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ANISOTROPIC SELF-AVOIDING WALKS

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Abstract. We consider a model of self-avoiding walks on the lattice Z^d with different weights for steps in each of the $2d$ lattice directions. We find that the direction-dependent mass for the two-point function of this model has three phases: mass positive in all directions; mass identically $-\infty$; and masses of different signs in different directions. The final possibility can only occur if the weights are asymmetric, i.e. in at least one coordinate the weight in the positive direction differs from the weight in the negative direction. The boundaries of these phases are determined exactly. We also prove that if the weights are asymmetric then a typical N-step self-avoiding walk has order N distance between its endpoints.

1. Introduction

The self-avoiding walk has long been a standard model of a long linear polymer molecule in a good solvent (de Gennes, 1979; Madras and Slade, 1993; Vanderzande, 1998). The polymer is represented by a sequence of steps in a lattice; in the usual isotropic model, steps in each lattice direction receive the same weight. However, there are situations in which the isotropic model is not appropriate. One basic example occurs when a polymer chain contains a dipole on each repeat unit, and the polymer is subject to an external electric field. If each dipole is rigidly attached to the polymer backbone and parallel to it, then the individual dipole units add vectorially to create a single large end-to-end dipole. Then it is easy for an external field to stretch and orient the polymer, when dissolved in a good solvent of low molecular weight (see Section 3.5 of Blythe, 1979). The ability to orient polymers in

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX solution by applying an electric field can greatly improve the information obtained by light scattering experiments (Jennings, 1972; Lipson and Stockmayer, 1989).

Anisotropic self-avoiding walks have also been used as models of flux lines in Borgs *et al.* (1999). This application will be described in more detail later in this section. But first we shall make precise definitions of our general model and some key quantities.

An *N-step self-avoiding walk* (SAW) on the d-dimensional hypercubic lattice Z^d is a sequence $\omega = (\omega(0), \omega(1), \dots, \omega(N))$ of $N+1$ *distinct* sites of Z^d, such that $\omega(i)$ and $\omega(i-1)$ are nearest neighbours for each $i = 1, \dots, N$. The vector $\omega(i) - \omega(i-1)$ is called the i^{th} step of ω. We write $|\omega| = N$ to denote the *length* of the SAW $\omega = (\omega(0), \dots, \omega(N))$.

In the standard isotropic SAW model, all SAWs of a given length are weighted equally. In this paper, we consider the case where steps in different directions can have different weights. We first specify a vector of $2d$ nonnegative weights $z = (z_{1+}, z_{1,-}, \dots, z_{d-})$. Then we define the weight of the N-step SAW ω to be

$$z^\omega := \prod_{i=1}^{d} z_{i+}^{N_{i+}(\omega)} z_{i-}^{N_{i-}(\omega)}, \tag{1.1}$$

where $N_{i+}(\omega)$ and $N_{i-}(\omega)$ denote the number of steps that the walk ω takes in the positive and negative i-direction, respectively.

Our approach will focus on certain fundamental generating functions. For u and v in Z^d, let $G_z(u,v)$ be the generating function of all SAW's of all lengths that start at u and end at v:

$$G_z(u,v) \;=\; \sum_{\omega : u \to v} z^\omega \tag{1.2}$$

(where $\omega : u \to v$ means $\omega(0) = u$ and $\omega(|\omega|) = v$). We call G_z the *two-point function.*

For $N \geq 0$, we define $\chi_N(z)$ to be the generating function of all N-step SAW's starting at the origin $\vec{0} \in Z^d$:

$$\chi_N(z) \;=\; \sum_{\substack{\omega : \omega(0) = \vec{0} \\ |\omega| = N}} z^\omega \,. \tag{1.3}$$

The *susceptibility* is defined as

$$\chi(z) = \sum_{N=0}^{\infty} \chi_N(z) = \sum_{v \in Z^d} G_z(\vec{0}, v). \tag{1.4}$$

In Proposition 2.2 we shall show that the limit

$$\lambda(z) \;:=\; \lim_{N \to \infty} \chi_N(z)^{1/N} \tag{1.5}$$

exists, and that $\chi(z) < \infty$ if and only if $\lambda(z) < 1$.

The *mass* of the model is the exponential rate of decay of the two-point function. Because of the anisotropy of the z vector, we define the mass to be explicitly dependent upon direction. Let $||\cdot||$ be a norm on R^d. Let $\{v^{(n)}\}$ be a sequence of vectors in Z^d with norms tending to infinity. Assume that $v^{(n)}/||v^{(n)}||$ converges to a vector $\alpha \in R^d$. Then the mass in the direction α should be defined as

$$m[\alpha; z] = \lim_{n\to\infty} \frac{-\log G_z(\vec{0}, v^{(n)})}{||v^{(n)}||}. \tag{1.6}$$

For technical reasons, we shall usually work with the following simpler definition. Let v be a non-zero vector in Z^d; then we define

$$m[v; z] = \lim_{L\to\infty} \frac{-\log G_z(\vec{0}, Lv)}{L}. \tag{1.7}$$

The existence of this limit will be proven in Theorem 3.4, and the equivalence of (1.6) and (1.7) is the subject of Corollary 3.6.

At this point let us review what is known for the isotropic case (see Sections 1.2, 1.3 and 3.1 of Madras and Slade (1993) for more details). Suppose that all $2d$ components of z are identical, i.e. $z_{i+} = z_{i-} = z_0$ for every i for some positive real number z_0. Then the weight (1.1) of a SAW ω is simply $z_0^{|\omega|}$. We shall write $\chi(z_0)$, $G_{z_0}(\vec{0}, v)$, etc. when discussing this case; this notation conforms to that of Madras and Slade (1993). Then $\chi_N(z_0) = c_N z_0^N$, where c_N is the number of N-step SAW's that start at the origin. Since $c_N^{1/N}$ converges to a constant μ as $N \to \infty$, we see that $\lambda(z_0) = \mu z_0$. Therefore $\lambda(z_0) = 1$ if and only if $z_0 = \mu^{-1}$; this value μ^{-1} is the "critical point" of the isotropic SAW model, often written z_c. The case $z_0 < z_c$ (corresponding to $\lambda(z_0) < 1$) is the "subcritical" case; here $\chi(z_0)$ converges, and the two-point function $G_{z_0}(\vec{0}, v)$ decays exponentially in v in all directions (i.e., the mass $m[v; z_0]$ is strictly positive for $z_0 < z_c$). Furthermore, the mass decreases to 0 as z_0 increases to z_c. In the "supercritical" case $z_0 > z_c$ (corresponding to $\lambda(z_0) > 1$), the susceptibility is infinite and all masses are $-\infty$ (corresponding to two-point functions that do not converge; see Chayes and Chayes (1986)).

We now return to the anisotropic model. If all components of z are small, then $\chi_N(z)$ decays exponentially in N; in this case, long SAW's are rare in the ensemble of all SAW's, and the masses are all positive. At the opposite extreme, if the components of z are sufficiently large, then the two-point functions are infinite, and the mass is $-\infty$. In the case of asymmetric weights, it turns out that there is an intermediate third possibility: the masses are finite, but they are *negative* in some directions. This indicates exponential growth of the two-point function with distance (at least in some directions). We shall derive a precise description of the boundaries of these three phases. To describe them, we need to introduce symmetrized weights. Suppose all $2d$ components of z are strictly positive. Then we define $\bar{z}$, the *symmetrized weight vector*, to be the vector $(\bar{z}_{1+}, \bar{z}_{1-}, \dots, \bar{z}_{d-})$ whose components are given by

$$\bar{z}_{i+} = \bar{z}_{i-} = \sqrt{z_{i+} z_{i-}}\,.$$

The definition becomes a bit complicated if some of the components of z are zero (see Lemma 2.6), but that is a special case that need not be considered yet.

It is not hard to show that $\lambda(\bar{z}) \le \lambda(z)$ (see Lemma 2.6). It turns out that the λ function describes the phase boundaries as follows.

Theorem 1.1. *(i) If $\lambda(z) < 1$, then $m[v; z]$ is positive and finite for every nonzero v in Z^d.*
(ii) If $\lambda(z) > 1$ and $\lambda(\bar{z}) < 1$, then $m[v; z]$ is finite for every v in Z^d, but $m[v; z]$ is positive for some vectors v and negative for others.
(iii) Finally, if $\lambda(\bar{z}) > 1$, then $m[v; z]$ is $-\infty$ for every nonzero v.

Note that case (ii) cannot occur if $z_{i+} = z_{i-}$ for every i (in particular, in the isotropic case). Theorem 1.1 will follow immediately from Theorem 2.3 and Corollary 3.5 below.

Anisotropic self-avoiding walks were considered in Borgs *et al.* (1999) as models of flux lines. In that paper, collections of mutually avoiding SAWs that begin and end on the boundary of a large region modelled the penetration of flux lines through the region. In the model of primary interest there, the anisotropy in the weights modelled the influence of an external magnetic field. It turns out that the flux lines model also has three phases. One phase boundary was the equation $\lambda(z) = 1$; the region $\lambda(z) < 1$ was shown to be a "Meissner phase" in which flux lines were unlikely to be long enough to penetrate the interior of the region. However, we could find no nonanalytic behaviour of the flux line model on the surface $\lambda(\bar{z}) = 1$, which is a phase boundary for the single SAW in the present paper. Rather, when $\lambda(z) > 1$ in the flux line model, we found many SAWs crossing the region in a direction in which the mass was negative; the second phase boundary described a transition to a maximally packed configuration of straight parallel lines.

In the present paper we also address the question of what a typical SAW looks like when the weights are not symmetric, i.e. when $z_{i+} \ne z_{i-}$ for at least one i. By comparison with ordinary random walks, one expects linear drift; i.e., the end-to-end distance of a typical N-step SAW (with weights given by z) is of the order N. We prove the following version of this result in Section 4.

Theorem 1.2. *Fix a weight vector z with all weights nonzero. Assume that z is not symmetric (i.e. $z \ne \bar{z}$). For each N, let v_N^* be the most likely endpoint of an N-step SAW. (That is, v_N^* is the vector $v \in Z^d$ for which the generating function of all N-step SAWs from $\vec{0}$ to v is maximized.) Then (for any norm $||\cdot||$)*

$$\liminf_{N\to\infty} \frac{||v_N^*||}{N} \;>\; 0.$$

In the context of a linear polymer with repeated attached dipoles parallel to its backbone, Theorem 1.2 says that even a small external electric field is enough to stretch and orient the polymer so that its end-to-end distance is proportional to its molecular weight N, instead of obeying the usual isotropic scaling behaviour (which is $\approx N^{0.6}$ in three dimensions). This result was predicted by Manna and Chakrabarti (1984) on the basis of exact enumeration data as well as a real space renormalization group analysis. We remark that symmetric models (satisfying $z_{i+} = z_{i-}$ for every i) should be in the same universality class as the isotropic model.

The paper is organized as follows. Section 2 begins with some elementary results about anisotropic SAWs, and then explores properties of a slightly different mass function, $M[v; z]$. This mass function is for SAWs that start at the origin and end at Lv, with the additional condition that they always stay between two parallel hyperplanes that pass through the origin and Lv respectively. We use the term "slab" to denote the region between two such hyperplanes. Such SAWs in slabs are easier to work with then unrestricted SAWs, primarily because two such SAWs can always be concatenated. SAWs in slabs play a role analogous to bridges in Madras and Slade (1993, especially Section 4.1) and in Chayes and Chayes (1986) (the latter reference uses the term "cylinder walks" for bridges). The main results about the mass function M are stated in Theorem 2.3. The rest of Section 2 is devoted to the proof of the various parts of this theorem. Section 3 proves that the masses M and m are equal in general, except perhaps on the critical surface $\lambda(\bar{z}) = 1$. Thus most of the results of Theorem 2.3 immediately extend to the mass m, and in particular this verifies the description of the three phases described in Theorem 1.1. The anisotropic "bubble diagram" plays an important role here. Finally, Section 4 uses the results of Sections 2 and 3 to prove Theorem 1.2.

2. Basic results and masses of walks in slabs

In this section we state and prove some basic results that generalize the well known properties of isotropic self-avoiding walks (SAWs) to anisotropic SAWs. Then we consider SAWs in slabs (which generalizes the concept of "bridges" or "cylinder walks" that have been used elsewhere; see Madras and Slade (1993) or Chayes and Chayes (1986), as well as the proof of Lemma 2.8 below), and prove some important properties of their masses.

Notation: For a vector $v = (v_1, \ldots, v_d)$ in R^d, we write $||v||_1 = |v_1| + \cdots + |v_d|$ to denote the L_1 norm of v. If $\omega = (\omega(0), \ldots, \omega(N))$ is a SAW, then $-\omega = (-\omega(0), \ldots, -\omega(N))$. We write $\omega_i(j)$ to denote the i^{th} coordinate of the j^{th} site $\omega(j)$. If $\psi = (\psi(0), \ldots, \psi(M))$ is another SAW, then the concatenation $\omega \circ \psi$ is defined to be the $(N + M)$-step SAW

$$(\omega(0), \ldots, \omega(N), \omega(N) + \psi(1) - \psi(0), \ldots, \omega(N) + \psi(M) - \psi(0)).$$

Let $z_{\max}$ (respectively, $z_{\min}$) denote the maximum (respectively, minimum) of the $2d$ weights $\{z_{1+}, z_{1-}, \ldots, z_{d-}\}$. We then write $\log z$ to denote the vector

$(\log z_{1+}, \log z_{1-}, \dots, \log z_{d-})$. To avoid some trivial remarks, we shall generally assume that $z_{i+} + z_{i-} > 0$ for every $i = 1, \dots, d$.

Lemma 2.1. *Let U be a fixed set of vectors in R^d. For each $u \in U$, let a_u be a nonnegative real number. Then the function*

$$\log\left(\sum_{u\in U} a_u e^{u\cdot\beta}\right)$$

is a convex function of $\beta \in R^d$.

Proof: This is a well-known consequence of Hölder's inequality. For $\beta', \beta'' \in R^d$ and $0 < \lambda < 1$, we have

$$\sum_{u\in U} a_u e^{u\cdot[\lambda\beta' + (1-\lambda)\beta'']} \leq \left(\sum_{u\in U} a_u e^{u\cdot\beta'}\right)^{\lambda} \left(\sum_{u\in U} a_u e^{u\cdot\beta''}\right)^{1-\lambda}.$$

The lemma follows upon taking log of both sides. □

Proposition 2.2. *Let $0 \leq z_{\min} \leq z_{\max} < \infty$. Then*
(i) The limit

$$\lambda(z) := \lim_{N\to\infty} \chi_N(z)^{1/N} \tag{2.1}$$

exists in $[0, \infty)$, and

$$\lambda(z) = \inf_{N\geq 1} \chi_N(z)^{1/N}. \tag{2.2}$$

(ii) $\lambda(z)$ is a log-convex function of $\log z$. Hence $\lambda(z)$ is continuous on $(0,\infty)^{2d}$.
(iii) $\chi(z) < \infty$ if and only if $\lambda(z) < 1$.
(iv) $\chi(z) \uparrow \infty$ as $\lambda(z) \uparrow 1$.
(v) For any $t > 0$, $\lambda(tz) = t\lambda(z)$.

Proof: (i): Any $(M + N)$-step SAW can be expressed as the concatenation of an M-step and an N-step self-avoiding walk. Therefore

$$\chi_{M+N}(z) \leq \chi_M(z)\chi_N(z) \quad \text{for all } M, N \geq 1. \tag{2.3}$$

Part (i) now follows from the usual subadditivity (submultiplicativity) property (see Section 1.2 of Madras and Slade 1993), together with the observation that

$$z_{\max} \leq \lambda(z) \leq 2dz_{\max}, \tag{2.4}$$

which follows from $z_{\max}^N \leq \chi_N(z) \leq (2dz_{\max})^N$. In particular, we note that $\lambda(z)$ is nonzero unless z is identically 0.
(ii): The convexity of $\log \chi_N(z)$, and hence of $\log \lambda(z)$, follows from Lemma 2.1. Convexity implies continuity on the interior of the set where the function is finite.

(*iii*): If $\lambda(z) < 1$, then clearly $\chi(z) = \sum_N \chi_N(z)$ converges. And if $\lambda(z) \geq 1$, then $\chi_N(z) \geq 1$ for every N by (2.2).
(*iv*): If $\lambda(z) < 1$, then by (2.2) we have

$$\chi(z) = \sum_{N=0}^{\infty} \chi_N(z) \geq \sum_{N=0}^{\infty} \lambda(z)^N = \frac{1}{1-\lambda(z)}. \tag{2.5}$$

Part (*iv*) follows.
(*v*): This follows from $\chi_N(tz) = t^N \chi_N(z)$. $\square$

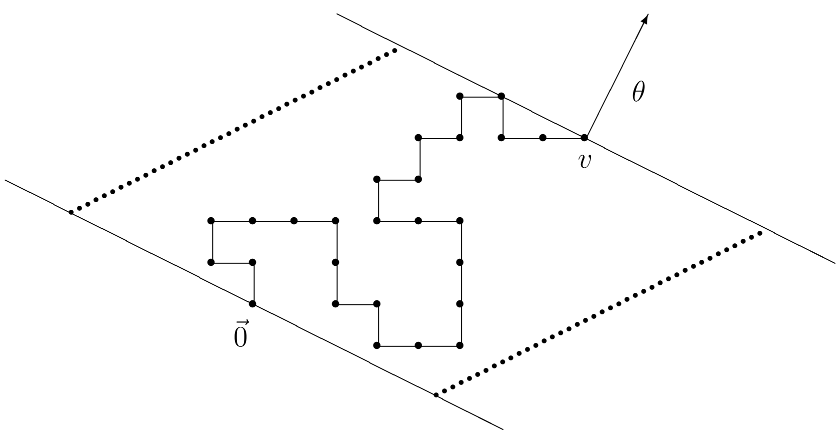

Figure 1: A self-avoiding walk in Z^2 from $\vec{0}$ to v that lies in $\mathrm{Slab}^T(v|\theta)$. The boundary of $\mathrm{Slab}(v|\theta)$ is indicated by the two angled solid lines, which are perpendicular to the vector θ. The dotted lines denote the boundary of $\mathrm{Tube}^T(v)$; they are parallel to the line segment from $\vec{0}$ to v (not shown).

We now need to introduce a direction-dependent mass for a restricted set of SAWs. Consider a weight vector z and a (nonzero) lattice point $v \in Z^d$. Also, let θ be a vector in R^d such that $\theta \cdot v > 0$. Let $\mathrm{Slab}(v|\theta)$ be the set of lattice points between the two hyperplanes through $\vec{0}$ and v that are normal to θ; that is,

$$\mathrm{Slab}(v|\theta) \;=\; \{u \in Z^d : 0 < u \cdot \theta \leq v \cdot \theta\}\,.$$

See Figure 1. Let $B_z(\vec{0}, v|\theta)$ be the generating functions of all SAWs that start at $\vec{0}$, end at v, and lie entirely in $\vec{0} \cup \mathrm{Slab}(v|\theta)$. Then for all integers $j, k \geq 1$, we have

$$B_z(\vec{0}, jv|\theta) B_z(\vec{0}, kv|\theta) \leq B_z(\vec{0}, (j+k)v|\theta). \tag{2.6}$$

Again, we can use subadditivity to define the mass $M[v;z]$ (actually, $M[v;z|\theta]$, but we shall show in Lemma 2.4 that M is independent of θ) via

$$M[v;z] \equiv M[v;z|\theta] = \lim_{L\to\infty} \frac{-\log B_z(\vec{0}, Lv|\theta)}{L} = \inf_{L\geq 1} \frac{-\log B_z(\vec{0}, Lv|\theta)}{L} \tag{2.7}$$

(analogously to Proposition 4.1.8 of Madras and Slade 1993). In particular, for every $L \geq 1$ we have

$$B_z(\vec{0}, Lv|\theta) \leq e^{-LM[v;z]}. \tag{2.8}$$

If $z_{\min} = 0$, then we often have to restrict our choices of v and θ. Let

$$W(z) = \{w \in R^d \setminus \{\vec{0}\} : \ w_i \leq 0 \text{ if } z_{i+} = 0, \text{ and } w_i \geq 0 \text{ if } z_{i-} = 0\}.$$

Thus a nonzero vector $v \in Z^d$ is in $W(z)$ if and only if there exists a SAW ω from $\vec{0}$ to v such that $z^\omega > 0$. It is easy to see that for $v \in Z^d \setminus \{\vec{0}\}$, the mass $M[v;z|v]$ equals $+\infty$ if and only if v is not in $W(z)$ (Lemma 2.4(i)). To avoid some trivialities in the statements of some theorems, we shall often require $v \in W(z) \cap Z^d$, in addition to the condition that $\theta \cdot v > 0$. Of course, if $z_{\min} > 0$, then $W(z) = R^d \setminus \{\vec{0}\}$, so there is no such restriction.

Before we state Theorem 2.3, which includes the main results of this section, we require two more definitions. Using the fact that these masses do not depend on θ, we define

$$M_0(z) := \inf_{v\in Z^d\setminus\{0\}} \frac{M[v;z]}{||v||_1}. \tag{2.9}$$

If z satisfies $0 < z_{\min} \leq z_{\max} < \infty$, then we define $\bar{z}$ to be the "symmetrized" weight vector, whose components are

$$\bar{z}_{i+} = \bar{z}_{i-} = \sqrt{z_{i+}z_{i-}}\,.$$

(See Lemma 2.6 for the definition of $\bar{z}$ when $z_{\min} = 0$.)

Theorem 2.3. *(i) M_0 is a concave function of $\log z$, finite on*

$$\{z : 0 < \lambda(\bar{z}) \leq 1\}$$

(which contains $\{z : 0 < \lambda(z) \leq 1\}$, since $\lambda(\bar{z}) \leq \lambda(z)$);
(ii) M_0 is identically $-\infty$ on $\{z : z_{\min} > 0, \lambda(\bar{z}) > 1\}$. In fact, for every z in this set, $M[v;z] = -\infty$ for every nonzero v;
(iii) $M_0(z) > 0$ if $\lambda(z) < 1$;
(iv) $M_0(z) < 0$ if $\lambda(z) > 1$;
(v) $M_0(z) = 0 = \lim_{t\uparrow 1} M_0(tz)$ if $\lambda(z) = 1$.

This theorem will follow from several intermediate results, which will be collected in the proof that appears at the end of this section.

We remark that (ii) above is false in some cases if we omit the condition $z_{\min} > 0$. For example, suppose $z_{i+} = 1$ and $z_{i-} = 0$ for every i. Then, by the definition of $\bar{z}$ in Lemma 2.6, $\bar{z} = z$ so that $\lambda(\bar{z}) = \lambda(z) = d > 1$, but $B_z(\vec{0}, v|v) \leq d^{||v||_1}$ for every v, so $M_0(z) \geq -\log d > -\infty$.

We now define "truncated" masses. For each nonzero $v \in Z^d$, let $\overline{\vec{0}v}$ be the (infinite) line that passes through the points $\vec{0}$ and v. For each positive integer $T > 0$, let $\mathrm{Tube}^T(v)$ be the set of points in R^d whose (Euclidean) distance from $\overline{\vec{0}v}$ is at most T. Next, for each $\theta \in R^d$ such that $v \cdot \theta > 0$, let $\mathrm{Slab}^T(v|\theta) = \mathrm{Slab}(v|\theta) \cap \mathrm{Tube}^T(v)$. Let $B_z^T(\vec{0}, v|\theta)$ be the generating function of all SAWs that start at $\vec{0}$, end at v, and lie entirely in $\vec{0} \cup \mathrm{Slab}^T(v|\theta)$. As above, we can use subadditivity to define the truncated mass

$$M^T[v;z] \;\equiv\; M^T[v;z|\theta] \;=\; \lim_{L\to\infty} \frac{-\log B_z^T(\vec{0}, Lv|\theta)}{L} \;=\; \inf_{L\geq 1} \frac{-\log B_z^T(\vec{0}, Lv|\theta)}{L}. \tag{2.10}$$

Observe that $M^T[v;z]$ is decreasing in T and is bounded below by $M[v;z]$.

The following lemma describes some basic properties of these masses. After proving this lemma we shall generally suppress the θ in the notation for the masses M.

Lemma 2.4. *(i) For each T, v and θ such that $\theta \cdot v > 0$, $M^T[v;z|\theta]$ is a finite concave (and hence continuous) function of* $\log z$*, for $z \in (0,\infty)^{2d}$. If we fix some components of z to be 0, then $M^T[v;z|\theta]$ is a finite concave function of the logarithms of the nonzero components of z (provided that v is in the appropriate $W(z)$). Hence $M[v;z|\theta] < +\infty$ under these conditions.*
(ii) For every $T > ||v||_1$, $M^T[v;z|\theta]$ does not depend on θ, subject to the constraint $\theta \cdot v > 0$.
(iii) $M[v;z|\theta] = \lim_{T\to\infty} M^T[v;z|\theta]$. Hence, $M[v;z|\theta]$ does not depend on θ.
(If $z_{\min} = 0$ in parts (ii) and (iii), then we add the condition that v is in $W(z)$.)

Corollary 2.5. *$M_0(z) < +\infty$ for every nonzero z. For $z \in (0,\infty)^{2d}$, $M_0(z)$ is a concave function of* $\log z$*. If we fix some components of z to be 0, then $M_0(z)$ is a concave function of the logarithms of the nonzero components of z.*

Proof of Lemma 2.4: Concavity is again the result of Lemma 2.1. The proof of finiteness in (i) and the proof of (iii) are straightforward adaptations of the proof of Lemma 4.1.11 in Madras and Slade (1993); see also the proof of Lemma 4.2 in Borgs *et al.* (1999).

For part (ii): Fix v and θ such that $\theta \cdot v > 0$, and fix $T > ||v||_1$. (If $z_{\min} = 0$, then v must be in $W(z)$.) We shall prove that $M^T[v;z|\theta] = M^T[v;z|v]$.

First, choose a positive integer K such that the translated slab $Kv + \mathrm{Slab}^T(v|v)$ lies completely in the half-space $\{w \in R^d : \theta \cdot w > 0\}$. Then we see that

$$\begin{aligned} kv + \mathrm{Slab}^T(jv|v) \;&\subset \mathrm{Slab}^T\big((2k+j)v|\theta\big) \\ &\qquad \text{for every } j = 1,2,\ldots \text{ and } k = K, K+1,\ldots \end{aligned} \tag{2.11}$$

Let v^* be the site in Z^d whose coordinates are

$$v_i^* = \begin{cases} v_i & \text{if } \theta_i v_i > 0 \\ 0 & \text{if } \theta_i v_i \le 0 \end{cases} \qquad (i = 1, \dots, d). \tag{2.12}$$

Let ψ be a $||v||_1$-step self-avoiding walk that starts at $\vec{0}$, ends at v, and passes through the site v^* (necessarily at the $||v^*||_1^{th}$ step). Let $\hat{\psi}$ be the SAW from $\vec{0}$ to v obtained by taking the steps of ψ in the reverse order: i.e. the SAW whose sites are $\hat{\psi}(j) = v - \psi(||v||_1 - j)$, $j = 0, \dots, ||v||_1$. Also, for $s = 1, 2, \dots$, let $\psi^{(s)}$ (respectively, $\hat{\psi}^{(s)}$) denote the concatenation of s copies of ψ (respectively, $\hat{\psi}$).

First observe that $v \cdot \psi(j)$ is strictly increasing in j for $0 \le j \le ||v||_1$, so ψ and $\hat{\psi}$ are both SAWs from $\vec{0}$ to v that lie entirely in $\vec{0} \cup \mathrm{Slab}^T(v|v)$. In particular, we see that $\psi^{(k)}$ and $\hat{\psi}^{(k)}$ are both SAWs. Next, the definition of v^* shows that $\theta \cdot \psi(j)$ is strictly increasing in j for $0 \le j \le ||v^*||_1$, and nonincreasing for $||v^*||_1 \le j \le ||v||_1$. Since $\theta \cdot \psi(0) = 0$, $\theta \cdot \psi(||v^*||_1) = \theta \cdot v^* > 0$, and $\theta \cdot \psi(||v||_1) = \theta \cdot v > 0$, we conclude that

$$\theta \cdot v^* \ge \theta \cdot \psi(j) > 0 \qquad \text{for every } j = 1, \cdots, ||v||_1 \tag{2.13}$$

and hence

$$\theta \cdot v - \theta \cdot v^* \le \theta \cdot \hat{\psi}(j) < \theta \cdot v \qquad \text{for every } j = 0, \cdots, ||v||_1 - 1. \tag{2.14}$$

Suppose that I is an integer greater than $(\theta \cdot v^*)/(\theta \cdot v)$. Then ψ (respectively, $(I-1)v + \hat{\psi}$) is a SAW from $\vec{0}$ to v (respectively, from $(I-1)v$ to Iv) which lies entirely in $\vec{0} \cup \mathrm{Slab}^T(Iv|\theta)$. Furthermore, for any integer $s \ge 1$, $\psi^{(s)}$ lies entirely in $\vec{0} \cup \mathrm{Slab}^T\big((I-1+s)v|\theta\big)$, as does $(I-1)v + \hat{\psi}^{(s)}$. Now fix an integer I which is greater than K and $(\theta \cdot v^*)/(\theta \cdot v)$. Let $\zeta = \psi^{(I-1)} \circ \hat{\psi}^{(I-1)}$, and let $I' = 2(I-1)$. Then we see that ζ is a SAW from $\vec{0}$ to $I'v$ which lies entirely in $\vec{0} \cup \mathrm{Slab}^T(I'v|\theta)$. Also, ζ lies in $\vec{0} \cup \mathrm{Slab}^T(I'v|v)$.

Now let j be any positive integer, and consider any SAW ω from $\vec{0}$ to jv that lies entirely in $\vec{0} \cup \mathrm{Slab}^T(jv|v)$. Then $\zeta \circ \omega \circ \zeta$ is a SAW from $\vec{0}$ to $(2I'+j)v$ that lies entirely in $\vec{0} \cup \mathrm{Slab}^T\big((2I'+j)v|\theta\big)$ [using (2.11) with $k = I'$]. Therefore

$$B_z^T\big((2I'+j)v|\theta\big) \ \ge\ \big(z^{\zeta}\big)^2 B_z^T(jv|v)\,. \tag{2.15}$$

Since (2.15) holds for every $j \ge 1$, we conclude that $M^T[v; z|\theta] \le M^T[v; z|v]$.

We now need to show $M^T[v; z|\theta] \ge M^T[v; z|v]$. This is based on the following claim: If ρ is a SAW from $\vec{0}$ to jv that lies in $\vec{0} \cup \mathrm{Slab}^T(jv|\theta)$, then $\zeta \circ \rho \circ \zeta$ is a SAW from $\vec{0}$ to $(2I'+j)v$ that lies entirely in $\vec{0} \cup \mathrm{Slab}^T\big((2I'+j)v|v\big)$. The analogue of (2.15) and the rest of the proof follow from this claim. To prove the claim, it suffices to show that if $w \in \mathrm{Slab}^T(jv|\theta)$, then $I'v + w \in \mathrm{Slab}^T\big((2I'+j)v|v\big)$. Choose $w \in \mathrm{Slab}^T(jv|\theta)$; we need to show

$$0 < (I'v + w) \cdot v \le (2I'+j)v \cdot v. \tag{2.16}$$

The point $(2I'+j)v+w$ is in $\mathrm{Tube}^T(v)$ but not in $\mathrm{Slab}^T\big((2I'+j)v|\theta\big)$, so it follows from (2.11) that $(2I'+j)v+w$ is not in $I'v+\mathrm{Slab}^T(jv|v)$; in fact, we must have $\big((2I'+j)v+w\big)\cdot v > (I'v+jv)\cdot v$, from which we obtain the left inequality of (2.16). Similarly, $w-jv$ is not in $\mathrm{Slab}^T\big((2I'+j)v|\theta\big)$, so it is not in $I'v+\mathrm{Slab}^T(jv|v)$; we deduce that $(w-jv)\cdot v \le I'v\cdot v$. This gives the right inequality of (2.16). This completes the proof of the lemma. $\square$

Lemma 2.6. *For a given weight vector z, let*

$$I(z) \;=\; \big\{ i\in\{1,\dots,d\} : \min\{z_{i+},z_{i-}\}>0\big\}\,.$$

Let $\bar z$ be the symmetrized weight vector with components

$$\bar z_{i+}=\bar z_{i-}=\sqrt{z_{i+}z_{i-}}\ \text{ if } i\in I(z), \qquad \text{and } \bar z_{i+}=z_{i+} \text{ and } \bar z_{i-}=z_{i-} \text{ if } i\notin I(z)\,.$$

Then
(i) $\lambda(\bar z)\le\lambda(z)$.
(ii) Suppose ω is an N-step self-avoiding walk, and let $\omega(N)-\omega(0)=v$. Then

$$z^\omega \;=\; \bar z^\omega \prod_{i\in I(z)}\left(\frac{z_{i+}}{z_{i-}}\right)^{v_i/2}.$$

(iii) Fix $v\in Z^d$. Then for any θ, with $\theta\cdot v>0$,

$$B_z(\vec 0,v|\theta) \;=\; B_{\bar z}(\vec 0,v|\theta)\prod_{i\in I(z)}\left(\frac{z_{i+}}{z_{i-}}\right)^{v_i/2}.$$

The same equation holds if we replace $B_\cdot(\vec 0,v|\theta)$ by $B^T_\cdot(\vec 0,v|\theta)$ or by $G_\cdot(\vec 0,v)$. Also,

$$M[v;\bar z] \;=\; M[v;z]+\frac12\sum_{i\in I(z)} v_i\log(z_{i+}/z_{i-})\,.$$

In particular, $M[v;z]$ is finite if and only if $M[v;\bar z]$ is finite.

Proof: (i): Fix z. For a point $u\in R^d$, let $r[u]$ be the point whose i^{th} coordinate is $-u_i$ if $i\in I(z)$ and is u_i if $i\notin I(z)$. For an N-step SAW ω, let $r[\omega]$ be the SAW $(r[\omega(0)],\dots,r[\omega(N)])$. Observe that

$$\begin{aligned} z^\omega z^{r[\omega]} &= \prod_{i=1}^d z_{i+}^{N_{i+}(\omega)+N_{i+}(r[\omega])}\, z_{i-}^{N_{i-}(\omega)+N_{i-}(r[\omega])} \\ &= \prod_{i\in I(z)} (z_{i+}z_{i-})^{N_{i+}(\omega)+N_{i-}(\omega)} \prod_{i\notin I(z)} z_{i+}^{2N_{i+}(\omega)} z_{i-}^{2N_{i-}(\omega)} \qquad (2.17)\\ &= (\bar z^\omega)^2. \end{aligned}$$

Therefore, using the arithmetic-geometric mean inequality,

$$\begin{aligned}
\chi_N(z) &= \sum_{\omega:|\omega|=N,\,\omega(0)=\vec{0}} z^{\omega} \\
&= \sum_{\omega:|\omega|=N,\,\omega(0)=\vec{0}} \left(\frac{z^{\omega}+z^{r[\omega]}}{2}\right) \\
&\geq \sum_{\omega:|\omega|=N,\,\omega(0)=\vec{0}} \sqrt{z^{\omega} z^{r[\omega]}} \\
&= \sum_{\omega:|\omega|=N,\,\omega(0)=\vec{0}} \bar{z}^{\omega} \;=\; \chi_N(\bar{z}).
\end{aligned} \tag{2.18}$$

Part (i) follows from Equation (2.18).
(ii): Let ω be an N-step SAW such that $\omega(0)-\omega(N)=v$. Notice that $N_{i+}(\omega)-N_{i-}(\omega)=v_i$ for each i. Therefore

$$\begin{aligned}
z^{\omega} &= \prod_{i=1}^{d} z_{i+}^{N_{i+}(\omega)} z_{i-}^{N_{i-}(\omega)} \\
&= \prod_{i\in I(z)} (z_{i+}z_{i-})^{[N_{i+}(\omega)+N_{i-}(\omega)]/2} z_{i+}^{[N_{i+}(\omega)-N_{i-}(\omega)]/2} z_{i-}^{[N_{i-}(\omega)-N_{i+}(\omega)]/2} \\
&\qquad\qquad\qquad \times \prod_{i\notin I(z)} z_{i+}^{N_{i+}(\omega)} z_{i-}^{N_{i-}(\omega)} \\
&= \prod_{i=1}^{d} \bar{z}_{i+}^{N_{i+}(\omega)} \bar{z}_{i-}^{N_{i-}(\omega)} \prod_{i\in I(z)} \left(\frac{z_{i+}}{z_{i-}}\right)^{v_i/2},
\end{aligned}$$

and part (ii) follows.
(iii): This follows immediately from (ii). $\square$

Lemma 2.7. *If $\lambda(z)<1$, then $M[v;z]\geq -||v||_1\log\lambda(z)>0$ for every non-zero v. That is, $M_0(z)\geq -\log\lambda(z)$.*

Proof: Fix a nonzero v and a vector θ such that $\theta\cdot v>0$. Choose D such that $\lambda(z)<D<1$, and choose $A>0$ such that $\chi_n(z)\leq AD^n$ for every $n\geq 1$. Then, for any integer $L\geq 1$,

$$\begin{aligned}
B_z(\vec{0},Lv|\theta) &\leq \sum_{n=L||v||_1}^{\infty} \chi_n(z) \\
&\leq \frac{AD^{L||v||_1}}{1-D}.
\end{aligned}$$

Take $-\log$, divide by L and let $L\to\infty$ to obtain $M[v;z]\geq -||v||_1\log D$. Since this holds for every D such that $\lambda(z)<D<1$, the result follows. $\square$

A weaker version of the following result (namely, that $M_0(z) < 0$ if $\lambda(z) > 1$) can be deduced from the proof of Lemma 4.2 of Borgs et al. (1999). The extension to the case $\lambda(z) = 1$ does not follow simply from continuity, since $M_0(z)$ can equal $-\infty$ for some z's such that $\lambda(z) > 1$.

Lemma 2.8. *If $\lambda(z) \geq 1$, then $M_0(z) \leq 0$.*

Proof: Assume without loss of generality that $z_{1+} \geq z_{1-}$ and $z_{1+} > 0$. We begin with some notation, following Madras and Slade (1993). Let ω be an N-step SAW ($N \geq 0$). We say that ω is a *half-space walk* if $\omega(0) = 0$ and $0 < \omega_1(j)$ for every $j = 1, \ldots, N$. We say that ω is a *bridge* if $\omega(0) = 0$ and $0 < \omega_1(j) \leq \omega_1(N)$ for every $j = 1, \ldots, N$. If ω is a bridge, then $\omega_1(N)$ is the *span* of ω.

Let ω be an N-step half-space walk. The Hammersley-Welsh argument (see Hammersley and Welsh 1962, or Section 3.1 of Madras and Slade, 1993) constructs a finite sequence of bridges $\omega^{(1)}, \ldots, \omega^{(k)}$, having spans $A_1, \ldots, A_k$, such that

$$A_1 > A_2 > \cdots > A_k \qquad \text{and}$$
$$\omega = \omega^{(1)} \circ (-\omega^{(2)}) \circ \omega^{(3)} \circ (-\omega^{(4)}) \circ \cdots \circ ((-1)^{k+1} \omega^{(k)}) .$$

Since $z_{1+} \geq z_{1-}$, we have that $z^{\omega^{(i)}} \geq z^{-\omega^{(i)}}$ for each bridge $\omega^{(i)}$, and hence

$$z^{\omega} = \prod_{i=1}^{k} z^{(-1)^{i+1}\omega^{(i)}} \leq \prod_{i=1}^{k} z^{\omega^{(i)}} \qquad \text{for the half-space walk } \omega. \tag{2.19}$$

Let $B_{z,A}$ be the generating function of all bridges of span A; let B_z and H_z be the generating functions for all bridges and half-space walks, respectively. (So $B_z = \sum_{A=0}^{\infty} B_{z,A}$). Then, by (2.19),

$$H_z \leq \prod_{A=1}^{\infty} \big(1 + B_{z,A}\big) \leq \prod_{A=1}^{\infty} \exp(B_{z,A}) = e^{B_z - 1}.$$

Let ξ be the one-step SAW from $\vec{0}$ to $(1, 0, \ldots, 0)$. For any N-step SAW ψ, there is an n in $\{0, 1, \ldots, N\}$ such that $\big(\psi(n), \ldots, \psi(N)\big) - \psi(n)$ and $\xi \circ \big(\psi(n), \ldots, \psi(0)\big)$ are both half-space walks. Therefore

$$\chi(z) z_{1+} \leq (H_z)^2 \leq e^{2B_z - 2}.$$

So if $\lambda(z) \geq 1$, then $\chi(z)$ diverges, and hence B_z diverges.

Next we have

$$\begin{aligned} B_z &= 1 + \sum_{y: y_1 > 0} B_z(\vec{0}, y | (1, 0, \ldots, 0)) \\ &\leq 1 + \sum_{y: y_1 > 0} e^{-M[y;z]} \\ &\leq 1 + \sum_{y: y_1 > 0} e^{-||y||_1 M_0(z)}, \end{aligned}$$

which is finite whenever $M_0(z) > 0$. So if $\lambda(z) \geq 1$, then the conclusion of the preceding paragraph shows that $M_0(z) \leq 0$. $\square$

To state the next lemma, we introduce, for every z with $0 < \lambda(z) < \infty$, its dual $z^* = z/\lambda(z)^2$. Note that z^* is defined in such a way that

$$\lambda(z^*)\lambda(z) = 1, \tag{2.20}$$

see Lemma 2.2(v).

Lemma 2.9. *If $\lambda(z) > 1$, then*

$$M_0(z) \leq -M_0(z^*), \tag{2.21}$$

implying in particular that $M_0(z) \leq -\log\lambda(z) < 0$.

Proof: The concavity property of M_0 (Corollary 2.5) shows that for every nonzero z' and every $t > 0$, we have

$$M_0(z') \geq \frac{1}{2}M_0(tz') + \frac{1}{2}M_0\Big(\frac{1}{t}z'\Big).$$

Assume $\lambda(z) > 1$. Let $t = \lambda(z)^{-1}$ and $z' = tz$. Then $\lambda(z') = 1$ by Proposition 2.2(v), so $M_0(z') \leq 0$ by Lemma 2.8. Therefore

$$M_0(z) = M_0\Big(\frac{1}{t}z'\Big) \leq 2M_0(z') - M_0(tz') \leq -M_0(z^*),$$

where we have used that $tz' = t^2z = z^*$ in the last step. Also, $\lambda(z^*) = 1/\lambda(z) < 1$ by (2.20), so

$$M_0(z) \leq -M_0(z^*) \leq \log\big(\lambda(z^*)\big) = -\log\big(\lambda(z)\big)$$

by (2.21) and Lemma 2.7. $\square$

Lemma 2.10. *Assume z satisfies $z_{i+} = z_{i-} > 0$ for every i. If $M[v;z] < 0$ for some nonzero v, then $M[u;z] = -\infty$ for every nonzero u.*

Proof: Assume $M[v;z] < 0$ for some nonzero v, and choose i such that $v_i \neq 0$. By the symmetry assumption on z, we know $M[v;z] = M[-v;z]$, so we can assume without loss of generality that $v_i > 0$.

For each $i = 1, \ldots, d$, let $e^{(i)}$ be the unit vector in the positive i^{th} coordinate direction. Let v' be the vector whose coordinates agree with those of $-v$ except that $v'_i = v_i$. Then $v + v' = 2v_ie^{(i)}$, and for all $k = 1, 2, \ldots$

$$B_z(\vec{0}, kv|e^{(i)})B_z(\vec{0}, kv'|e^{(i)}) \;\leq\; B_z(\vec{0}, 2kv_ie^{(i)}|e^{(i)})\,. \tag{2.22}$$

Therefore $M[v;z] + M[v';z] \;\geq\; 2v_iM[e^{(i)};z]$. The symmetry of z implies that $M[v';z] = M[v;z]$, and so $M[e^{(i)};z] < 0$.

Our main step is to show the following assertion:

$$\text{If } M[e^{(k)};z] \in [-\infty, 0) \text{ for some } k, \text{ then } M[e^{(j)};z] = -\infty \text{ for all } j \neq k. \tag{2.23}$$

After (2.23) has been proven, then we deduce from the preceding paragraph that $M[e^{(j)};z] = -\infty$ for all $j \neq i$; another application of (2.23) shows that $M[e^{(i)};z] = -\infty$ also. Then, for any nonzero vector $u = \sum_i u_i e^{(i)}$, we have

$$B_z(\vec{0}, Lu|u) \geq \prod_{k:u_k \neq 0} B_z(\vec{0}, Lu_k e^{(k)}|u),$$

which implies

$$M[u;z] \;\leq\; \sum_{k:u_k \neq 0} |u_i| M[e^{(i)};z] \;=\; -\infty.$$

So we see that the Lemma follows once (2.23) has been proven.

To prove (2.23), assume $M[e^{(k)};z] < 0$, and fix $j \neq k$. By Lemma 2.4(iii), there exists a T such that $M^T[e^{(k)};z|e^{(k)}] < 0$; and by Equation (2.10), there exists an L such that

$$B_z^T(\vec{0}, Le^{(k)};z|e^{(k)}) > 2\,.$$

For each integer $r \geq 1$, consider the collection of all SAWS that go:
from $\vec{0}$ to $2Te^{(j)}$ in $2T$ steps, and then
from $2Te^{(j)}$ to $2Te^{(j)} + rLe^{(k)}$ inside $2Te^{(j)} + \mathrm{Slab}^T(rLe^{(k)}|e^{(k)})$, and then
to $2Te^{(j)} + (rL+1)e^{(k)}$ in 1 step, and then
to $5Te^{(j)} + (rL+1)e^{(k)}$ in $3T$ steps, and then
to $5Te^{(j)} + e^{(k)}$ inside $5Te^{(j)} + (rL+1)e^{(k)} + \mathrm{Slab}^T(-rLe^{(k)}| - e^{(k)})$, and then
to $5Te^{(j)}$ in 1 step, and then
to $7Te^{(j)}$ in $2T$ steps.
The generating function of all such SAWs is less than $B_z(\vec{0}, 7Te^{(j)}|e^{(j)})$ and greater than

$$\begin{aligned} z_{j+}^{7T} z_{k+}^2 B_z^T(\vec{0}, rLe^{(k)}|e^{(k)}) B_z^T(\vec{0}, -rLe^{(k)}| - e^{(k)}) &\geq\; z_{j+}^{7T} z_{k+}^2 B_z^T(\vec{0}, Le^{(k)}|e^{(k)})^{2r} \\ &>\; z_{j+}^{7T} z_{k+}^2 2^{2r}\,. \end{aligned}$$

Therefore $B_z(\vec{0}, 7Te^{(j)}|e^{(j)}) \;>\; z_{j+}^{7T} z_{k+}^2 2^{2r}$ for every $r \;\geq\; 1$, which implies that $B_z(\vec{0}, 7Te^{(j)}|e^{(j)}) \;=\; +\infty$, and hence $M[e^{(j)};z] \;=\; -\infty$. This proves (2.23) and the Lemma. □

The following result follows immediately from Lemmas 2.9 and 2.10.

Corollary 2.11. *Assume z satisfies $z_{i+} = z_{i-} > 0$ for every i and $\lambda(z) > 1$. Then $M[v;z] = -\infty$ for every nonzero v.*

We are finally ready to put the pieces together to complete the proof of the main theorem of this section.

Proof of Theorem 2.3: (i) Corollary 2.5 gives concavity and the fact that $M_0(z) < +\infty$ for every nonzero z. If $\lambda(\bar{z}) < 1$, then Lemma 2.7 shows that $M[v;\bar{z}] \geq -||v||_1 \log \lambda(\bar{z})$ for every nonzero v; then Lemma 2.6(iii) tells us that

$$M_0(z) \geq -\log \lambda(\bar{z}) - \frac{1}{2} \max_{i \in I(z)} \left\{ \left| \log(z_{i+}/z_{i-}) \right| \right\}. \tag{2.24}$$

Therefore M_0 is finite on $\{z : \lambda(\bar{z}) < 1\}$. It remains to show that $M_0(z) > -\infty$ when $\lambda(\bar{z}) = 1$; this will follow from (v) below and Lemma 2.6(iii).
(ii): This follows from Corollary 2.11 and Lemma 2.6(iii).
(iii): This is Lemma 2.7.
(iv): This follows from Lemma 2.9.
(v): Fix z such that $\lambda(z) = 1$. For every $t > 0$, define

$$F(t) \;:=\; M_0(tz) \;=\; \inf_{v,T} \frac{M^T[v;tz]}{||v||_1} \,.$$

For each T and v (with $v \in W(z)$ if $z_{\min} = 0$), $M^T[v;tz]$ is decreasing in t, and by Lemma 2.4(i), it is continuous in t; therefore F is left-continuous. Since $F(t) > 0$ whenever $0 < t < 1$ (by part (iii) above), we conclude that $M_0(z) \geq 0$. Finally, $M_0(z) \leq 0$ by Lemma 2.8. □

3. The mass of the full two-point function

In this section we shall prove that the mass $m[v;z]$ of the full two-point function, as defined in Equation (1.7), is well-defined, and that it equals the slab mass $M[v;z]$ except perhaps when $\lambda(\bar{z}) = 1$. The analogue of this for isotropic walks was proved in Chayes and Chayes (1986).

A key quantity in our analysis is the *bubble diagram* $\mathcal{B}(z)$, which is defined as follows:

$$\mathcal{B}(z) \;=\; \sum_{v \in Z^d} G_z(\vec{0}, v) G_z(v, \vec{0}) \,.$$

This is an extension of the isotropic definition, which is usually written $\mathcal{B}(z_0) = \sum_v G_{z_0}(\vec{0}, v)^2$. See Section 1.5 of Madras and Slade (1993) for a discussion of the role of the bubble diagram, particularly in high dimensions.

Proposition 3.1. *(i) $\mathcal{B}(z) = \mathcal{B}(\bar{z})$ for every z.*
(ii) If $\lambda(\bar{z}) < 1$, then $\mathcal{B}(z)$ is finite.
(iii) If $\lambda(\bar{z}) > 1$ and $z_{\min} > 0$, then $\mathcal{B}(z)$ is infinite.

Proof: (i): For any $v \in Z^d$, we have

$$\begin{aligned} G_z(\vec{0}, v) G_z(v, \vec{0}) &= G_z(\vec{0}, v) G_z(\vec{0}, -v) \\ &= G_{\bar{z}}(\vec{0}, v) G_{\bar{z}}(\vec{0}, -v) \qquad\qquad \text{(by Lemma 2.6}(iii)) \\ &= G_{\bar{z}}(\vec{0}, v) G_{\bar{z}}(v, \vec{0}), \end{aligned}$$

and part (i) follows.
(ii): If $\lambda(\bar{z}) < 1$, then $\chi(\bar{z}) < \infty$ (by Proposition 2.2(iii)). Moreover, using (i), we have

$$\mathcal{B}(z) = \sum_{v \in Z^d} G_{\bar{z}}(\vec{0}, v)^2 \leq \chi(\bar{z})^2 < \infty.$$

(iii): This follows from Theorem 2.3(ii), which tells us that $G_z(\vec{0}, v)$ does not tend to 0 as v tends to infinity. □

The proof of the following lemma is identical to the proof of its isotropic analogue, which is Lemma 4.1.4 of Madras and Slade (1993).

Lemma 3.2. *For any $x, y \in Z^d$, and any z,*

$$G_z(\vec{0}, x) G_z(x, y) \;\leq\; \mathcal{B}(z) G_z(\vec{0}, y)\,.$$

Lemma 3.3. *For every $v, w \in Z^d$ (and in $W(z)$ if $z_{\min} = 0$),*

$$M[v + w; z] \;\leq\; M[v; z] + M[w; z].$$

Proof: The result is obvious if v or w is the zero vector, so assume both are nonzero. If $v = tw$ for some positive rational number t, then in fact v and w are both integer multiples of some vector $u \in Z^d$, and the result follows easily since $M[ku; z] = kM[u; z]$ for every positive integer k. If $v = -tw$ for some positive rational number t, then the previous sentence proves the result for $\bar{z}$ in place of z; to derive the result for nonsymmetric z, simply use the formula of Lemma 2.6(iii).

Now assume that v and w are linearly independent. Let θ be the vector $v/||v||_2 + w/||w||_2$ (where $||\cdot||_2$ denotes Euclidean norm). Therefore $\theta \cdot v > 0$ and $\theta \cdot w > 0$, and so a concatenation argument shows that

$$B_z(\vec{0}, Lv|\theta) B_z(\vec{0}, Lw|\theta) \leq B_z(\vec{0}, Lv + Lw|\theta).$$

The lemma now follows from the definition of the mass M and the fact that M does not depend upon θ (Lemma 2.4(iii)). □

Theorem 3.4. *Assume that $\lambda(\bar{z}) < 1$, or that $z_{\min} = 0$. Then for every nonzero $v \in Z^d$, the limit $m[v; z]$ (of Equation (1.7)) exists and equals $M[v; z]$. Moreover, if $\mathcal{B}(z) < \infty$ (as is always the case when $\lambda(\bar{z}) < 1$), then*

$$G_z(\vec{0}, v) \;\leq\; \mathcal{B}(z) e^{-m[v; z]}. \tag{3.1}$$

Proof: We shall first dispense with the case $z_{\min} = 0$. Assume without loss of generality that $z_{1+} > z_{1-} = 0$. Then every SAW from $\vec{0}$ to v must stay between the hyperplanes $x_1 = 0$ and $x_1 = v_1$. Therefore $B_z(-e^{(1)}, v|e^{(1)}) = z_{1+} G_z(\vec{0}, v)$, so the existence of the limit $M[v; z]$ implies that of $m[v; z]$, as well as their equality.

Now assume $\mathcal{B}(z) < \infty$ (whether or not $z_{\min}$ is 0). The existence of the limit (1.7) and the inequality (3.1) follow from Lemma 3.2 and subadditivity, exactly as in the proof of Theorem 4.1.18 of Madras and Slade (1993).

It remains to prove the equality of the masses when $\lambda(\bar{z}) < 1$ and $z_{\min} > 0$. Since $B_z(\vec{0}, Lv|v) \leq G_z(\vec{0}, Lv)$, we obviously have $M[v; z] \geq m[v; z]$. Therefore we only need to prove $m[v; z] \geq M[v; z]$.

By Lemma 2.6(iii), it suffices to consider the case $z = \bar{z}$, i.e. $z_{i+} = z_{i-} > 0$ for every i. Observe that $M[v; \bar{z}] > 0$ for every nonzero $v \in Z^d$ (Lemma 2.7), and that

$$M[kv; \bar{z}] = |k| M[v; \bar{z}] \qquad \text{for every integer } k. \tag{3.2}$$

We can use this equation to extend the definition of $M[v; \bar{z}]$ to all vectors v in Q^d (the set of vectors in R^d with rational coordinates). Then for every $u, w \in Q^d$,

$$|M[u; \bar{z}] - M[w; \bar{z}]| \;\leq\; M[u - w; \bar{z}] \;\leq\; -||u - w||_1 \log z_{\min} \tag{3.3}$$

(the first inequality comes from Lemma 3.3, and the second from the fact that $z_{\min}^{||v||_1} \leq B_{\bar{z}}(\vec{0}, v|\theta) \leq \exp(-M[v; \bar{z}])$). Thus $M[\cdot; \bar{z}]$ is uniformly continuous on Q^d, and so it extends to a continuous function on all of R^d, which we shall also write $M[\cdot; \bar{z}]$. This function is a norm on R^d (by (3.2) and Lemma 3.3).

Fix a nonzero $v \in Z^d$ and let

$$\mathcal{S}[v] \;=\; \{w \in R^d \;:\; M[w; \bar{z}] \leq M[v; \bar{z}]\}.$$

This is a convex set with v on its boundary, so there exists a "supporting hyperplane" of $\mathcal{S}[v]$ at v; that is, there exists a $\theta \in R^d$ such that $\theta \cdot v > 0$ and $\mathcal{S}[v]$ is contained in the half-space $\mathcal{H} = \{w \in R^d \;:\; \theta \cdot w \leq \theta \cdot v\}$. Now consider any SAW ω from $\vec{0}$ to Lv (where L is a positive integer). We can break ω into three subwalks, as follows. Let i_0 be the largest i at which $\min\{\theta \cdot \omega(i) : 0 \leq i \leq |\omega|\}$ is attained, and let j_0 be the largest j at which $\max\{\theta \cdot \omega(j) : i_0 \leq j \leq |\omega|\}$ is attained. (Observe that $\theta \cdot \omega(i_0) \leq 0 < \theta \cdot (Lv) \leq \theta \cdot \omega(j_0)$, and that $\theta \cdot \omega(i_0) < \theta \cdot \omega(k) \leq \theta \cdot \omega(j_0)$ for every k between i_0 and j_0.) Let the first subwalk be the part of ω from $\vec{0}$ to $\omega(i_0)$, the second from $\omega(i_0)$ to $\omega(j_0)$, and the third from $\omega(j_0)$ to $\omega(|\omega|)$. This decomposition implies the following inequality:

$$G_{\bar{z}}(\vec{0}, Lv) \leq \sum_{u : \theta \cdot u \leq 0} \; \sum_{y : \theta \cdot y \geq \theta \cdot (Lv)} G_{\bar{z}}(\vec{0}, u) B_{\bar{z}}(u, y|\theta) G_{\bar{z}}(y, Lv). \tag{3.4}$$

In the above sum, we know that $\theta \cdot (y - u) \geq \theta \cdot (Lv)$, so the vector $(y - u)/L$ is not in the in the interior of the half-space $\mathcal{H}$. Therefore $(y - u)/L$ is not in in the interior of $\mathcal{S}[v]$, which implies that $M[(y - u)/L; \bar{z}] \geq M[v; z]$. From this we deduce

$$e^{-LM[v;\bar{z}]} \;\geq\; e^{-M[y-u;\bar{z}]} \;\geq\; B_{\bar{z}}(u, y|\theta)$$

(where the last inequality follows from Equation (2.8)). Next we use this bound on $B_{\bar{z}}(u, y|\theta)$ in (3.4), and then bound the double sum in (3.4) by including all u's and y's in Z^d, obtaining

$$G_{\bar{z}}(\vec{0}, Lv) \leq \chi(\bar{z})^2 e^{-LM[v;\bar{z}]}.$$

This implies that $m[v; \bar{z}] \geq M[v; \bar{z}]$, and so the proof is complete. $\square$

Corollary 3.5. *For every $v \in Z^d$, we have $m[v;z] = M[v;z]$, except perhaps when $\lambda(\bar{z}) = 1$ and $z_{\min} > 0$.*

Proof: By Theorem 3.4, the only case not yet proven is $\lambda(\bar{z}) > 1$ and $z_{\min} > 0$. But in this case, Theorem 2.3(ii) tells us that $M[v;z] = -\infty$, so the corollary follows because $m[v;z] \leq M[v;z]$. $\square$

Corollary 3.6. *Assume $\lambda(\bar{z}) < 1$ and $z_{\min} > 0$. Let $\{v^{(n)}\}$ be a sequence of vectors in Z^d and let $\{t_n\}$ be a sequence of positive numbers tending to infinity, such that $v^{(n)}/t_n$ converges to a vector α in R^d. Then*

$$\lim_{n\to\infty} \frac{-\log G_z(\vec{0}, v^{(n)})}{t_n} \;=\; m[\alpha; z],$$

where $m[\cdot; z]$ is the extension of the mass function to all of R^d, as described in the proof of Theorem 3.4.

In particular, taking $t_n = ||v^{(n)}||$ gives Equation (1.6), and shows that the definition (1.6) does not really depend on the choice of norm.

Proof: The proof is similar to that of Theorem 4.1.18 in Madras and Slade (1993), which is the isotropic analogue of this result. By (3.1), we know

$$-\log G_z(\vec{0}, v^{(n)}) \geq -\log \mathcal{B}(z) + m[v^{(n)}; z],$$

and therefore

$$\liminf_{n\to\infty} \frac{-\log G_z(\vec{0}, v^{(n)})}{t_n} \geq m[\alpha; z] \tag{3.5}$$

by the continuity of $m[\cdot; z]$. So it suffices to prove the reverse inequality for the lim sup.

Fix $\epsilon > 0$. Choose a vector $v \in Z^d$ and a positive integer J such that $||\alpha - J^{-1}v||_1 \leq \epsilon$. Now choose a sequence of positive integers k_n such that k_n/t_n converges to J^{-1}. By Lemma 3.2 we have

$$G_z(\vec{0}, k_n v) G_z(k_n v, v^{(n)}) \;\leq\; \mathcal{B}(z) G_z(\vec{0}, v^{(n)}). \tag{3.6}$$

Then we have

$$\begin{aligned} \lim_{n\to\infty} \frac{-\log G_z(\vec{0}, k_n v)}{t_n} &= J^{-1} m[v; z] \\ &= m[J^{-1}v; z] \\ &\leq m[\alpha; z] - \epsilon \log z_{\min} \qquad\qquad \text{(by (3.3))}. \end{aligned} \tag{3.7}$$

Next, using the trivial bound $G_z(u,w) \geq z_{\min}^{||w-u||_1}$, we obtain

$$\limsup_{n\to\infty} \frac{-\log G_z(k_n v, v^{(n)})}{t_n} \;\leq\; -||\alpha - J^{-1}v||_1 \log z_{\min} \;\leq\; -\epsilon \log z_{\min}\,. \tag{3.8}$$

Finally, we combine (3.6), (3.7), and (3.8) to obtain

$$\limsup_{n\to\infty} \frac{-\log G_z(\vec{0}, v^{(n)})}{t_n} \le m[\alpha; z] - 2\epsilon \log z_{\min} .$$

Since $\epsilon > 0$ is arbitrary, this shows that $\limsup_n -\log G_z(\vec{0}, v^{(n)})/t_n \le m[\alpha; z]$. Together with (3.5), this completes the proof. $\square$

4. Linear extension of SAW's with drift

In this section we shall assume $z_{\min} > 0$.

A natural question is the following: Suppose that $z_{1+} > z_{1-} > 0$. This gives a directional preference among steps parallel to the x_1-axis. Consider all N-step SAW's starting at $\vec{0}$ weighted according to z. Does the distance between the endpoints of a typical N-step SAW grow linearly with N?

There are several ways to formulate this question, and we shall discuss one way here. For nonnegative integers N and vectors $v \in Z^d$, let $G_z^{(N)}(\vec{0}, v)$ be the generating function of all N-step SAWs from $\vec{0}$ to v. Let v_N^* be the v for which $G_z^{(N)}(\vec{0}, v)$ is maximized. (We suppress the dependence of v_N^* on z.) We would like to know whether

$$\liminf_{N\to\infty} \frac{||v_N^*||_1}{N} \; > \; 0. \tag{4.1}$$

Theorem 1.2 says that (4.1) holds provided that $z_{\min} > 0$ and z is not symmetric (i.e. $z \neq \bar{z}$). This will be a direct consequence of Propositions 4.1 and 4.2 below.

Proposition 4.1. *Assume $\lambda(z) > \lambda(\bar{z})$. Then (4.1) holds.*

Proof: Observe that the definition of v_N^* does not change if we multiply the vector z by a positive scalar t. Also, $\overline{(tz)} = t\bar{z}$ for any scalar $t > 0$, so by Proposition 2.2(v) we can and shall assume that $\lambda(\bar{z}) < 1 < \lambda(z)$.

There at most $(2N+1)^d$ vectors v for which $G_z^{(N)}(\vec{0}, v)$ is non-zero, so we have $G_z^{(N)}(\vec{0}, v_N^*) \le \chi_N(z) \le (2N+1)^d G_z^{(N)}(\vec{0}, v_N^*)$. Therefore

$$\lim_{N\to\infty} G_z^{(N)}(\vec{0}, v_N^*)^{1/N} \; = \; \lambda(z). \tag{4.2}$$

We also know

$$\begin{aligned} G_z^{(N)}(\vec{0}, v_N^*) &\le G_z(\vec{0}, v_N^*) \\ &\le \mathcal{B}(z) \exp(-M[v_N^*; z]) \qquad \text{(by Theorem 3.4)} \\ &\le \mathcal{B}(z) \exp(-||v_N^*||_1 M_0(z)). \end{aligned} \tag{4.3}$$

We know that $\mathcal{B}(z) < \infty$ (Proposition 3.1(ii)) and $-\infty < M_0(z) < 0$ (Theorem 2.3(i, iv). Taking N^{th} roots in (4.3) and using (4.2) as well as $\lambda(z) > 1$, we get

$$\liminf_{N\to\infty} \frac{||v_N^*||_1}{N} \ge \frac{\log \lambda(z)}{|M_0(z)|} \; > \; 0. \qquad \square$$

Proposition 4.2. *Assume* $z_{\min} > 0$ *and* $z_{i+} \neq z_{i-}$ *for some* i*. Then* $\lambda(z) > \lambda(\bar{z})$*.*

Proof: Without loss of generality, assume $z_{1+} > z_{1-} > 0$. Also, scaling z as in the proof of Proposition 4.1, assume that $\lambda(\bar{z}) = 1$. Our goal is to prove that $\lambda(z) > 1$. To do this, it suffices to prove that $M_0(z) < 0$ (by Theorem 2.3(iii, v)).

We claim that $M[e^{(1)}; \bar{z}] = 0$ (where $e^{(1)}$ is the unit vector in the $+x_1$ direction). From this claim, Lemma 2.6(iii) shows that $M[e^{(1)}; z] = -\log(z_{1+}/z_{1-}) < 0$; this immediately implies $M_0(z) < 0$.

So to prove the proposition, it suffices to prove the claim of the previous paragraph. Since $\lambda(\bar{z}) = 1$, Theorem 2.3(v) tells us that $M_0(\bar{z}) = 0$; hence $M[e^{(1)}; \bar{z}] \geq 0$. We must show that $M[e^{(1)}; \bar{z}]$ cannot be strictly positive.

Consider the proof of Lemma 2.8, using our $\bar{z}$ instead of the z there. Everything in that proof applies because $\lambda(\bar{z}) \geq 1$, $\bar{z}_{1+} \geq \bar{z}_{1-}$, and $\bar{z}_{1+} > 0$ (in fact, the first two hold with equality). In particular, the generating function $B_{\bar{z}}$ of all bridges diverges at $\bar{z}$. We shall use this fact and a new version of the final paragraph of that proof to complete the present proof.

We define the mass

$$\widetilde{M}_1(\bar{z}) = \lim_{L \to \infty} \frac{-\log B_{\bar{z},L}}{L}$$

where $B_{\bar{z},L}$ is the generating function of bridges of span L, as defined in the proof of Lemma 2.8. By subadditivity, this limit exists and satisfies

$$B_{\bar{z},L} \leq e^{-L\widetilde{M}_1(\bar{z})} \qquad \text{for every } L \geq 1,$$

exactly as in the anisotropic case; see Proposition 4.1.8 of Madras and Slade (1993). This mass satisfies $\widetilde{M}_1(\bar{z}) = M[e^{(1)}; \bar{z}]$; the proof of this relation is identical to the proof of the isotropic case, which is Lemma 4.1.12 of Madras and Slade (1993). We remark that one property required for the proof to carry over is that $B_{\bar{z}}(\vec{0}, v|e^{(1)}) = B_{\bar{z}}(\vec{0}, -v|e^{(1)})$, which follows from the fact that $\bar{z}_{i+} = \bar{z}_{i-}$ for every i.

Finally, we have

$$\begin{aligned} B_{\bar{z}} = 1 + \sum_{L=1}^{\infty} B_{\bar{z},L} &\leq 1 + \sum_{L=1}^{\infty} e^{-L\widetilde{M}_1(\bar{z})} \\ &= 1 + \sum_{L=1}^{\infty} e^{-LM[e^{(1)};\bar{z}]}. \end{aligned}$$

Since $B_{\bar{z}}$ diverges, we conclude that $M[e^{(1)}; \bar{z}]$ cannot be strictly positive. This concludes the proof. □

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