Analysis of piecewise linear (PWL) feedback systems

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Problem Statement

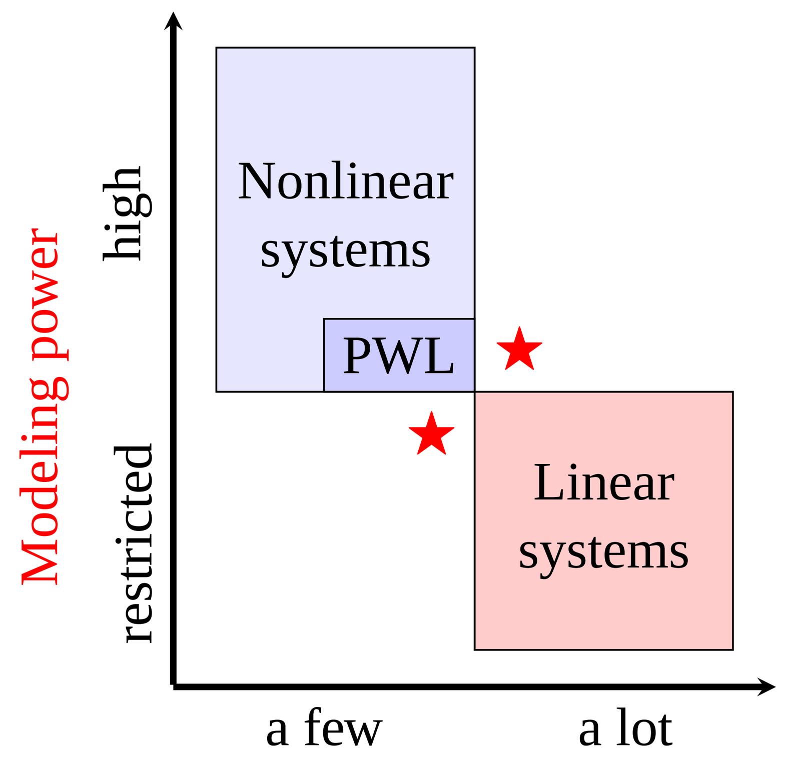
To provide sufficient conditions which guarantee the stability of oscillations in a class of dynamical systems known as piecewise linear feedback systems.

Why?

There exist numerous examples where assessing the stability of oscillations is important, e.g.

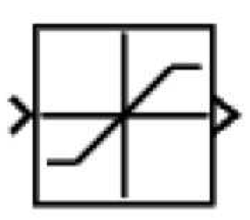
- Planetary movement
- Cardiac rhythms
- Circadian rhythms
- AC electrical power
- Cristal oscillators (which provide the clock signal for digital microprocessors)

Why piecewise linear?



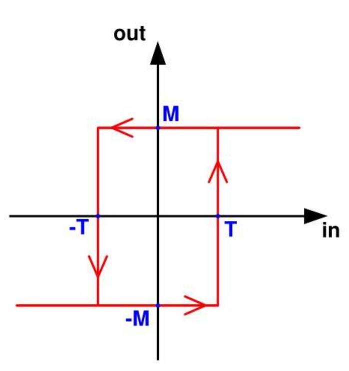
Furthermore, PWL models are ideally suited for certain common nonlinear phenomena:

- Saturation.



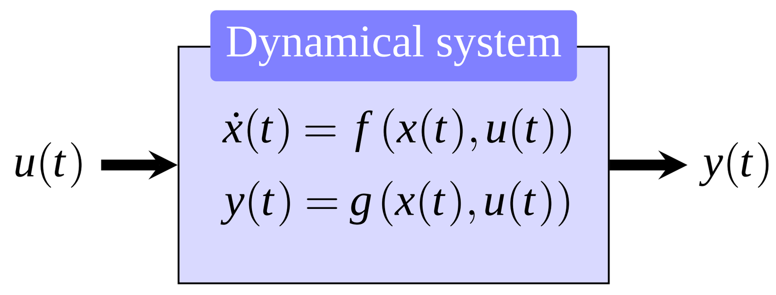
- Multiple modes of operation (e.g. on/off).

- Hysteresis (magnetic, electrical, elastic)



Background information

PWL dynamical systems



A dynamical system renders a time-dependent relationship between the inputs $u(t) \in \mathbb{R}^m$ and the outputs $y(t) \in \mathbb{R}^p$ via a set of quantities $x(t) \in \mathbb{R}^n$ called states which characterise the system's internal circumstances or attributes.

The class of PWL systems considered here are defined by a set of affine linear systems

$$\begin{cases} \dot{x}(t) = A_q x(t) + B_q u(t) \\ y(t) = C x(t) \end{cases} \qquad (1.a)$$

where $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^n$ and $C^\top \in \mathbb{R}^n$, together with a logical rule to switch among them

$$q(x) \in \{1, \ldots, M\} \qquad (1.b)$$

which depends on present values of $x(t)$.

Switching surfaces

We define the switching surfaces S_i as the regions of the state space where the mathematical description of the PWL system switches from

$$\dot{x}(t) = A_j x(t) + B_j u(t)$$

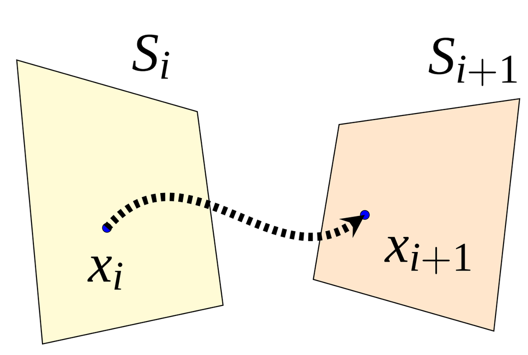
to

$$\dot{x}(t) = A_k x(t) + B_k u(t)$$

where $j \neq k$.

Impact maps

The impact maps $H_i(\cdot)$ show how the trajectory of the system evolves from one switching surface to the next one.



$$x_{i+1} = H_i(x_i) \in S_{i+1} \text{ for all } x_i \in S_i$$

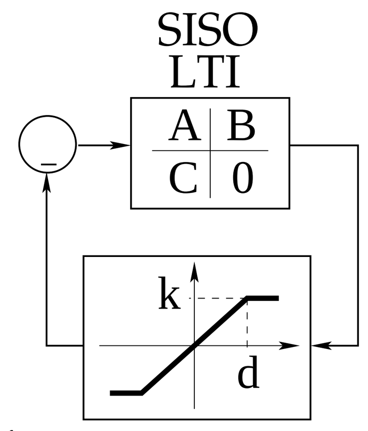
Lyapunov functions

Let $V(\cdot)$ be a nonnegative function such that

$$V(x) > 0 \text{ for all } x \neq x^\circ \text{ and } V(x^\circ) = 0$$
$$V(x_{k+1}) < V(x_k)$$

Then $x_k \to x^\circ$ as $k \to \infty$

System to be considered



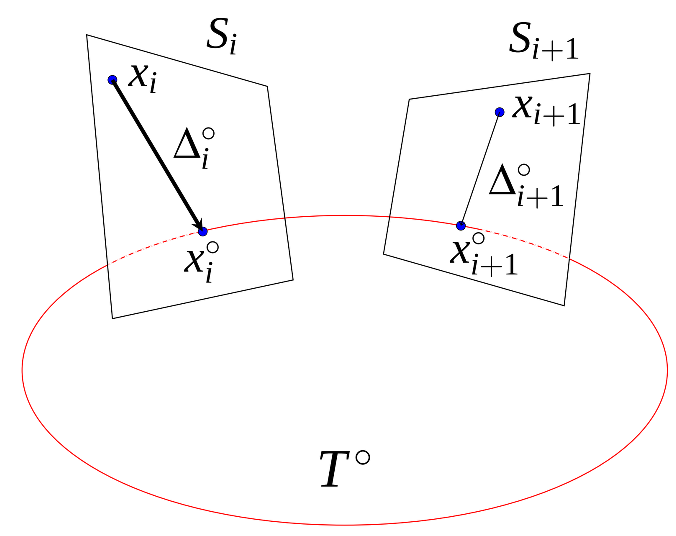
$$\dot{x} = \begin{cases} Ax - \kappa B, & Cx > d \\ (A - \frac{\kappa}{d} BC)x, & |Cx| \leq d \\ Ax + \kappa B, & Cx < -d \end{cases}$$

$A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^3$, $C^\top \in \mathbb{R}^3$, $\kappa > 0$, $d > 0$

In this case, the switching surfaces are given by

$$S := \{x \in \mathbb{R}^n \mid Cx = d\}$$
$$\underline{S} := \{x \in \mathbb{R}^n \mid Cx = -d\}$$

The oscillatory trajectory of the system, denoted T°, is assumed to intersect the switching surfaces S_i at the points x_i°.



Proposed methodology

Finding Lyapunov functions V_i on the switching surfaces S_i such that

$$V_{i+1}(x_{i+1}) < V_i(x_i) \text{ for all } x_{i+1} = H_i(x_i)$$

would guarantee that $x_i \to x_i^\circ$ as time elapses, thus guaranteeing the stability of the oscillatory trajectory.

Results

- A constructive methodology for finding the aforementioned Lyapunov functions has been proposed [2] as an extension of the work presented in [1].
- Extensions which consider a wider class of PWL systems than the one considered here are still pending.

References

[1] J. M. Gonçalves. *Constructive global analysis of hybrid systems*. PhD thesis, Massachusetts Institute of Technology, September 2000.

[2] A. Salinas-Varela, G. Stan, and J. Gonçalves. Global asymptotic stability of the limit cycle in piecewise linear versions of the goodwin oscillator. In *17th IFAC World Congress (IFAC 2008)*, July 2008.

Research supported by the Mexican Council of Science and Technology (CONACyT), the Cambridge Overseas Trust, and the Overseas Research Studentship Programme.