A Geometric approach to Chip Firing games

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Chip firing games

A Chip firing game played on an undirected connected graph is defined as follows:
Each vertex of the graph is given some chips (positive or negative). At each step of the game, a vertex is allowed to fire some k (positive or negative) chips to each of its neighbors.

1 Problem Statement

A natural question one could ask on a chip firing game is:

Question 1 *Given an initial configuration, can we reach an "effective" configuration i.e., a configuration in which no vertex has a negative number of chips after a finite sequence of chip firings?*

More generally,

Question 2 *Given a configuration, what is the rank of the configuration i.e., the minimum number of chips we should remove from the system such that the resulting configuration is not effective?*

2 Riemann-Roch formula of Baker and Norine

Let D be a configuration. Denote by $r(D)$ the rank of the configuration and by $deg(D)$ the total number of chips in the configuration. For a connected graph G with n vertices and m edges, we have the following theorem:

Theorem 1 *For every connected graph G, there exists a configuration K with $deg(K) = 2g - 2$ such that for every configuration D we have:*

$$r(D) - r(K - D) = deg(D) - (g - 1).$$

where g is the genus of G given by $m - n + 1$.

3 Our Results

The study of chip firing games on graphs can be reduced to the study of Voronoi diagrams of lattices under the distance function defined by a regular simplex. In particular to the study of local maxima induced by the distance function on the lattice, see Figure 1.
In this direction, we introduce two geometric invariants of a lattice namely:

1. Uniformity.
2. Reflection Invariance.

Under some technical conditions, we show that:

Theorem 2 *A lattice admits a Riemann-Roch formula if and only if it is uniform and reflection-invariant.*

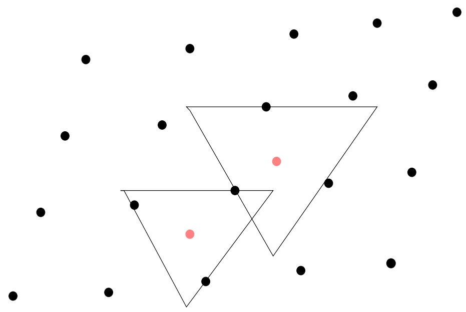

Figure 1 *The local maxima induced by the simplicial distance function on a lattice are shown in pink.*

4 Open problem

Our results show that the Laplacian lattice is uniform and strongly reflection-invariant. This raises the following question:

Question 3 *Do the properties uniformity and reflection-invariance completely charecterise the Laplacian lattice?*

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