# Provably optimal decentralised broadcast algorithms

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July 21, 2006

Technical Report
MSR-TR-2006-105

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## ABSTRACT

In this paper we consider the problem of broadcasting information from a source node to a set of receiver nodes. In the context of edge-capacitated networks, we consider the "random useful" packet forwarding algorithm. We prove that it yields a stable system, provided the data injection rate at the source node is less than the (min(min-cut)) of the graph. As a corollary we retrieve a famous theorem of Edmonds.

We next consider node-capacitated networks. In this context we introduce the "random useful to most deprived neighbour" packet forwarding scheme. We show that it yields a stable system in the particular case where the network graph is the complete graph, whenever the node capacities are large enough for centralised schemes to achieve successful broadcast of the data injection rate.


## I. INTRODUCTION

The broadcast problem consists in relaying data to a set of nodes (or users) constituting a system. The present work deals with the case where data is made available at some given rate at a single source node. In this context we want to determine distributed data forwarding algorithms such that data is eventually received by all system nodes. We consider two distinct scenarios.

In the context of edge-capacitated networks, we consider the "random useful" packet forwarding algorithm. We prove that it yields a stable system, provided the data injection rate at the source node is less than the (min(min-cut)) of the graph. As a corollary we retrieve a famous theorem of Edmonds.

We next consider node-capacitated networks. In this context we introduce the "random useful to most deprived neighbour" packet forwarding scheme. We show that it yields a stable system in the particular case where the network graph is the complete graph, whenever the node capacities are large enough for centralised schemes to achieve successful broadcast of the data injection rate.

The paper is organised as follows. Section II describes the model of an edge-capacitated network, as well as the "random useful" packet forwarding algorithm. Section III contains the main result for edge capacitated networks, while Section IV deals with node capacitated networks.

## II. EDGE CAPACITATED NETWORKS: MODEL AND ALGORITHM

### A. System model

A directed, edge-capacitated graph $G = (V, E)$ is given. Source node $s$ wants to send data to all other nodes, i.e. broadcast. Data transfers consist of packet replication from some node $u$ to a node $v$ such that $(u, v) \in E$. The *injection rate* at the source is denoted by $\lambda$, and is bey definition the rate at which the source gets new packets.

Once injected at the source, a packet $p$ can be in a number of different states. It can be replicated at all nodes in the system, hence successfully broadcast. Alternatively, it can be *idle*, that is not actively transferred, and replicated at nodes $u$ in some set

$S \subset V$[1]. The subsets over which packets can be replicated is not arbitrary: it must contain a spanning tree rooted at $s$, and hence in particular it must contain $s$. We shall denote by $\mathcal{S}$ the collection of strict subsets of $V$ that contain the source node $s$.

Alternatively, it can be replicated at some nodes $u \in S$, for some subset $S \in \mathcal{S}$, but also actively transferred along some edges $e \in F$, for some subset $F \subset E$.

We shall adopt the following description of the system state: for all $S \in \mathcal{S}$, $X_S$ denotes the number of idle packets, that are replicated exactly at the nodes $u \in S$. In addition, an unordered list of subgraphs $A = \{G_1 = (W_1, F_1), \ldots, G_m = (W_m, F_m)\}$ is maintained, describing the "active packets": $W_i$ is the set of nodes at which the $i$-th active packet is currently replicated; $F_i$ is the set of edges along which the $i$-th active packet is currently transferred.

We shall assume that at any given time, at most one packet is transferred along a given edge. Thus, the total number of active packets is at most $|E|$. We shall further assume that the following constraints are met: for an active packet with description $(W, F)$, for each $(u, v) \in F$, then $u \in W$, $v \notin W$, and there is no other edge $e \in F$ that points towards $v$.

Introduce the notation:

$$X_{+u-v} = \sum_{S \in \mathcal{S}: u \in S, v \notin S} X_S.$$

This counts the number of idle packets that are present at node $u$ and absent at node $v$.

Let also $X^a_{+u-v}$ denote the number of active packets that could possibly be forwarded along edge $(u, v)$, given the above constraints. That is to say, let

$$X^a_{+u-v} = \sum_{(W,F) \in A} \mathbf{1}_{u \in W} \mathbf{1}_{v \notin W} \mathbf{1}_{\forall u' \in V, (u',v) \notin F}.$$

The following **activity condition** will be enforced at all times: for any edge $(u, v)$, either there is an active packet that is actively transferred along edge $(u, v)$, or:

$$X_{+u-v} = 0 \text{ and } X^a_{+u-v} = 0.$$

In words, if there is no ongoing transfer along some edge $(u, v)$, then necessarily no packet present in the system could be transferred along this edge.

The system evolution is determined by the following transition mechanism.

## B. The scheduling algorithm

*a) Primary transitions:* The first type of primary transitions is due to a fresh packet arrival at the source. After such a transition, the state variable $X_{\{s\}}$ is updated to $X_{\{s\}} + 1$.

[1]In this paper the symbol $\subset$ is used to denote strict subsets; the symbol $\subseteq$ is used to denote non strict subsets.

The second type of primary transitions is due to completion of transfer of an active packet along some edge. Let this packet be represented by $(W, F)$, and let $e = (u, v) \in F$ be the edge along which replication has just completed. Then $W$ is updated to $W \cup \{v\}$, and $F$ is updated to $F \setminus e$.

*b) Secondary transitions:* These happen subsequently to primary transitions, to ensure that the activity condition is met. If, after a primary transition, there is an edge $(u, v)$ for which the activity condition is not met, this means that this edge is not actively used, while the number of packets $X_{+u-v} + X^a_{+u-v}$ which could potentially be transferred along that edge is positive. In this case, one of these $X_{+u-v} + X^a_{+u-v}$ packets will be selected uniformly at random, and start being replicated along edge $(u, v)$.

More precisely, for each $S \in \mathcal{S}$ such that $u \in S$, $v \notin S$, with probability

$$\frac{X_S}{X_{+u-v} + X^a_{+u-v}},$$

the following state updates are made:

$$\begin{aligned} X_S &\leftarrow X_S - 1, \\ A &\leftarrow A \cup (S, (u, v)). \end{aligned}$$

For each active packet $(W, F)$ such that $u \in W$, $v \notin W$, and for all $u' \in V$, $(u', v) \notin F$, then with probability $1/(X_{+u-v} + X^a_{+u-v})$, the active set $A$ is updated as follows:

$$A \leftarrow A \setminus (W, F) \cup (W, F \cup (u, v)).$$

Note that all these transition probabilities sum to 1, as required.

This secondary transition mechanism corresponds to what we shall call the "random useful" packet forwarding strategy: when a new useful packet transfer along a given edge $(u, v)$ can start, the packet that is actually transferred is selected uniformly at random from the total collection of packets present at $u$ and not at $v$, and not currently transferred towards $v$.

### *C. A Markovian special case*

A general version of the model would assume that the time intervals between fresh packet arrivals at the source are i.i.d. random variables, and that packet transfer times along a given edge are also i.i.d. random variables. Under these assumptions, the model we just described is a Markov process, provided we augment the state space to keep track of the residual times till (i) arrival of the next fresh packet, and (ii) completion of transmission along a given edge. Of particular interest is the case where these i.i.d. random variables are in fact deterministic.

A treatment of the general i.i.d. case will be considered in future work. In the present work, we focus on the special case where the i.i.d. random variables involved are Exponential random variables, where the mean inter-packet arrival at the source equals $\lambda^{-1}$, and the mean packet transfer time along edge $(u, v)$ is $c_{uv}^{-1}$. In this particular case, the evolution of the state variables described above is Markovian, without the adjunction of residual time variables. In the sequel we focus on this particular setup.

## III. Edge capacities: main result

The main result in the present context is the following

*Theorem 1:* The Markov process $((X_S)_{S\in\mathcal{S}}, A)$ corresponding to random useful packet forwarding is ergodic under the condition

$$\lambda < \min_{S\in\mathcal{S}} \sum_{u\in S} \sum_{v\notin S} c_{uv}. \tag{1}$$

This result will be established by using the so-called "fluid limits" approach, introduced and popularised by [4] and [1]. Informally, the approach consists in first establishing that trajectories of the original Markov process, after joint rescaling of both time and space, evolve according to some simpler, "fluid" dynamics, and then to prove that trajectories of the fluid dynamics converge to zero in finite time.

### A. Fluid dynamics: characterization and convergence

Let us introduce the following definition.

*Definition 1:* The real-valued non-negative functions $t \to y_S(t)$, $S \in \mathcal{S}$, are called fluid trajectories of the above Markov process if they satisfy the following conditions.

For all $S \in \mathcal{S}$, all $u \in S$, all $v \notin S$, there exist non-negative functions $t \to \phi_{S,(uv)}(t)$ such that

$$\begin{array}{rcl} y_{\{s\}}(t) & = & y_s(0) + \lambda t - \sum_{v\in V\setminus\{s\}} \phi_{\{s\},(sv)}(t) \\ S \neq \{s\}: \; y_S(t) & = & y_S(0) + \sum_{u\in S}\sum_{v\in S\setminus\{u\}} \phi_{S\setminus\{v\},(uv)}(t) \\ & & - \sum_{u\in S}\sum_{v\notin S} \phi_{S,(uv)}(t), \end{array} \tag{2}$$

and that are non-decreasing, Lipschitz continuous with Lipschitz constants $c_{uv}$. In addition, for all $(u,v)\in E$, it holds that:

$$\sum_{S\in\mathcal{S}: u\in S, v\notin S} \phi_{S,(uv)} \text{ is } c_{uv}\text{-Lipschitz.}$$

Moreover at almost every point $t$, the function $\phi_{S,(uv)}$ is differentiable, and the following holds:

$$y_{+u-v}(t) > 0 \Rightarrow \frac{d}{dt}\phi_{S,(uv)}(t) = c_{uv}\frac{y_S(t)}{y_{+u-v}(t)}, \tag{3}$$

where we have used the notation

$$y_{+u-v}(t) := \sum_{S'\in\mathcal{S}: u\in S', v\notin S'} y_{S'}(t). \tag{4}$$

$\diamondsuit$

The following result shows in what sense such fluid trajectories describe the dynamics of the original Markov process after spatial and temporal rescaling:

*Theorem 2:* Consider a sequence of initial conditions $(X^N(0), A^N(0))$, $N > 0$, such that

$$\lim_{N\to\infty} \frac{1}{N} X_S^N(0) = x_S(0), \; S \in \mathcal{S}.$$

Introduce the rescaled process

$$Y_S^N(t) := \frac{1}{N} X_S^N(Nt),$$

where $X_S^N(t)$ represents the state of the Markov process with initial conditions $(X^N(0), A^N(0))$ at time $t$. Then for any subsequence of indices $N$, there exists a further subsequence, denoted $N'$, such that, for some fluid trajectory $(y_S)$ as per Definition 1, with initial conditions $(x_S(0))$, the following uniform convergence takes place:

$$\lim_{N'\to\infty} \sup_{t\in[0,T]} |Y_S^{N'}(t) - y_S(t)| = 0, \quad S \in \mathcal{S},\ T \in \mathbb{R}_+. \tag{5}$$

*Proof:* It will be more convenient to work with the state variables $\tilde{X}_S$, which count the total number of packets, active or idle, present at nodes $u \in S$. That is:

$$\tilde{X}_S = X_S + \sum_{(W,F)\in A} \mathbf{1}_{W=S}.$$

We shall thus consider the rescaled processes

$$\begin{aligned} \tilde{Y}_S^N(t) &:= \tfrac{1}{N} \tilde{X}^N(Nt) \\ &= Y^N(t) + \tfrac{1}{N} \textstyle\sum_{W\in A^N} \mathbf{1}_{W=S}. \end{aligned}$$

Since they differ from $Y_S^N$ by at most $|E|/N$, the processes agree in the limit $N \to \infty$. Let $P_{uv}$, $(u,v) \in E$, be independent unit rate Poisson processes. The Poisson process $P_{uv}$ will be used to determine the instants at which packet transfers along edge $(u,v)$ complete. Introduce the notation:

$$\Phi_{S,(uv)}^N(t) = P_{uv}\left( c_{uv} \int_0^t \sum_{(W,F)\in A^N(s-)} \mathbf{1}_{W=S,(u,v)\in F} ds \right).$$

This process keeps track of the number of completions of packet transfers along edge $(u,v)$, for packets that were previously present at node set $S$.

We thus have the following, for all $S \in \mathcal{S}$, $S \neq \{s\}$:

$$\tilde{X}_S^N(t) = \tilde{X}_S^N(0) + \sum_{u\in S, v\in S\setminus\{u\}} \Phi_{S\setminus\{v\},(uv)}^N(t) - \sum_{u\in S, v\notin S} \Phi_{S,(uv)}^N(t).$$

In the particular case where $S = \{s\}$, we use another unit rate Poisson process $P_0$ to count fresh arrivals at the source, and write:

$$\tilde{X}_{\{s\}}^N(t) = \tilde{X}_{\{s\}}^N(0) + P_0(\lambda t) - \sum_{v\neq s} \Phi_{\{s\},(sv)}^N(t).$$

We now show that, for any subsequence, there exists a further subsequence $N'$ along which the rescaled processes $t \to \frac{1}{N'} \Phi_{S,(uv)}^{N'}(N't)$ converge uniformly on any compact interval $[0,T]$ to a non-decreasing function $\phi_{S,(uv)}$, that is moreover Lipschitz-continuous with Lipschitz constant $c_{uv}$[2].

[2]Similar arguments can be used to establish that $(1/N)P_0(\lambda_N t)$ converges uniformly on $[0,T]$ to $\lambda t$, and are thus omitted.

To this end, fix some $T > 0$, and write

$$\sup_{t\in[0,T]}\left|\frac{1}{N}\Phi^N_{S,(uv)}(Nt) - c_{uv}\int_0^t \sum_{(W,F)\in A^N(Ns-)} \mathbf{1}_{W=S,(u,v)\in F} ds\right| \leq \sup_{t\in[0,c_{uv}T]}\left|\frac{1}{N}P_{uv}(Nt) - t\right|.$$

Using for instance the following lemma, which is a classical result on the maximal deviation of a Poisson process from its mean, in conjunction with Borel-Cantelli's lemma, it can be shown that the right-hand side of the last display converges almost surely to zero as $N \to \infty$.

*Lemma 1:* Let $\Xi$ be a unit rate Poisson process. Then for all $T > 0$, $N > 0$, and all $\epsilon > 0$, it holds that

$$\mathbf{P}(sup_{0\leq t\leq T}|\Xi(Nt) - Nt| \geq \epsilon NT) \leq e^{-NTh(\epsilon)} + e^{-NTh(-\epsilon)}, \tag{6}$$

where

$$h(\lambda) := (1+\lambda)\log(1+\lambda) - \lambda \tag{7}$$

is the Cramér transform of a unit mean, centered Poisson random variable. In the above formula, it is understood that $h(-\lambda) = +\infty$ if $\lambda > 1$.

To establish the claimed convergence of the rescaled processes $\frac{1}{N}\Phi^N_{S,(uv)}(Nt)$ to Lipschitz-continuous, non-decreasing functions $\phi_{S,(uv)}$ along subsequences, it is therefore sufficient to establish that such convergence holds for the functions:

$$t \to c_{uv}\int_0^t \sum_{(W,F)\in A^N(Ns-)} \mathbf{1}_{W=S,(u,v)\in F} ds. \tag{8}$$

To this end, we use the following lemma, taken from Ye et al. [6]:

*Lemma 2:* (Lemma 6.3, Ye et al. [6]) Suppose that a sequence of functions $f_k : [0,T] \to \mathbb{R}$ has the following properties:

(i) $\{f_k(0)\}_{k\geq 0}$ is bounded;

(ii) there is a constant $M > 0$, and a sequence of positive numbers $\sigma_k$, with $\sigma_k \to 0$ as $k \to \infty$, such that

$$|f_k(t) - f_k(s)| \leq M(t-s) + \sigma_k, \quad k \geq 0,\ s,t \in [0,T].$$

Then the sequence admits a subsequence that converges uniformly on $[0,T]$ to a Lipschitz continuous function $f : [0,T] \to \mathbb{R}$ with Lipschitz constant $M$.

Clearly the conditions of the Lemma are met for the functions (8), with as a Lipschitz constant $M = c_{uv}$. Moreover, any limiting function must be non-decreasing since the functions (8) are all non-decreasing.

Note now that for $t < t'$,

$$\sum_{S\in\mathcal{S}:u\in S,v\notin S} c_{uv}\int_t^{t'} \sum_{(W,F)\in A^N(Ns-)} \mathbf{1}_{W=S,(u,v)\in F} ds \leq c_{uv}(t'-t).$$

This readily implies that for any given $(u,v) \in E$, the limiting functions $\phi_{S,(uv)}$ summed over $S \in \mathcal{S}$ such that $u \in S$ and $v \notin S$ are $c_{uv}$-Lipschitz.

It now remains to establish the last property in the definition of fluid trajectories, that is: at almost every $t$, the function $\phi_{S,(uv)}(t)$ is differentiable, and provided $y_{+u-v}(t) > 0$, then:

$$\frac{d}{dt}\phi_{S,(uv)}(t) = c_{uv}\frac{y_S(t)}{y_{+u-v}(t)}\cdot$$

By Rademacher's theorem, a Lipschitz-continuous function is differentiable almost everywhere. Let thus $t$ be a point where $\phi_{S,(uv)}(t)$ is differentiable. Consider first the case where $y_S(t) > 0$. Fix some $h > 0$. We want to evaluate the following quantity:

$$\frac{1}{h}c_{uv}\int_t^{t+h}\sum_{(W,F)\in A^N(Ns-)}\mathbf{1}_{W=S,(u,v)\in F}ds.$$

Note that on the interval $\tau \in [t, t+h]$, $N'^{-1}X_S^{N'}(N'\tau)$ equals $y_S(t)+0(h)+\epsilon_{N'}$, where $\epsilon_{N'} \to 0$ as $N' \to \infty$, by convergence of the rescaled trajectories, and by Lipschitz continuity of the limiting trajectories.

Thus, after each completion of a transfer along edge $(u,v)$ during the interval $[Nt, N(t+h)]$, the probability that the next packet selected for transmission along edge $(u,v)$ is a previously idle packet, replicated at nodes $w \in S$ is asymptotic to $y_S(t)/y_{+u-v}(t) + 0(h)$. Furthermore, once such a transfer is started, the probability that the packet under consideration is elected for transmission along another edge converges to zero as $N \to \infty$, since there are close to $Ny_S(t)$ other idle packets that could alternatively have been selected for such a transmission. Together these arguments ensure that

$$\lim_{N\to\infty}\frac{1}{h}c_{uv}\int_t^{t+h}\sum_{(W,F)\in A^N(Ns-)}\mathbf{1}_{W=S,(u,v)\in F}ds = c_{uv}\frac{y_S(t)}{y_{+u-v}(t)} + O(h)\cdot$$

However, the left-hand side of this expression also reads

$$\frac{1}{h}\left(\phi_{S,(uv)}(t+h) - \phi_{S,(uv)}(t)\right),$$

and thus the derivative of $\phi_{S,(uv)}$ at $t$ must equal $c_{uv}\frac{y_S(t)}{y_{+u-v}(t)}$ as announced.

Finally, consider the case where $y_S(t) = 0$, and choose a particular $t$ at which all $S'$ with $u \in S'$, $v \notin S'$ are such that $\phi_{S',(uv)}(t)$ are differentiable. We know that almost everywhere, the sum of these derivatives can not exceed $c_{uv}$, because it is a Lipschitz constant for the sum of these functions. However, the sum of the derivatives for those $S'$ such that $y_{S'}(t) > 0$ equals $c_{uv}$, therefore the derivatives for those $S$ such that $y_S(t) = 0$ must equal zero. ■

### B. Fluid dynamics: stability

In the present section, we establish that any fluid trajectories as per Definition 1 satisfy a suitable stability property:

*Theorem 3:* Assume that Condition (1) holds. Let $(y_S)_{S\in\mathcal{S}}$ denote fluid trajectories as per Definition 1. For all $S \subset V$, define:

$$y_{\subseteq S} = \sum_{S'\in\mathcal{S}, S'\subseteq S} y_{S'}.$$

Then there exist positive parameters $\beta_1, \ldots, \beta_{|V|-1}$, and $\epsilon > 0$ such that the function

$$L(\{y_S\}_{S\in\mathcal{S}}) := \sup_{S\subset V} \beta_{|S|} y_{\subseteq S}$$

verifies:

$$L(y(t)) \leq \max\left(0, L(y(0)) - \epsilon t\right). \tag{9}$$

Denote by $K$ the total number of nodes, that is $K = |V|$. The proof will rely on the following lemma:

*Lemma 3:* Let $\alpha > 0$ be fixed. For given $\delta, A > 0$, define:

$$\begin{array}{ll} \epsilon_{K-1} = \delta; & \\ \epsilon_{K-1-i} = \delta A(1+A)^{i-1}, & i = 1\ldots, K-2, \\ \beta_{K-1} = 1; & \\ \beta_{K-i} = \prod_{j=K-i+1}^{K-1}\left(\frac{1}{1-\epsilon_j}\right), & i = 2, \ldots, K-1. \end{array} \tag{10}$$

Then $A$ and $\delta$ can be chosen so that the following properties hold for any $(y_S)_{S\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{S}}$, $(y_S)_{S\in\mathcal{S}} \neq 0$. For all $S \subset V$, all $u \in S$, $v \notin S$:

$$y_{+u-v} < \epsilon_{|S|} y_{\subseteq S} \Rightarrow \beta_{|S|-1} y_{\subseteq S\setminus\{u\}} > \beta_{|S|} y_{\subseteq S}. \tag{11}$$

Moreover for all $S \subset V$ such that, for all $u \in S$, all $v \notin S$, $y_{+u-v} \geq \epsilon_{|S|} y_{\subseteq S}$, assuming there exist $u \in S$ and $v \notin S$ such that for some $S' \not\subseteq S$: $u \in S'$, $v \notin S'$ and $y_{S'} > \alpha y_{+u-v}$, then it holds that:

$$\beta_{|S\cup S'|} y_{\subseteq S\cup S'} > \beta_{|S|} y_{\subseteq S}. \tag{12}$$

*Proof:* (of Lemma 3) Let us first establish sufficient conditions on the parameters $\epsilon_i$, $\beta_i$ for the conclusions of the Lemma to hold. Consider the first requirement (11), and let thus $S$ be such that for some $u \in S$ and $v \notin S$, one has

$$y_{+u-v} < \epsilon_{|S|} y_{\subseteq S}.$$

Write now:

$$\begin{aligned} y_{\subseteq S} &= y_{\subseteq S\setminus\{u\}} + \textstyle\sum_{S'\in\mathcal{S}: u\in S', S'\subseteq S} y_{S'} \\ &\leq y_{\subseteq S\setminus\{u\}} + y_{+u-v} \\ &< y_{\subseteq S\setminus\{u\}} + \epsilon_{|S|} y_{\subseteq S}. \end{aligned}$$

It thus follows that

$$y_{\subseteq S\setminus\{u\}} > (1-\epsilon_{|S|}) y_{\subseteq S}.$$

Thus the desired conclusion (11) will follow provided:

$$\beta_{i-1}(1-\epsilon_i) \geq \beta_i, \quad i=2,\dots,K-1. \tag{13}$$

Clearly, this condition will be satisfied with the particular choice of coefficients $\beta_i$ as in (10), provided the $\epsilon_i$ lie in the interval $(0,1)$, which will be ensured by taking $\delta > 0$ sufficiently small.

Let us now turn to Condition (12). Let thus $S \subset V$ be such that for all $u \in S$ and $v \notin S$, $\epsilon_{|S|} y_{\subseteq S} \leq y_{+u-v}$. Assume moreover the existence of $u \in S$, $v \notin S$, and $S' \not\subseteq S$ such that $u \in S'$, $v \notin S'$, and satisfying moreover:

$$y_{S'} > \alpha y_{+u-v}.$$

Then necessarily, one has:

$$y_{S'} > \alpha \epsilon_{|S|} y_{\subseteq S}.$$

The left-hand side of Condition (12) then verifies:

$$\begin{array}{ll} \beta_{|S\cup S'|} y_{\subseteq S\cup S'} & \geq \beta_{|S\cup S'|}\left(y_{S'} + y_{\subseteq S}\right) \\ & > \beta_{|S\cup S'|}(1+\alpha\epsilon_{|S|}) y_{\subseteq S}. \end{array}$$

Therefore, (12) will hold provided

$$\beta_{|S\cup S'|}(1+\alpha\epsilon_{|S|}) \geq \beta_{|S|}.$$

For sufficiently small $\delta > 0$, the coefficients $\epsilon_i$ as in (10) will be strictly less than 1, and hence the coefficients $\beta_i$ as in (10) will be decreasing with $i$. Thus, the above condition will be satisfied provided:

$$\beta_{K-1}(1+\alpha\epsilon_i) \geq \beta_i, \quad i=1,\dots,K-2. \tag{14}$$

For $i = K-2$, this condition reads $1+\alpha\epsilon_{K-2} \geq 1/(1-\epsilon_{K-1})$. Recalling from (10) that $\epsilon_{K-1} = \delta$, the right-hand side reads $1+\delta+o(\delta)$, while the left-hand side reads $1+\alpha\delta A$. Thus, this particular condition is met provided $A\alpha > 1$, and $\delta > 0$ is small enough.

Let us now consider $i \in \{1,\dots,K-3\}$. Note that the right-hand side of (14) is equivalent to, for small $\delta > 0$:

$$\begin{array}{ll} \beta_i & = \prod_{j=i+1}^{K-1}\left(\frac{1}{1-\epsilon_j}\right) \\ & = 1+\sum_{j=i+1}^{K-1}\epsilon_j + o(\delta) \\ & = 1+\delta+\sum_{j=i+1}^{K-2}\delta A(1+A)^{K-2-j} + o(\delta) \\ & = 1+\delta+\delta A\sum_{j=0}^{K-3-i}(1+A)^j + o(\delta) \\ & = 1+\delta+\delta A\frac{(1+A)^{K-3-i+1}-1}{A} + o(\delta) \\ & = 1+\delta(1+A)^{K-2-i} + o(\delta). \end{array}$$

On the other hand, the left-hand side of (14) equals $1+\alpha\delta A(1+A)^{K-2-j}$. Thus, provided $\alpha A > 1$, and $\delta > 0$ is small enough, the announced properties hold. ■

*Proof:* (of Theorem 3) Consider the particular parameters $\beta_i$, $\epsilon_i$ as in Lemma 3. Clearly, for a vector $y \in \mathbb{R}_+^{\mathcal{S}}$ that is non-zero, any set $S^* \subset V$ achieving the maximum in $\max_{S \subset V} \beta_{|S|} y_{\subseteq S}$ is such that $y_{\subseteq S^*} > 0$. Moreover, for all $u \in S^*$, $v \notin S^*$, one must have:

$$y_{+u-v} \geq \epsilon_{|S^*|} y_{\subseteq S^*} > 0,$$

for otherwise optimality of the set $S^*$ would be contradicted by Condition (11). In addition, for all $u \in S^*$, $v \notin S^*$, and all $S' \not\subseteq S^*$ such that $u \in S'$, $v \notin S'$, necessarily $y_{S'} \leq \alpha y_{+u-v}$, for otherwise optimality of $S^*$ would be contradicted by (12).

One thus has the following evaluation:

$$\begin{aligned} \tfrac{d}{dt} y_{\subseteq S^*} &= \textstyle\sum_{S \subseteq S^*} \tfrac{d}{dt} y_S \\ &= \lambda - \textstyle\sum_{u \in S^*, v \notin S^*} \sum_{S \subseteq S^*, u \in S} \tfrac{d}{dt} \phi_{S,(uv)} \\ &= \lambda - \textstyle\sum_{u \in S^*, v \notin S^*} c_{uv} \left[ 1 - \sum_{S' \not\subseteq S^*, u \in S', v \notin S'} \tfrac{y_{S'}}{y_{+u-v}} \right] \\ &\leq \lambda - \textstyle\sum_{u \in S^*, v \notin S^*} c_{uv} + \sum_{u \in S^*, v \notin S^*} c_{uv} \sum_{S' \not\subseteq S^*, u \in S', v \notin S'} \alpha \\ &\leq \lambda - \textstyle\sum_{u \in S^*, v \notin S^*} c_{uv} + \max_{e \in E} c_e |E| 2^K \alpha. \end{aligned}$$

In the above, we have used the expression (3) for the derivative of the functions $\phi_{S,e}$, and the bound of $\alpha$ on the ratio $y_{S'}/y_{+u-v}$ previously established.

Furthermore, the conditions (13) and (14) used in the proof of Lemma 3 can be shown to imply the following. For a set $S$ such that $\beta_{|S|} y_{\subseteq S} \geq (1-r) \beta_{|S^*|} y_{\subseteq S^*}$, where $r > 0$ is some small positive constant, necessarily for all $u \in S$, $v \notin S$,

$$y_{+u-v} \geq \left( 1 - \frac{1 - \epsilon_{|S|}}{1 - r} \right) y_{\subseteq S}.$$

In addition, for $u \in S$, $v \notin S$ and $S' \not\subseteq S$ such that $u \in S'$, $v \notin S'$, then one has:

$$y_{S'} \leq \left( \frac{1 + \alpha \epsilon_{|S|}}{1 - r} - 1 \right) \frac{1}{1 - \frac{1 - \epsilon_{|S|}}{1 - r}} y_{+u-v} = (\alpha + O(r))\, y_{+u-v}.$$

Thus, for such $S$, one has the similar evaluation

$$\frac{d}{dt} y_{\subseteq S} \leq \lambda - \sum_{u \in S, v \notin S} c_{uv} + \max_{e \in E} c_e |E| 2^K \alpha \left( 1 + O(r) \right). \tag{15}$$

Note that the choice of $\alpha > 0$ in Lemma 3 was arbitrary. For definiteness, set

$$\alpha = \frac{1}{2} \frac{\min_{S \subset V} \sum_{u \in S, v \notin S} c_{uv} - \lambda}{|E| 2^K \max_{e \in E} c_e}$$

This is positive, under the stability condition (1). Then from the above evaluation (15), it follows that necessarily, almost everywhere the Lipschitz continuous function $L(y(t))$ must satisfy:

$$\frac{d}{dt} L(y(t)) \leq -\epsilon \mathbf{1}_{y(t) \neq 0},$$

where

$$\epsilon := \frac{1}{2}\left(\min_{S\subset V}\sum_{u\in S, v\notin S} c_{uv} - \lambda\right).$$

The result of Theorem 3 follows. ■

### C. Proof of Theorem 1

The proof of Theorem 1 will require to combine Theorems 2, 3 and the following ergodicity criterion, which is a direct consequence of Theorem 8.13, p.224 in Robert [3]:

*Theorem 4:* Let $Z(t)$ be a Markov jump process on a countable state space $\mathcal{Z}$. Assume there exists a function $L : \mathcal{Z} \to \mathbb{R}_+$ and constants $M$, $\epsilon$, $\tau > 0$ such that for all $z \in \mathcal{Z}$:

$$L(z) > M \Rightarrow \frac{1}{L(z)}\mathbf{E}_z L\left(Z(L(z)\tau)\right) \leq 1 - \epsilon. \tag{16}$$

If in addition the set $\{z : L(z) \leq M\}$ is finite, and $\mathbf{E}_z L(Z(1)) < +\infty$ for all $z \in \mathcal{Z}$, then the process $Z(t)$ is ergodic.

Let us show how this result applies in the present context. Here we have $Z(t) = (X(t), A(t))$, and our candidate Lyapunov function takes as argument the $X$-component only, and reads

$$L(Z) = \sup_{S\subset V} \beta_{|S|} X_{\subseteq S}.$$

Let us set $\tau = 1$, where $\epsilon$ is as in Theorem 3, and establish that (16) holds by contradiction. Assuming it fails, there must exist a sequence of initial conditions $Z^N(0)$ such that $L(Z^N(0)) \to \infty$, and such that

$$\lim_{N\to\infty} \frac{1}{L(Z^N(0))}\mathbf{E}L\left(Z^N(L(Z^N(0))\tau)\right) > 1 - \epsilon. \tag{17}$$

However, by Theorem 1, any accumulation point of the sequence

$$\frac{1}{L(Z^N(0))} X^N(L(Z^N(0))\tau)$$

must be equal to $y(\tau)$ for some fluid trajectory $y$ issued from an initial condition $y(0)$ such that $L(y(0)) = 1$. Furthermore, this family of random vectors is uniformly integrable: indeed, writing

$$\frac{1}{L(Z^N(0))} X_S^N(L(Z^N(0))\tau) \leq \frac{X_S^N(0)}{\beta_{|S|} X_S^N(0)} + \frac{1}{L(Z^N(0))}\sum_{e\in E} P_e(L(Z^N(0))c_e\tau),$$

where the $P_e$ are the Poisson processes previously introduced, uniform integrability can be readily checked. Since the function $L$ grows not faster than linearly, the family of random variables

$$\frac{1}{L(Z^N(0))} L\left(X^N(L(Z^N(0))\tau)\right)$$

is also uniformly integrable. Since the function $L$ is continuous, accumulation points of this sequence must be of the form $L(y(\tau))$, for some fluid trajectory $y$ issued from an initial condition $y(0)$ such that $y(0) = 1$. By Theorem 3, all such accumulation points are less than, or equal to $1-\epsilon$. This together with uniform integrability ensures that

$$\limsup_{N\to\infty} \frac{1}{L(Z^N(0))} \mathbf{E} L\left(Z^N(L(Z^N(0))\tau)\right) \leq 1-\epsilon,$$

which contradicts (17). The proof is concluded by verifying the other assumptions of Theorem 4, i.e. that $\{z : L(z) \leq M\}$ is finite for sufficiently large $M$. This holds trivially, because for any $X$-component the number of potential $A$-components is bounded (say by $|E|$ times the number of subgraphs of $G$).

Finally, one must check that $\mathbf{E}_z L(Z(1)) < +\infty$ for all $z$; this is easily verified, once more by bounding $X_S(1)$ by its initial value plus increments of Poisson processes.

## IV. Node-capacitated networks

### A. Model and Algorithm

*c) Neighbour selection:* Here, the system is also described by a a graph $G = (V, E)$. However, the capacities are now associated with nodes rather than with edges. We shall denote by $c_u$ the capacity of node $u$, and assume that each node devotes its capacity to one of its "most deprived neighbours". By this, the following is meant. For each of its neighbours $v$, node $u$ evaluates the number $Z_{+u-v}$ of packets that it could usefully forward to node $v$. Using the same notation as before, this reads:

$$Z_{+u-v} = X_{+u-v} + X^a_{+u-v}.$$

It then elects one neighbour $v$ for which the corresponding quantity $Z_{+u-v}$ is maximal. Ties can be broken either at random, or in a systematic manner. Once the target neighbour $v$ is chosen, then one of the $Z_{+u-v}$ packets held by $u$ and useful to $v$ is chosen, and forwarded from $u$ to $v$, at rate $c_u$.

*d) Packet selection:* We now describe how packets are elected for transmission once a node's capacity becomes available. For non-source nodes $u$, who have chosen to transmit to some most deprived neighbour $v$, then the packet to be transmitted is selected at random among all the possible $Z_{+u-v}$ possible choices.

For the source node $s$, having chosen to transmit to some most deprived neighbour $v$, the following strategy is used: if the source has a packet that it has not sent to anyone before (a *fresh* packet), that is if $X_{\{s\}} > 0$, then one such fresh packet is forwarded to node $v$; if no such fresh packets are available, then the packet to be forwarded is selected uniformly at random from the $Z_{+s-v}$ possible choices.

As in the edge capacitated case, the state space consists in the collection of variables $X_S$, for all $S \in \mathcal{S}$, and the collection of active packet states $A = ((W_1, F_1), \ldots, (W_m, F_m))$. The constraints on these active packet states are different though: we now assume that each node forwards a packet to only one of its neighbours at a given time. Thus for each node $u$, there is at most one edge $(u,w)$ appearing in the sets $F_i$, $i=1,\dots,m$. Otherwise the same constraints apply: for a given active packet $(W,F)$, and each edge $(u,v)\in F$, necessarily, $u\in W$ and $v\notin W$; also, there is no other edge $(u',v)$ pointing towards $v$ in $F$.

We shall assume that packet transmissions are not preempted, even if a neighbour of some node $u$ becomes more deprived than the neighbour $v$ to which node $u$ is currently transmitting.

As in the edge-capacitated case, in the present work we focus on the case where completion of a packet transmission by node $u$ is an Exponential random variable with mean $1/c_u$, and where fresh packets arrive at the source node $s$ at the instants of a Poisson process with rate $\lambda$.

### *B. Fluid limits*

We first define the candidate fluid trajectories for the system under consideration:
*Definition 2:* The real-valued, non-negative functions $(y_S)_{S\in\mathcal{S}}$ are called fluid trajectories of the node-capacitated system if the following properties hold.

For all $S\in\mathcal{S}$, $u\in S$, $v\notin S$ such that $(u,v)\in E$, there exist non-decreasing, Lipschitz-continuous functions $\phi_{S,(uv)}$ with Lipschitz constant $c_u$, such that Equations (2) hold. Furthermore, using notation

$$y_{+u-v}:=\sum_{S\in\mathcal{S}:u\in S,v\notin S} y_S,$$

for all $S\in\mathcal{S}$, $u\in S$, the functions $\{\phi_{S,(uv)}\}_{v\notin S,(uv)\in E}$ are differentiable at almost every $t$, and if $\sum_{v:(u,v)\in E} y_{+u-v}(t)>0$, their derivatives satisfy:

$$\frac{d}{dt}\phi_{S,(uv)}(t)=0 \text{ if } y_{+u-v}(t)<\max_{v':(u,v')\in E}\left(y_{+u-v'}(t)\right), \tag{18}$$

$$\sum_{v:(uv)\in E}\ \sum_{S:u\in S,v\notin S}\frac{d}{dt}\phi_{S,(uv)}(t)=c_u. \tag{19}$$

If $u\neq s$, that is for a non-source node, one also has, for all $v$ such that $(uv)\in E$ and assuming the condition

$$\sum_{S:u\in S,v\notin S}\frac{d}{dt}\phi_{S,(uv)}(t)>0$$

holds, the following equation:

$$\forall S/u\in S, v\notin S,\ \frac{d}{dt}\phi_{S,(uv)}(t)=\frac{y_S(t)}{\sum_{S':u\in S',v\notin S'}y_{S'}(t)}\sum_{S':u\in S',v\notin S'}\frac{d}{dt}\phi_{S',(uv)}(t)\cdot \tag{20}$$

For the source node $s$, one has the following:

$$y_{\{s\}}>0\Rightarrow\sum_{v\neq s}\frac{d}{dt}\phi_{\{s\},(sv)}(t)=c_s. \tag{21}$$

In the case where $y_{\{s\}} = 0$, one then has for all $v$ such that $(sv) \in E$, assuming the condition

$$\sum_{S \in \mathcal{S}: S \neq \{s\}, v \notin S} \frac{d}{dt}\phi_{S,(sv)}(t) > 0$$

holds, the following:

$$\forall S \in \mathcal{S}/S \neq \{s\}, v \notin S: \; \frac{d}{dt}\phi_{S,(sv)}(t) = \frac{y_S(t)}{\sum_{S' \in \mathcal{S}: S' \neq \{s\}, v \notin S'} y_{S'}(t)} \sum_{S' \in \mathcal{S}: S' \neq \{s\}, v \notin S'} \frac{d}{dt}\phi_{S',(sv)}(t). \tag{22}$$

$\Diamond$

We now establish the following

*Theorem 5:* Consider a sequence of initial conditions $(X^N(0), A^N(0))$, $N > 0$, such that the limit

$$\lim_{N \to \infty} \frac{1}{N} X^N(0) = y(0) \in \mathbb{R}_+^{\mathcal{S}}$$

exists, with $y(0) \neq 0$. Then for any subsequence, there exists a further subsequence, that we denote by $N'$, and a fluid trajectory with initial condition $y(0)$, such that for all $T > 0$, all $S \in \mathcal{S}$,

$$\lim_{N' \to \infty} \sup_{t \in [0,T]} \left| \frac{1}{N'} X_S^{N'}(Nt) - y_S(t) \right| = 0. \tag{23}$$

*Proof:* Introduce the functions

$$\Phi_{S,(uv)}^N(t) := P_u \left( c_u \int_0^t \sum_{(W,F) \in A^N(s-)} \mathbf{1}_{W=S,(u,v) \in F} ds \right),$$

where $P_u$ are independent, unit rate Poisson processes. The existence of functions $\phi_{S,(uv)}$ that are non-increasing and Lipschitz continuous with Lipschitz constant $c_u$, and such that for functions $y_S$ given by (2), the above uniform convergence (23) holds, is established exactly as in the proof of Theorem 2, and hence the detailed argument is omitted.

It only remains to establish properties (18–22) of the derivatives $\frac{d}{dt}\phi_{S,(uv)}(t)$. Fix thus $h > 0$, and consider the quantity

$$\frac{1}{h}\left( \frac{1}{N}\Phi_{S,(uv)}^N(N(t+h)) - \frac{1}{N}\Phi_{S,(uv)}^N(Nt) \right). \tag{24}$$

Assume that the node $u$ is such that the limiting processes $(y)$ satisfy

$$\sum_{v' \neq u} y_{+u-v'}(t) > 0. \tag{25}$$

Then, provided $y_{+u-v}(t) < \max_{v' \neq v} y_{+u-v}(t)$, by Lipschitz continuity of the limiting trajectories, the same inequality holds throughout the interval $[t, t+h]$. Thus, by convergence of the rescaled trajectories to the fluid limits, for large enough $N$, neighbour

$v$ is never selected for transmission by node $u$ over the whole interval $[Nt, N(t+h)]$. It then follows that the term (24) converges to 0 as $N \to \infty$. This establishes (18).

Note next that, when (25) holds, for large enough $N$ one has the following equality:

$$\sum_{v \neq u, S \in \mathcal{S}: u \in S, v \notin S} \frac{1}{h} \left( \frac{1}{N} \Phi^N_{S,(uv)}(N(t+h)) - \frac{1}{N} \Phi^N_{S,(uv)}(Nt) \right) = \frac{1}{N} \left( P_u(N(t+h)) - P_u(Nt) \right).$$

This is because node $u$'s capacity is always used when there are packets that node $u$ can usefully transmit. This identity guarantees that

$$\lim_{N \to \infty} \sum_{v \neq u, S \in \mathcal{S}: u \in S, v \notin S} \frac{1}{h} \left( \frac{1}{N} \Phi^N_{S,(uv)}(N(t+h)) - \frac{1}{N} \Phi^N_{S,(uv)}(Nt) \right) = c_u,$$

from which (19) follows.

Assume now that for non-source node $u$, node $v$ is such that

$$\sum_{S: u \in S, v \notin S} \frac{d}{dt} \phi_{S,(uv)}(t) > 0.$$

Then for all $S$ such that $u \in S$, $v \notin S$, of all the instants during the interval $[Nt, N(t+h)]$ at which node $u$ chooses to send a packet to node $v$, a fraction $y_S(t)/y_{+u-v}(t) + 0(h) + 0(1/N)$ of these choices is towards an idle packet previously replicated at all nodes in $S$. Furthermore, once transfer of such previously idle packets has started, such a packet is elected for transmission by some other node with probability $0(1/N)$. This thus shows that

$$\lim_{N \to \infty} \frac{1}{N} \left[ \Phi^N_{S,(uv)}(N(t+h)) - \Phi^N_{S,(uv)}(Nt) \right] = \left( \frac{y_S(t)}{y_{+u-v}(t)} + 0(h) \right) \sum_{S': u \in S', v \notin S'} \left[ \phi_{S',(uv)}(t+h) - \phi_{S',(uv)}(t \right.$$

Dividing by $h$ and letting $h$ tend to zero establishes (20).

Equation (21) follows by similar arguments, relying on the fact that the source node $s$ forwards fresh packets, whenever there are some available. Equation (22) is also established by similar arguments, now relying on the fact that the source, when sending non-fresh packets, selects such packets uniformly at random. ■

### C. *Stability for the complete graph*

The main result we shall establish is in the case of the complete graph, that is all edges $(u, v)$, $u \neq v$, are present in $E$. We then have the following

*Theorem 6:* Assume that the graph $G = (V, E)$ is complete, and that the injection rate $\lambda$ verifies:

$$\lambda < \min \left( c_s, \frac{\sum_{u \in V} c_u}{K-1} \right), \tag{26}$$

where $K = |V|$. Then the Markov process keeping track of the system state under "random useful to most deprived neighbour" scheduling strategy is ergodic.

The proof of Theorem 6 parallels exactly that of Theorem 1, relying on a combination of Theorem 4 with Theorem 5 (taking the role played by Theorem 2 in the proof of Theorem 1) and of Theorem 7 below (taking the role played by Theorem 3 in the proof of Theorem 1). We shall not reproduce the whole argument, but shall instead only detail the proof of the following result on stability of fluid trajectories:

*Theorem 7:* For any $y = (y_S)_{S\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{S}}$, define the *workload* function $w(y)$ as:

$$w(y) = \sum_{S\in\mathcal{S}} y_S \left(K - |S|\right), \tag{27}$$

where $K = |V|$. Under the assumption (26), when the graph $G$ is complete, any fluid trajectory $y$ as per Definition 2 is such that, for some $\epsilon > 0$,

$$w(y(t)) \leq \max(0, w(y(0)) - \epsilon t). \tag{28}$$

*Proof:* To establish (28), it suffices to show that, for all fluid trajectory $y$, at a point $t$ where $y(t)$ is differentiable and $y(t) \neq 0$, one has

$$\frac{d}{dt} w(y(t)) \leq -\epsilon.$$

This is true because the function $t \to w(y(t))$ is Lipschitz-continuous, which follows from Lipschitz continuity of the individual functions $t \to y_S(t)$.

We distinguish two cases. First, consider the case where at $t$, for all $u \in V$, one has

$$\sum_{v\neq u} y_{+u-v}(t) > 0. \tag{29}$$

Write then, using (2):

$$\begin{aligned}
\tfrac{d}{dt} w(y(t)) &= \textstyle\sum_{S\in\mathcal{S}} (K-|S|) \tfrac{d}{dt} y_S(t)\\
&= \lambda(K-1) - \textstyle\sum_{S\in\mathcal{S}} \sum_{u\in S, v\notin S} \tfrac{d}{dt}\phi_{S,(uv)}(t)\\
&= \lambda(K-1) - \textstyle\sum_{u\in V} \sum_{v:(uv)\in E} \sum_{S: u\in S, v\notin S} \tfrac{d}{dt}\phi_{S,(uv)}(t)\\
&= \lambda(K-1) - \textstyle\sum_{u\in V} c_u,
\end{aligned}$$

where the last equality follows from (19), which is applicable in view of Assumption (29). Thus in the present case, under Assumption (26), the time derivative $(d/dt)w(y(t))$ decreases at a constant speed as desired.

Consider now the case where for a non-empty set $S^*$, all $u \in S^*$ are such that

$$\sum_{v\neq u} y_{+u-v}(t) = 0.$$

Equivalently, for all $S \in \mathcal{S}$ such that $u \in S$, one has $y_S(t) = 0$. It readily follows that for any node $u \in V$, the set of most deprived neighbours consists precisely of those nodes $v \in S^*$.

Distinguish now according to whether $y_{\{s\}}(t) = 0$ or not. In the first case where $y_{\{s\}}(t) = 0$, necessarily there must exist $T \in \mathcal{S}$, $T \neq \{s\}$ for which $y_T(t) > 0$, by the assumption that $y(t) \neq 0$. Note now that, by non-negativity of the function $t \to y_{\{s\}}(t)$, one must necessarily have:

$$\frac{d}{dt} y_{\{s\}}(t) = 0, \tag{30}$$

and by the same argument, for all $S$ such that $S \cap S^* \neq \emptyset$, one also has

$$\frac{d}{dt} y_S(t) = 0. \tag{31}$$

On the other hand, it follows from Equation (18) that the left-hand side of (30) also reads

$$\lambda - \sum_{v \in S^*} \frac{d}{dt} \phi_{\{s\},(sv)}(t) \cdot$$

It thus follows from (19) that

$$\sum_{S \in \mathcal{S}, S \neq \{s\}} \sum_{v \in S^*} \frac{d}{dt} \phi_{S,(sv)}(t) = c_s - \lambda > 0.$$

Using (30–31), write then

$$\begin{array}{ll} \frac{d}{dt} w(y(t)) & = \sum_{S \in \mathcal{S}: S \neq \{s\}, S \cap S^* = \emptyset} (K - |S|) \frac{d}{dt} y_S(t) \\ & = - \sum_{S \in \mathcal{S}: S \neq \{s\}, S \cap S^* = \emptyset} (K - |S|) \sum_{u \in S, v \in S^*} \frac{d}{dt} \phi_{S,(uv)}(t) \\ & \leq - \sum_{S \in \mathcal{S}: S \cap S^* = \emptyset} (K - |S|) \sum_{v \in S^*} \frac{d}{dt} \phi_{S,(sv)}(t) \\ & = -(c_s - \lambda). \end{array}$$

In the above, we have used the fact that the most deprived nodes are those in $S^*$, and hence by (18), for all $S$ such that $S \cap S^* = \emptyset$, all $u \in S$, $v \in S \setminus \{u\}$, necessarily

$$\frac{d}{dt} \phi_{S \setminus \{v\},(uv)}(t) = 0,$$

for the capacity of node $u$ is fully targeted towards nodes in $S^*$.

The last case to consider is when $y_{\{s\}}(t) > 0$. Then in view of (21),

$$\frac{d}{dt} y_{\{s\}}(t) = \lambda - c_s.$$

This entails that

$$\begin{array}{ll} \frac{d}{dt} w(y(t)) & = -(K-1)(c_s - \lambda) + \sum_{S \in \mathcal{S}: S \neq \{s\}, S \cap S^* = \emptyset} (K - |S|) \frac{d}{dt} y_S(t) \\ & = -(K-1)(c_s - \lambda) - \sum_{S \in \mathcal{S}: S \neq \{s\}, S \cap S^* = \emptyset} (K - |S|) \sum_{u \in S, v \in S^*} \frac{d}{dt} \phi_{S,(uv)}(t) \\ & \leq -(c_s - \lambda)(K-1). \end{array}$$

Thus, it follows that (28) holds, with $\epsilon = \min(c_s - \lambda, \sum_{u \in V} c_u - (K-1)\lambda)$. ■

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