging to $B_k(n)$, the k-th stage of the lower central series of the braid group B_n, determines a braid closure $\hat{b}$ whose finite type invariants vanish through type $k+1$. Since the groups $SU(m)$ are simple, if $\rho : B_n \to SU(m)$ is dense then the restriction $\rho : B_k(n) \to SU(m)$ is also dense. Thus link invariants with vanishing invariants of type $\leq k+1$ can approximate the non-perturbative Jones invariants of an arbitrary link. It would be nice to follow this with a uniformity (in measure) statement, but this seems to lie outside the scope of the ergodic theorem we know since in the free group F_n, which we use to parameterize the braid group, the k-th term of the lower central series $F_k(n)$ is infinitely generated.

Let us now come to the question of the rate of approximation. Here to have any kind of general positive answer, one must restrict to semisimple Lie groups (which fortunately is where the Jones representations we have studied take their values). To see this, consider $G = S^1$ and the Liouville number $\gamma = (\sum_n 10^{-n!})2\pi$, while γ generates a dense subgroup and the atomic measure on its partial orbit converges to the rotationally invariant measure, one must wait an exceptionally long time for the orbit to come near certain points. In contrast semisimple groups have a distinctly limited supply of finite subgroups and nothing similar can occur. A theorem to this effect can be found in [Ki][So] and appears in its best form in [NC].

Theorem 5.6. *Let X be a set closed under inverse in a compact semisimple Lie group G (with Killing metrics) such that the group closure $\langle X\rangle$ is dense in G. Let X_l be the words of length $\leq l$ in X, then X_l is an ϵ-net in G for $l = \mathcal{O}(log^2(\frac{1}{\epsilon}))$, i.e., for all $g \in G$, $dist(g, X_l) < \epsilon$.*

Conjecturally the theorem should still hold for $l = \mathcal{O}(log(\frac{1}{\epsilon}))$ and there are some number theoretically special generating sets of $SU(2)$ [GJS] for which such an estimate for l can in fact be obtained. Such results now translate into topological statements:

Corollary 5.7. *Given a "conceivable" value v for the evaluation of Jones polynomial of $\hat{b}$ at a root of unity, i.e., one that lies in the computed support of the limiting distribution for $b \in B_n$, the n-string braids, to approximate v by v', $||v - v'|| < \epsilon$, it is sufficient to consider braids $b'_l \in B_n$ of length $l = \mathcal{O}(log^2(\frac{1}{\epsilon}))$ with Jones evaluations $b'_l = v'$, $||v - v'|| < \epsilon$.*

6 Fibonacci representations

In this section, we apply the techniques of sections 2 and 4 to prove a density theorem for a different class of representations. These arise from Chern-Simons theory for $r = 5$ and $G = SO(3)$, what G. Kuperberg calls the *Fibonacci TQFT* [KK].

We briefly recall the setup. The geometric objects we consider are compact oriented surfaces with boundary, not necessarily connected, endowed with a parameterization of each boundary component, i.e., a homeomorphism from S^1. Each boundary component is labeled with an element of $\{0, 2\}$. To each labeled surface Σ there is an associated finite-dimensional Hilbert space V_Σ such that

$$V_{\Sigma_1 \coprod \Sigma_2} = V_{\Sigma_1} \otimes V_{\Sigma_2}.$$

If Σ is a labeled surface and $f : S^1 \to \Sigma$ is a simple closed curve, we can cut Σ along $f(S^1)$. We call the resulting labeled surface $\Sigma_{f,a}$ if the two new boundary components are labeled a, and

$$V_\Sigma = V_{\Sigma_{f,0}} \oplus V_{\Sigma_{f,2}}. \tag{5}$$

If $Aut(\Sigma)$ denotes the group of isotopy classes of orientation, label, and parameterization preserving homeomorphisms $\Sigma \to \Sigma$, there is a natural projective unitary action on V_Σ, provided the Hilbert space in question is nonzero. The restriction of this action to the subgroup stabilizing the points of $f(S^1)$ decomposes according to equation (5). When Σ has genus 0, the projective representation lifts canonically to a linear representation.

If Σ is a disk with label a, then $\dim V_\Sigma = \delta_{0a}$. If Σ is an annulus with labels a and b, then $\dim V_\Sigma = \delta_{ab}$. When $a = b$, it makes sense to ask for the scalar given by the Dehn twist. If $a = 0$, it is 1: if $a = 2$, it is $\omega = e^{\frac{4\pi i}{5}}$. If Σ has genus 0 and 3 boundary components with labels $a, b, c \in \{0, 2\}$, then

$$\dim V_\Sigma = \begin{cases} 0 & \text{if } a+b+c = 2, \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

Lemma 6.1. *If $\Sigma_{g,m,n}$ has genus g and m (resp. n) boundary components labelled 0 (resp. 2), then*

$$\dim V_{\Sigma_{g,m,n}} = 5^{\frac{g-1}{2}} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^{g+n-1} + (-1)^{g-1} \left(\frac{1-\sqrt{5}}{2}\right)^{g+n-1} \right\}.$$

Proof: Immediate by induction. □

Note that the dimension does not depend on m: we can "cap off" a boundary component with label 0 by gluing on a disk with label 0. To simplify bookkeeping, we regard each V_Σ as a projective representation space for $P_{g,m+n}$, the pure mapping class group for a surface of genus g with $m + n$ boundary components. The representation factors through $P_{g,n}$ and is independent of m. Without abuse of notation, we may therefore denote it $\rho_{g,n}$.

Theorem 6.2. *Except when $g+n = 1$, $\rho_{g,n}(P_{g,n})$ is dense in $PU(\dim V_{\Sigma_{g,n}})$.*

The exceptional pairs $(1, 0)$ and $(0, 1)$ arise in different ways. In the first case, there is a two-dimensional projective representation whose image is known to be the icosahedral group; in the second case, there is no representation since V_Σ is 0-dimensional. The rest of this section is devoted to the proof of the theorem.

Lemma 6.3. *Theorem 6.2 holds for $(g, n) = (0, 4)$.*

Proof: We first compute explicitly the representation of this case using [KL]. The representation of a braid generator (in an appropriate basis) is $\begin{pmatrix} e^{\frac{4\pi i}{5}} & 0 \\ 0 & -e^{\frac{2\pi i}{5}} \end{pmatrix}$. The fusion matrix is $\begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\sqrt{\frac{\sqrt{5}-1}{2}} \\ -\sqrt{\frac{\sqrt{5}-1}{2}} & \frac{\sqrt{5}-1}{2} \end{pmatrix}$. It follows that any finite subgroup of $PU(2) = SO(3)$ can be ruled out quickly except the icosahedral group. For this, we compute the trace of the product of two consecutive braid generators. This trace cannot arise as the trace of an element of the binary icosahedral group in the 2-dimensional representation. Therefore, the image must be dense in $PU(2)$. □

Proposition 6.4. *If* $\dim V_{\Sigma_{g,n}} > 0$*, then* $\rho_{g,n}$ *is irreducible.*

Proof: First let $g = 0$. The proposition holds for $n \leq 4$. For $n = 5$, we have a 3-dimensional representation, so it is reducible only if it has an invariant line. Regarding $P_{0,5}$ as a quotient of the braid group B_5, we observe that σ_1, σ_2, and σ_4 must all fix the line, and all three eigenvalues must be the same, either 1 or ω. In the first case, the line is precisely the subspace of $V_{\Sigma_{0,5}}$ associated to a loop with label 0 enclosing the first two boundary components of $\Sigma_{0,5}$; it is also the subspace associated to a loop with label 0 enclosing the last two boundary components of $\Sigma_{0,5}$. However, if we cut along both loops, we are left with a pair of pants whose labels sum to 2. This is impossible by (6). On the other hand, if the eigenvalue is ω, the line in question lies in the 2-dimensional space associated to a loop with label 2 enclosing the last two boundary components of $\Sigma_{0,5}$, and this line is fixed by σ_1 and σ_2, contrary to Lemma 6.3.

Now we use induction on n. The dimension of $V_{\Sigma_{0,n}}$ is F_{n-1}, where F denotes the Fibonacci sequence. We can divide $\Sigma_{0,n}$ by a loop enclosing the last two boundary components or by a loop enclosing the last three. In the first case, we obtain a representation of the loop stabilizer which, by the induction hypothesis, is a sum of irreducible pieces of dimensions F_{n-2} and F_{n-3}. In the second case, we obtain a representation of the (different) loop stabilizer which decomposes into irreducible pieces of dimension F_{n-4} and $2F_{n-3}$. As

$$F_{n-4} < F_{n-3} < F_{n-2} < 2F_{n-3},$$

the representation of $P_{0,n}$ is irreducible.

For the higher genus case, we use a similar argument, but in this case, we choose a non-separating loop and a loop which splits off a $\Sigma_{1,1}$. In this way, we can write two different restrictions of $\rho_{g,n}$ as (projectivizations of) a direct sum of two irreducible representations in two different ways. The inequality

$$\dim V_{\Sigma_{g-1,n}} < \inf\big(\dim V_{\Sigma_{g-1,n+1}}, \quad 2\dim V_{\Sigma_{g-1,n}}\big)$$

gives the induction step whenever it holds, which means in every case except when $g + n \leq 3$. The case $(1,0)$ is well-known. For $(1,1)$ there is nothing to prove. For $(2,0)$ the decompositions $5 = 1 + 4 = 2 + 3$ are different. This leaves the cases $(1,2)$ and $(3,0)$ which can be handled in the same way as $(0,5)$ above . $\square$

We can now prove Theorem 6.2. We start with $g = 0$ and use induction. For $n = 5$, Theorem 1.1 implies the desired density. For $n \geq 6$, $F_{n-2} > \frac{F_{n-1}}{2}$, so any closed subgroup of $U(F_{n-1})$ acting irreducibly and containing $SU(F_{n-2})$ contains $SU(F_{n-1})$. Excluding the cases $(1,0)$, $(1,1)$, and $(1,2)$, in each case $g > 0$,

$$\dim V_{\Sigma_{g-1,n+2}} > \frac{\dim V_{\Sigma_{g,n}}}{2},$$

so the induction hypothesis together with irreducibility is enough to give density. For $(1,2)$, we use Theorem 1.1, and there is nothing to prove for $(1,0)$ or $(1,1)$.

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