

Tight Lower Bounds for Multiplicative Weights Algorithmic Families

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Abstract

We study the fundamental problem of prediction with expert advice and develop regret lower bounds for a large family of algorithms for this problem. We develop simple adversarial primitives, that lend themselves to various combinations leading to sharp lower bounds for many algorithmic families. We use these primitives to show that the classic Multiplicative Weights Algorithm (MWA) has a regret of $\sqrt{\frac{T \ln k}{2}}$, there by completely closing the gap between upper and lower bounds. We further show a regret lower bound of $\frac{2}{3} \sqrt{\frac{T \ln k}{2}}$ for a much more general family of algorithms than MWA, where the learning rate can be arbitrarily varied over time, or even picked from arbitrary distributions over time. We also use our primitives to construct adversaries in the geometric horizon setting for MWA to precisely characterize the regret at $\frac{0.391}{\sqrt{\delta}}$ for the case of 2 experts and a lower bound of $\frac{1}{2} \sqrt{\frac{\ln k}{2\delta}}$ for the case of arbitrary number of experts k .

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1 Introduction

In this paper we develop tight lower bounds on the regret obtainable by a broad family of algorithms for the fundamental problem of prediction with expert advice. Predicting future events based on past observations, a.k.a. prediction with expert advice, is a classic problem in learning. The experts framework was the first framework proposed for online learning and encompasses several applications as special cases. The underlying problem is an online optimization problem: a *player* has to make a decision at each time step, namely, decide which of the k experts' advice to follow. At every time t , an *adversary* sets gains for each expert: a gain of g_{it} for expert i at time t . Simultaneously, the *player*, seeing the gains from all previous steps except t , has to choose an action, i.e., decide on which expert to follow. If the player follows expert $j(t)$ at time t , he gains $g_{j(t),t}$. At the end of each step t , the gains associated with all experts are revealed to the player, and the player's choice is revealed to the adversary. In the *finite horizon model*, this process is repeated for T steps, and the player's goal is to perform (achieve a cumulative gain) as close as possible to the best single action (best expert) in hindsight, i.e., to minimize his *regret* $R_{T,k}$:

$$R_{T,k} = \max_{1 \leq i \leq k} \sum_{t=1}^T g_{it} - \sum_{t=1}^T g_{j(t),t}.$$

Apart from assuming that the g_{it} 's are bounded in $[0, 1]$, we don't assume anything else about the gains¹. Just as natural as the finite horizon model is the model with a *geometric horizon*: the stopping time is a geometric random variable with expectation $\frac{1}{\delta}$. In other words, the process ends at any given step with probability δ , independently of the past. Equivalently, both the player and the adversary discount the future with a $1 - \delta$ factor. In this paper, we study both the finite horizon model and the geometric horizon model. We begin with the discussion for finite horizon model below.

Main contribution. In this paper we develop simple adversarial primitives and demonstrate that, when applied in various combinations, they result in remarkably sharp lower bounds for a broad family of algorithms. We first describe the family of algorithms we study, and then discuss our main results.

Multiplicative Weights Algorithm. We begin with the Multiplicative Weights Algorithm, which is a simple, powerful and widely used algorithm for a variety of learning problems. In the experts problem, at each time t , MWA computes the cumulative gain $G_{it-1} = \sum_{s=1}^{t-1} g_{is}$ of each expert i accumulated over the past $t-1$ steps, and will follow expert i 's advice with probability proportional to $e^{\eta G_{it-1}}$. Namely, with probability $\frac{e^{\eta G_{it-1}}}{\sum_{j=1}^k e^{\eta G_{jt-1}}}$ where η is a parameter that can be tuned. The per-step computation of the algorithm is extremely simple and straightforward. The intuition behind the algorithm is to increase the weight of any expert that performs well by a multiplicative factor. Despite the simplicity and the heuristic origins of the algorithm, it is surprisingly powerful: the pioneering work of Cesa Bianchi et al. [7] showed that MWA obtains a sublinear regret of $\sqrt{\frac{T \ln k}{2}}$, and that this is asymptotically optimal as the number of experts k and the number of time steps T both tend to ∞ .

Families of algorithms. The MWA is a single-parameter family of algorithms, i.e., the learning rate parameter η is the only parameter available for the player. In general one could think of η being an arbitrary function of time t , i.e., at step t , algorithm follows expert i with probability $\frac{e^{\eta(t)G_{it-1}}}{\sum_{j=1}^k e^{\eta(t)G_{jt-1}}}$. Note that this is a T -parameter family of algorithms and is quite general. To see why this is general, note that after fixing G_{it-1}

¹As one might expect, it turns out that restricting the adversary to set gains in $\{0, 1\}$ instead of $[0, 1]$ is without loss of generality (see [10] or [18]). Henceforth, we restrict ourselves to the binary adversary, which just sets gains of 0 or 1.

for all i , any probability p_{it} of picking expert i at time t can be expressed as $\frac{e^{\eta(t)G_{it-1}}}{\sum_{j=1}^k e^{\eta(t)G_{jt-1}}}$, irrespective of what $p_{i,t-1}$ was — something that is certainly not possible when η is independent of t . The most general family of algorithms we study is when at each time t , the quantity $\eta(t)$ is drawn from an arbitrary distribution F_t over reals. Since F_t could be arbitrary, this is an infinite-parameter family of algorithms. We denote the

1. single parameter MWA family by $\mathcal{A}_{\text{single}}$;
2. family where $\eta(t)$ decreases with t by \mathcal{A}_{dec} ;
3. family where $\eta(t)$ is arbitrary function of t by \mathcal{A}_{arb} ;
4. family where $\eta(t)$ is drawn from F_t for each t by $\mathcal{A}_{\text{rand}}$.

It is straightforward to see that $\mathcal{A}_{\text{single}} \subseteq \mathcal{A}_{\text{dec}} \subseteq \mathcal{A}_{\text{arb}} \subseteq \mathcal{A}_{\text{rand}}$. The reason we start with $\mathcal{A}_{\text{single}}$ is that it is the classic MWA and precisely characterizing its regret is still open. We study \mathcal{A}_{dec} because often when MWA algorithms are working with unknown T , they employ a strategy where η decreases with time. We move on to further significantly generalize this by studying \mathcal{A}_{arb} , $\mathcal{A}_{\text{rand}}$.

Minimax regret, and Notation. We study the standard notion of minimax regret for each of the above family of algorithms. Formally, let $R_{T,k}(A, D)$ denote the regret achieved by algorithm A when faced with adversary D in the prediction with expert advice game with T steps and k experts. We use $R_k(A, D)$ to denote the asymptotic², in T , value of $R_{T,k}(A, D)$, i.e., $R_k(A, D) = \sqrt{T} \cdot \lim_{T \rightarrow \infty} \frac{R_{T,k}(A, D)}{\sqrt{T}}$. The minimax regret of a family $\mathcal{A}_{\mathcal{F}}$ of algorithms against a family $\mathcal{D}_{\mathcal{F}}$ of adversaries is given by $R_{T,k}(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}}) = \min_{A \in \mathcal{A}_{\mathcal{F}}} \max_{D(A) \in \mathcal{D}_{\mathcal{F}}} R_{T,k}(A, D)$. Let $\mathcal{D}_{\text{univ}}$ denote the universe of all adversaries. We use the shorthand $R_{T,k}(\mathcal{A}_{\mathcal{F}})$ for $R_{T,k}(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}) = \min_{A \in \mathcal{A}_{\mathcal{F}}} \max_{D(A) \in \mathcal{D}_{\text{univ}}} R_{T,k}(A, D)$. We use $R_k(\mathcal{A}_{\mathcal{F}})$ to denote the asymptotic, in T , value of $R_{T,k}(\mathcal{A}_{\mathcal{F}})$, i.e., $R_k(\mathcal{A}_{\mathcal{F}}) = \sqrt{T} \cdot \lim_{T \rightarrow \infty} \frac{R_{T,k}(\mathcal{A}_{\mathcal{F}})}{\sqrt{T}}$.

Goal. One of our goals in this paper is to compute the precise values of $R_k(\mathcal{A}_{\text{single}})$, $R_k(\mathcal{A}_{\text{dec}})$, $R_k(\mathcal{A}_{\text{arb}})$ and $R_k(\mathcal{A}_{\text{rand}})$ for each value of k , and, *describe and compute* the adversarial sequences that realize these regrets. For clarity, we compute the precise values of $R_k(\mathcal{A}_{\mathcal{F}})$ by:

1. computing the best-response adversary in $\mathcal{D}_{\text{univ}}$ for every algorithm in $\mathcal{A}_{\mathcal{F}}$;
2. computing $R_k(\mathcal{A}_{\mathcal{F}})$ the regret of the optimal algorithm in $\mathcal{A}_{\mathcal{F}}$ (i.e., the algorithm that gets the smallest regret w.r.t. its best-response adversary).

In many cases, the first step, namely computing the best-response adversary, is challenging. We find the best-response adversaries for the families $\mathcal{A}_{\text{single}}$ and \mathcal{A}_{dec} . For the families \mathcal{A}_{arb} and $\mathcal{A}_{\text{rand}}$, we perform the first step approximately, i.e., we compute a nearly best-response adversary, and thus we obtain lower bounds on $R_k(\mathcal{A}_{\text{arb}})$ and $R_k(\mathcal{A}_{\text{rand}})$.

What is known, and what to expect? It is well known that for $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{single}}, \mathcal{A}_{\text{dec}}, \mathcal{A}_{\text{arb}}, \mathcal{A}_{\text{rand}}$: $R_{T,k}(\mathcal{A}_{\mathcal{F}}) \leq \sqrt{(T \ln k)/2}$ for all T, k , and in the doubly asymptotic limit, as both T and k go to ∞ , the optimal regret of $\mathcal{A}_{\text{single}}$ is $\sqrt{(T \ln k)/2}$, i.e., $\lim_{T \rightarrow \infty, k \rightarrow \infty} \left(R_{T,k}(\mathcal{A}_{\text{single}}) / \sqrt{(T \ln k)/2} \right) = 1$. (see [7, 5]). While there are useful applications for $k \rightarrow \infty$, there are also several interesting use-cases of the experts problem with just a few experts (rain-or-shine ($k = 2$), buy-or-sell-or-hold ($k = 3$)). It seems like for small k such as 2, 3, 4 etc. $R_k(\mathcal{A}_{\text{single}})$ could be a significant constant factor smaller than $\sqrt{(T \ln k)/2}$. And given that families like \mathcal{A}_{dec} etc. are supersets of $\mathcal{A}_{\text{single}}$, it seems even more likely that $R_k(\mathcal{A}_{\text{dec}})$ etc. are constant factor smaller than $\sqrt{(T \ln k)/2}$. Surprisingly, we show that is not the case: the regret of $\sqrt{(T \ln k)/2}$ that is obtained as $k \rightarrow \infty$ is *already obtained at $k = 2$* . Thus our work completely closes the gap between upper and lower bounds for all k .

²Although R_k doesn't have a T in the subscript, R_k is still dependent on T . We suppress T merely to indicate asymptotics in T .

1.1 Main Results

Finite horizon model.

1. $R_k(\mathcal{A}_{\text{single}}) = R_k(\mathcal{A}_{\text{dec}}) = \sqrt{\frac{T \ln k}{2}}$ for even k , $R_k(\mathcal{A}_{\text{single}}) \geq R_k(\mathcal{A}_{\text{dec}}) \geq \sqrt{\frac{T \ln k}{2}(1 - \frac{1}{k^2})}$ for odd k .
2. $R_k(\mathcal{A}_{\text{arb}}) \geq R_k(\mathcal{A}_{\text{rand}}) \geq \frac{2}{3} \sqrt{\frac{T \ln k}{2}}$ for even k , $R_k(\mathcal{A}_{\text{arb}}) \geq R_k(\mathcal{A}_{\text{rand}}) \geq \frac{2}{3} \sqrt{\frac{T \ln k}{2}(1 - \frac{1}{k^2})}$ for odd k .

Geometric horizon model. In the geometric horizon model, the current time t is not relevant, since the expected remaining time for which the game lasts is the same irrespective of how many steps have passed in the past. Thus $\eta(t)$ is without loss of generality, independent of t . Nevertheless, η could still depend on other aspects of the history of the game, like the cumulative gains of all the experts etc. We establish some quick notation before discussing results. Let δ denote the probability that the game stops at any given step, independently of the past (and therefore the expected length of the game is $\frac{1}{\delta}$). Let $R_{\delta,k}(A, D)$ denote the regret achieved by algorithm A when faced with adversary D in the prediction with expert advice game with stopping probability δ and k experts. The minimax regret for a family $\mathcal{A}_{\mathcal{F}}$ of algorithms is given by $R_{\delta,k}(\mathcal{A}_{\mathcal{F}}) = R_{\delta,k}(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}) = \min_{A \in \mathcal{A}_{\mathcal{F}}} \max_{D \in \mathcal{D}_{\text{univ}}} R_{\delta,k}(A, D)$. Let³ $R_k(\mathcal{A}_{\mathcal{F}}) = \frac{1}{\sqrt{\delta}} \lim_{\delta \rightarrow 0} \sqrt{\delta} \cdot R_{\delta,k}(\mathcal{A}_{\mathcal{F}})$.

We show the following: (1) $R_2(\mathcal{A}_{\text{single}}) = \frac{0.391}{\sqrt{\delta}}$; (2) $R_k(\mathcal{A}_{\text{single}}) \geq \frac{1}{2} \sqrt{\frac{\ln k}{2\delta}}$ for all k .

The regret lower bound of $\frac{1}{2} \sqrt{\frac{\ln k}{2\delta}}$ we obtain is at most a factor 2 away from the regret upper bound of $\sqrt{\frac{\ln k}{2\delta}}$. Further, we show that the adversarial family that we use for the family of algorithms $\mathcal{A}_{\text{single}}$ to obtain the precise regret for 2 experts, also obtains the optimal regret for the universe $\mathcal{A}_{\text{univ}}$ of *all algorithms*. See Remark 4 for more on this result.

1.2 Simple adversarial primitives and families

While the optimal regret $R_k(\mathcal{A}_{\mathcal{F}})$ is defined by optimizing over the most general family $\mathcal{D}_{\text{univ}}$ of all adversaries, (i.e., $R_k(\mathcal{A}_{\mathcal{F}}) = R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}})$) one of our primary contributions in this work is to develop simple and analytically easy-to-work-with adversarial primitives that we use to construct adversarial families (call a typical such family $\mathcal{D}_{\text{simple}}$) such that:

- $\mathcal{D}_{\text{simple}}$ is simple to-describe and to-optimize-over, i.e., computing $\max_{D \in \mathcal{D}_{\text{simple}}} R_{T,k}(A, D)$ is much simpler than computing $\max_{D \in \mathcal{D}_{\text{univ}}} R_{T,k}(A, D)$.
- optimizing over $\mathcal{D}_{\text{simple}}$ is guaranteed to be as good (or approximately as good) as optimizing over $\mathcal{D}_{\text{univ}}$ for many algorithmic families $\mathcal{A}_{\mathcal{F}}$, i.e., $R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}) = R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{simple}})$ for many $\mathcal{A}_{\mathcal{F}}$. As $R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}) \geq R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{simple}})$, the non-trivial part is to prove (approximate) equality for $\mathcal{A}_{\mathcal{F}}$.

We demonstrate the versatility of our primitives by using simple combinations of them to develop sharp lower bounds to algorithmic families $\mathcal{A}_{\text{single}}$, \mathcal{A}_{dec} , \mathcal{A}_{arb} , and $\mathcal{A}_{\text{rand}}$. There is a lot of room for further combinations of primitives that might be useful to construct adversarial families tailored to other algorithmic families.

The “looping” and “straight-line” primitives. These primitives are best described by focusing on the case of $k = 2$ experts. In the two experts case, the algorithm makes its decision at step t , by just looking at the difference d of the cumulative gains of the leading and lagging experts’ cumulative gains. As such, the adversary has to simply control how the difference d evolves over time. The “looping” primitive simply loops

³Note that the notation $R_k(\cdot)$ is overloaded: it could refer to finite or geometric horizon setting depending on the context. But since the setting is clear from the context, we drop the δ vs T .



Figure 1: Optimal adversaries in finite horizon (left) and geometric horizon (right)

the value of d between 0 and 1 indefinitely (i.e., advances⁴ one expert in one step and advances the other in the next step and so on, so that d simply loops between 0 and 1). The “straight-line” adversary simply keeps advancing the value of d by 1 at each step. Interestingly, the worst-case adversary for each of the finite and geometric horizon settings is a composition of looping and straight-line primitives. Strikingly, despite the apparent similarity between two settings, the optimal adversaries in the two models turn out to be “mirror images” of each other. The optimal adversary in finite horizon loops first and then goes in straight-line, while the geometric horizon’s optimal does the reverse. The structure of these two families is depicted in Figure 1, that shows the evolution of the difference d between the cumulative gains of the leading and lagging experts. This fundamental difference between the structures of the optimal adversary in these two settings also manifests in the optimal regret values of these two settings (see Remark 5).

The generalizations of these primitives for arbitrary k is straightforward. The looping primitive partitions the set of experts into two teams, say A and B , and then it advances all experts in team A in one step and in team B in the other, and so on. The straight-line primitive picks an arbitrary expert and keeps advancing that expert by 1 in each step.

Combining the primitives. Here’s how we create effective adversarial families from these primitives. In fact the families are often trivial, i.e., they have only one member and therefore there’s nothing to optimize. We ignore the odd and even k distinctions here for ease of description and just focus on the even k case. Please see the technical sections for precise descriptions, which is only slightly different from what is here.

1. Perform $\frac{T-\ell}{2}$ loops and then ℓ straight-line steps, for $\ell = T^{3/4}$. Call this adversary D_{lsdet} (stands for loop-straight-deterministic). Clearly, this adversarial family is simple-to-describe and there is nothing to optimize here as there is only one member in the family. Most importantly, it gives the precisely optimal regret for algorithmic families $\mathcal{A}_{\text{single}}$ and \mathcal{A}_{dec} as $T \rightarrow \infty$. I.e.,

$$\text{For families } \mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{single}}, \mathcal{A}_{\text{dec}}: \quad R_k(\mathcal{A}_{\mathcal{F}}, D_{lsdet}) = \sqrt{\frac{T \ln k}{2}} = R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}).$$

The best known regret lower bound for $\mathcal{A}_{\text{single}}$ was $\frac{1}{4}\sqrt{T \log_2 k}$ [11], which leaves a factor 2.35 gap between upper and lower bounds, that our work closes. We are not aware of prior lower bounds for \mathcal{A}_{dec} .

2. Perform $\frac{T-r}{2}$ loops and then r straight-line steps, where r is chosen uniformly at random from $\{0, 1, \dots, T^{3/4}\}$. This family is simple and there is nothing to optimize here as well. Call this adversary D_{lsrand} (denoting loop, straight, uniformly random). We show that when $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{arb}}$ or when $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{rand}}$:

$$\text{For families } \mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{arb}}, \mathcal{A}_{\text{rand}}: \quad R_k(\mathcal{A}_{\mathcal{F}}, D_{lsrand}) \geq \frac{2}{3}\sqrt{\frac{T \ln k}{2}} \geq \frac{2}{3}R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}).$$

Note that while this lower bound doesn’t precisely match the upper bound, the upper bound $R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}}) \leq \sqrt{(T \ln k)/2}$ and is likely even smaller for small k (particularly for a large family of algorithms like \mathcal{A}_{arb} or $\mathcal{A}_{\text{rand}}$) — *thus our result shows that the ratio between upper and lower bounds is at most $\frac{3}{2}$ and likely even smaller.* To the best of our knowledge our lower bound is the first for the classes \mathcal{A}_{arb} and $\mathcal{A}_{\text{rand}}$.

⁴advances here refers to setting the gain of that expert to 1.

3. In geometric horizon, even for the family $\mathcal{A}_{\text{single}}$ and at $k = 2$ experts, instead of a single adversary working for all members of $\mathcal{A}_{\text{single}}$, we have a single-parameter family of adversaries to optimize over. Namely, follow the straight-line primitive for r steps and then the looping primitive for $\frac{T-r}{2}$ steps. Call this single-parameter family (parameterized by r) as \mathcal{D}_{sl} . The exact number r is determined by optimizing it as a function of the parameter η used by the algorithm in $\mathcal{A}_{\text{single}}$. Specifically, for the case of 2 experts we show that: $R_2(\mathcal{A}_{\text{single}}, \mathcal{D}_{sl}) = \frac{0.391}{\sqrt{\delta}} = R_2(\mathcal{A}_{\text{single}}, \mathcal{D}_{\text{univ}})$. Note that \mathcal{D}_{sl} is again simple-to-describe and straightforward-to-optimize over. Further, it is the precisely optimal adversary family for not just $\mathcal{A}_{\text{single}}$ but also the *universe of all algorithms* $\mathcal{A}_{\text{univ}}$ (see Remark 4), i.e., $R_2(\mathcal{A}_{\text{univ}}, \mathcal{D}_{\text{univ}}) = R_2(\mathcal{A}_{\text{univ}}, \mathcal{D}_{sl})$.
4. But in the geometric horizon setting, if we don't shoot for the precisely optimal adversary family, and aim for just approximately optimal, then we don't need a single-parameter family: just following one of the two looping/straight-line primitives gives a lower bound of $\frac{1}{2}\sqrt{\frac{\ln k}{2\delta}}$. Let D_ℓ, D_s be the looping and straight line primitives. Then: $R_k(\mathcal{A}_{\text{single}}, \{D_\ell, D_s\}) \geq \frac{1}{2}\sqrt{\frac{\ln k}{2\delta}} \geq \frac{1}{2}R_k(\mathcal{A}_{\text{single}}, \mathcal{D}_{\text{univ}})$. Note that while this lower bound doesn't precisely match the upper bound $R_k(\mathcal{A}_{\text{single}}, \mathcal{D}_{\text{univ}})$, the latter is at most⁵ $\sqrt{\frac{\ln k}{2\delta}}$, which is at most a factor 2 larger than lower bound. The only known regret lower bounds in the geometric horizon setting was what one could infer from the finite horizon setting lower bound of $\frac{1}{4}\sqrt{T \log_2 k}$ [11], and it is not even clear what this exactly translates to in the geometric horizon setting.

Remark 1 *To give a sense that the primitives offer enough variety in combination, here is a simple modification over the adversary $D_{l\text{rand}}$, that we call $D_{l\text{rand}++}$: use $D_{l\text{rand}}$ with probability p , and with probability $1 - p$ play the looping primitive D_l for all the T steps. This increases the lower bound from $\frac{2}{3}\sqrt{(T \ln k)/2}$ to $0.68\sqrt{(T \ln k)/2}$ (see Theorem 3). We believe that this can be increased further by picking the stopping time for looping from a non-uniform distribution etc.*

1.3 Motivation and discussion

In this work we seek to understand the structure of worst case input sequences for a broad family of algorithms and crisply expose their vulnerabilities. By identifying such structures, we also get the precise regret suffered by them. Our motivation in exploring this question includes the following.

1. After 25 years since MWA was introduced [17, 25], we do not have a sharp regret bound for it. $\mathcal{A}_{\text{single}}$ is known to suffer a regret of at most $\sqrt{\frac{T \ln k}{2}}$, but the best known lower bound on regret is $\frac{1}{4}\sqrt{T \log_2 k}$ [11], with a factor 2.35 gap between these two bounds. For larger families like \mathcal{A}_{arb} , $\mathcal{A}_{\text{rand}}$ no lower bounds were known. For an algorithm as widely used as MWA, it is fruitful to have a sharp regret characterization.
2. The patterns in the worst-case adversarial sequences that we characterize are simple to spot if they exist (or even if anything close exists), and make simple amends to the algorithm that result in significant gains.
3. The problem is theoretically clean and challenging: how powerful are simple input patterns beyond the typically used pure random sequences in inflicting regret?

Related Work. *Classic works:* The book by Cesa-Bianchi and Lugosi [6] is an excellent source for both applications and references for prediction with expert advice. The prediction with experts advice paradigm was introduced by Littlestone and Warmuth [17] and Vovk [25]. The famous multiplicative weights update algorithm was introduced independently by these two works: as the weighted majority algorithm by Littlestone and Warmuth and as the aggregating algorithm by Vovk. The pioneering work of Cesa-Bianchi et al. [7]

⁵This is a simple extension of the standard proof that MWA has a regret upper bound of $\sqrt{(T \ln k)/2}$ in the finite horizon setting with T steps and k experts, and the realization that in the geometric horizon setting, the expected stopping time is $\frac{1}{\delta}$.

considered $\{0, 1\}$ outcome space for nature and showed that for the absolute loss function $\ell(x, y) = |x - y|$ (or $g(x, y) = 1 - |x - y|$), the asymptotically optimal regret is $\sqrt{\frac{T \ln k}{2}}$. This was later extended to $[0, 1]$ outcomes for nature by Haussler et al. [13]. The asymptotic optimality of $\sqrt{\frac{T \ln k}{2}}$ for arbitrary loss (gain) functions follows from the analysis of Cesa-Bianchi [5]. When it is known beforehand that the cumulative loss of the optimal expert is going to be small, the optimal regret can be considerably improved, and such results were obtained by Littlestone and Warmuth [17] and Freund and Schapire [9]. With certain assumptions on the loss function, the simplest possible algorithm of following the best expert already guarantees sub-linear regret Hannan [12]. Even when the loss functions are unbounded, if the loss functions are exponential concave, sub-linear regret can still be achieved Blum and Kalai [4].

Recent works: Gravin et al. [10] give the minimax optimal algorithm, and the regret for the prediction with expert advice problem for the cases of $k = 2$ and $k = 3$ experts. The focus of [10] was providing a regret upper bound for the family of all algorithms, while the focus of this paper is to provide regret lower bounds for large families of algorithms. Luo and Schapire [18] consider a setting where the adversary is restricted to pick gain vectors from the basis vector space $\{e_1, \dots, e_k\}$. Abernethy et al. [2] consider a different variant of experts problem where the game stops when cumulative loss of any expert exceeds given threshold. Abernethy et al. [1] consider general convex games and compute the minimax regret exactly when the input space is a ball, and show that the algorithms of Zinkevich [26] and Hazan et al. [14] are optimal w.r.t. minimax regret. Abernethy et al. [3] provide upper and lower bounds on the regret of an optimal strategy for several online learning problems without providing algorithms, by relating the optimal regret to the behavior of a certain stochastic process. Mukherjee and Schapire [21] consider a continuous experts setting where the algorithm knows beforehand the maximum number of mistakes of the best expert. Rakhlin et al. [22] introduce the notion of sequential Rademacher complexity and use it to analyze the learnability of several problems in online learning w.r.t. minimax regret. Rakhlin et al. [23] use the sequential Rademacher complexity introduced in [22] to analyze learnability w.r.t. general notions of regret (and not just minimax regret). Rakhlin et al. [24] use the notion of conditional sequential Rademacher complexity to find relaxations of problems like prediction with static experts that immediately lead to algorithms and associated regret guarantees. They show that the random payout strategy has a sound basis and propose a general method to design algorithms as a random payout. Koolen [15] studies the regret w.r.t. every expert, rather than just the best expert in hindsight and considers tradeoffs in the Pareto-frontier. McMahan and Abernethy [19] characterize the minimax optimal regret for online linear optimization games as the supremum over the expected value of a function of a martingale difference sequence, and similar characterizations for the minimax optimal algorithm and the adversary. McMahan and Orabona [20] study online linear optimization in Hilbert spaces and characterize minimax optimal algorithms. Chaudhuri et al. [8] describe a parameter-free learning algorithm motivated by the cases of large number of experts k . Koolen and van Erven [16] develop a prediction strategy called Squint, and prove bounds that incorporate both quantile and variance guarantees.

2 Finite horizon

We begin our analysis of MWA by focusing on the simple case of $k = 2$ experts. We first identify the structure of the optimal adversary, and through it we obtain the tight regret bound as $T \rightarrow \infty$. Before proceeding further, it is useful to recall that when the gains of the leading and lagging experts are given by $g + d$ and g , the MWA algorithm follows these experts with probabilities $\frac{e^{\eta d}}{e^{\eta d} + 1}$ and $\frac{1}{e^{\eta d} + 1}$ respectively. Thus, when the adversary increases d by 1 i.e., increases the gain of the leading expert by 1, the regret benchmark (namely, the gains of the leading expert) increases by 1, where as MWA is correct only with probability $\frac{e^{\eta d}}{e^{\eta d} + 1}$, and this therefore inflicts a regret of $\frac{1}{e^{\eta d} + 1}$ on MWA. On the other hand, if the adversary decreases d by 1, then

the benchmark doesn't change, whereas MWA succeeds with probability $\frac{1}{e^{\eta d} + 1}$, and this therefore inflicts a regret of $\frac{-1}{e^{\eta d} + 1}$. When the adversary doesn't change d , the regret inflicted is 0.

Structure of the optimal adversary. Let η be the fixed update rate of the optimal MWA (the parameter in the exponent as explained in Section 1)⁶. Against a specific algorithm, an optimal adversary can always be found in the class of deterministic adversaries. The actions of the optimal adversary (against a specific MWA algorithm) depend only on the distance d between leading and lagging experts and time step t .

1. **Loop aggregation:** At each time step, the adversary may either increase or decrease the gap d by 1, or leave d unchanged. We denote these actions of the adversary by $d \xrightarrow{t} d + 1$, $d \xrightarrow{t} d - 1$, and $d \xrightarrow{t} d$. The respective regret values inflicted on the algorithm are given by $\frac{1}{e^{\eta d} + 1}$, $\frac{-1}{e^{\eta d} + 1}$, and 0, which are all independent of the time when an action was taken. This means that if the adversary loops between d and $d + 1$ at several disconnected points of time, it may as well aggregate all of them and complete all of them in consecutive time steps. I.e., the optimal adversary starts at $d = 0$ and then weakly monotonically increases d , stopping at various points $d = s$, looping for an arbitrary length of time between $d = s$ and $d = s - 1$ and then proceeding forward.
2. **Staying at same d is dominated:** It is not hard to see that any action $x \rightarrow x$ is dominated for the adversary as this wastes a time step and inflicts 0 regret on the algorithm. Thus the “weakly monotonically increases” in the previous paragraph can be replaced by “strictly monotonically increases” (except of course for the stopping points for looping).
3. **Loop(0) domination:** Define $\text{Loop}(d) \stackrel{\text{def}}{=} d \rightarrow d + 1 \rightarrow d$. It is easy to see that the regret inflicted by $\text{Loop}(d)$ is exactly $\frac{1}{e^{\eta d} + 1} - \frac{1}{e^{\eta(d+1)} + 1}$ and this quantity is maximized at $d = 0$. Thus, the optimal adversary should replace all loops by loops at 0. This gives us the structure claimed in Figure 1 for the optimal adversary.

Given the optimal adversary's structure (as described in Figure 1) w.l.o.g. we can assume it to be looping for $\frac{T-\ell}{2}$ steps at 0 and then monotonically increasing d for ℓ steps at which point the game ends. In the following we will analyze the regret inflicted by the optimal adversary (which we showed was optimal for the class of algorithm $\mathcal{A}_{\text{single}}$) against a broader class \mathcal{A}_{dec} of MWA. The regret of the adversary is:

$$\sum_{t=1}^{\frac{T-\ell}{2}} \left[\frac{1}{2} - \frac{1}{e^{\eta(2t)} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{1}{e^{\eta(T-\ell+d+1)d} + 1}. \quad (1)$$

Asymptotic regret of the optimal adversary. We first notice that for a fixed adversary with a given ℓ , the regret of MWA with decreasing $\eta(t)$ in (1) is greater than or equal to the regret of MWA with a constant $\eta' = \eta(T - \ell)$, i.e.,

$$\frac{T-\ell}{2} \left[\frac{1}{2} - \frac{1}{e^{\eta'} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{1}{e^{d\eta'} + 1}. \quad (2)$$

This is true as each individual term in (2) is equal to or smaller than the corresponding term in (1). In the following we are going to use $\ell = T^{3/4}$ for the adversary and for convenience, we write $e^{\eta'(T-\ell)} = \tau = 1 + \frac{\alpha}{\sqrt{T}}$. The two terms in (1) together place strong bounds on what α should be: they imply that $\alpha = \Theta(1)$. We show this in 2 steps: first we show that $\alpha = O(1)$, and then show that $\alpha = \Omega(1)$.

⁶In fact, we can identify the optimal adversary for a much broader family of algorithms (see Appendix A.3 for more details).

1. The first term in (2) forces α to be $O(1)$. The regret of MWA for $\ell = T^{3/4}$ is at least

$$\frac{T - \ell}{2} \left[\frac{1}{2} - \frac{1}{e^{\eta'} + 1} \right] \simeq \frac{T}{2} \left[\frac{1}{2} - \frac{1}{e^{\eta'} + 1} \right] = \frac{\alpha \sqrt{T}}{4(1 + e^{\eta'})} = \frac{\sqrt{T}}{4\left(\frac{2}{\alpha} + \frac{1}{\sqrt{T}}\right)}.$$

Since MWA's regret upper bound in the finite horizon model is $\Theta(\sqrt{T})$, α must be $O(1)$.

2. To show that $\alpha = \Theta(1)$ we argue that the regret from the second term of (2) is $\omega(\sqrt{T})$ when $\alpha = o(1)$. For all $d \leq \frac{\sqrt{T}}{\alpha}$, we have $\tau^d = (1 + \frac{\alpha}{\sqrt{T}})^d \leq e$. Thus MWA's regret for $k = \min(\frac{\sqrt{T}}{\alpha}, T^{3/4})$ is at least

$$\sum_{d=0}^{k-1} \frac{1}{\tau^d + 1} \geq \sum_{d=0}^{k-1} \frac{1}{e + 1} = \Omega(k) = \omega(\sqrt{T}).$$

Since MWA's regret upper bound in the finite horizon model is $\Theta(\sqrt{T})$, we get $\alpha = \Omega(1)$.

Now, we obtain the following asymptotic estimate for the second part of (2), where $\eta' \sim e^{\eta'} - 1 = \frac{\alpha}{\sqrt{T}}$.

$$\sum_{d=0}^{\ell-1} \frac{1}{\tau^d + 1} \sim \int_0^{\ell} \frac{dx}{e^{\eta'x} + 1} = \frac{1}{\eta'} \ln \left(2 \frac{e^{\ell\eta'}}{e^{\ell\eta'} + 1} \right) \sim \frac{\sqrt{T}}{\alpha} \left(\ln(2) - \ln(1 + e^{-\ell\eta'}) \right). \quad (3)$$

The first part of (2) can be estimated as follows

$$\frac{T - \ell}{2} \left[\frac{1}{2} - \frac{1}{e^{\eta'} + 1} \right] \sim \frac{T}{2} \left[\frac{e^{\eta'} - 1}{2(e^{\eta'} + 1)} \right] \sim \frac{T}{2} \cdot \frac{\eta'}{4} = \frac{\alpha \sqrt{T}}{8}. \quad (4)$$

As $e^{-\ell\eta'} = e^{-\alpha T^{1/4}} = o(1)$, (3) simplifies to $\frac{\ln(2)\sqrt{T}}{\alpha}$, while the estimate for (4) is $\frac{\alpha\sqrt{T}}{8}$. Now the estimate $\sqrt{T} \left[\frac{\ln(2)}{\alpha} + \frac{\alpha}{8} \right]$ for the regret in (2) is minimized for the choice of parameter $\alpha = \sqrt{8 \ln(2)}$. Then the regret of the optimal MWA is at least $\sqrt{T \cdot \frac{\ln(2)}{2}} (1 + o(1))$. It is known that there is MWA for $k = 2$ experts with regret at most $\sqrt{T \cdot \frac{\ln(2)}{2}}$ (asymptotic in T). Thus, we obtain the following claim 1 (in the claim below, by ‘‘optimal MWA’’ we mean the MWA with the optimally tuned $\eta(t) = \eta'$).

Claim 1 For $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{single}}, \mathcal{A}_{\text{dec}}$: $R_2(\mathcal{A}_{\mathcal{F}}, D_{\text{lsdet}}) = \sqrt{\frac{T \ln 2}{2}} = R_2(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}})$.

We generalize the adversary for $k = 2$ and obtain a tight lower bound for \mathcal{A}_{dec} matching the known upper bound for arbitrary *even* number k of experts and almost matching bound for *odd* number k of experts. Since $R_k(\mathcal{A}_{\text{single}}) \geq R_k(\mathcal{A}_{\text{dec}})$, the lower bound in Theorem 1 below applies to $\mathcal{A}_{\text{single}}$ as well.

Theorem 1 For $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{single}}, \mathcal{A}_{\text{dec}}$: *i) For even k :* $R_k(\mathcal{A}_{\mathcal{F}}, D_{\text{lsdet}}) = \sqrt{\frac{T \cdot \ln k}{2}} = R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}})$.

ii) For odd k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{\text{lsdet}}) \geq \sqrt{\frac{T \cdot \ln k}{2} \left(1 - \frac{1}{k^2}\right)} \geq \sqrt{1 - \frac{1}{k^2}} R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}})$.

Proof: Let $\eta(t)$ be the update rate of the optimal MWA, we define $\ell = T^{3/4}$ and $\eta' = \eta(T - \ell)$. We employ the following adversary for the even k number of experts:

1. Divide all experts into two equal parties, numbered A and B . For the first $\frac{T-\ell}{2}$ rounds ($\ell = T^{3/4}$), advance all the experts in party A in even numbered rounds, and all experts in party B in odd numbered rounds.
2. For the remaining ℓ steps, pick an arbitrary expert and keep advancing just that expert.

Similar to (1) this adversary obtains the regret of at least $\sum_{t=1}^{\frac{T-\ell}{2}} \left[\frac{1}{2} - \frac{1}{e^{\eta(2t)} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{k-1}{e^{d \cdot \eta(T-\ell+d+1)} + k-1}$. We further notice that similar to (2) the regret of MWA with decreasing $\eta(t)$ in the above expression is greater than or equal to the regret of MWA with a constant $\eta' = \eta(T - \ell)$, i.e., the previous expression is at least

$$\frac{T-\ell}{2} \left[\frac{1}{2} - \frac{1}{e^{\eta'} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{k-1}{e^{d \cdot \eta'} + k-1}. \quad (5)$$

We use (4) to estimate the first term of (5). We estimate the second term of (5) similar to (3) as follows.

$$\sum_{d=0}^{\ell-1} \frac{k-1}{e^{d\eta'} + k-1} \sim \int_0^\ell \frac{(k-1)dx}{e^{x\eta'} + k-1} = \frac{1}{\eta'} \ln \left(\frac{k \cdot e^{\ell\eta'}}{e^{\ell\eta'} + k-1} \right) \sim \frac{\sqrt{T} \ln(k)}{\alpha}. \quad (6)$$

Now, combining these two estimates the regret from (5) is at least

$$\frac{\alpha\sqrt{T}}{8} + \frac{\sqrt{T} \ln(k)}{\alpha} \geq 2 \cdot \sqrt{\frac{\alpha\sqrt{T}}{8} \cdot \frac{\sqrt{T} \ln(k)}{\alpha}} = \sqrt{\frac{T \ln(k)}{2}},$$

which precisely matches the upper bound on the regret of MWA[7].

For the odd k number of experts we employ almost the same adversary as for even k , although, since k now is odd, we split experts into two parties of *almost* equal sizes (see Appendix A.1 for full details). ■

2.1 General variations of MWA

We have seen that the best known MWA with a flat learning rate η achieves optimal (or almost optimal in the case of odd number of experts) regret among all MWAs with monotone decreasing learning rates $\eta(t)$. However, it seems that in the finite horizon model a better strategy for tuning parameters of MWA would be to use higher rates $\eta(t)$ towards the end T . In the following we study a broader family of MW algorithms \mathcal{A}_{arb} where learning parameter $\eta(t)$ can vary in an arbitrary way. In the following theorem we show that such adaptivity of MWA cannot decrease the regret of the algorithm by more than a factor of $2/3$.

Remark 2 *In fact, our analysis extends to the family $\mathcal{A}_{\text{rand}}$ where each $\eta(t)$ can be a random variable drawn from a distribution F_t . Effectively, with a random $\eta(t)$ the algorithm player can get any convex combination $f(G_{it-1}, t) = \mathbf{E}_{\eta(t)}[e^{\eta(t)G_{it-1}}]$ of $e^{\eta(t)G_{it-1}}$ in the vector of probabilities for following each expert i at time t . This constitutes a much richer family of algorithms compared to the standard single parameter MWA family.*

Theorem 2 *For $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\text{arb}}, \mathcal{A}_{\text{rand}}$: i) For even k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{\text{lsrand}}) \geq \frac{2}{3} \sqrt{\frac{T \cdot \ln k}{2}} \geq \frac{2}{3} R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}})$.*

ii) For odd k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{\text{lsrand}}) \geq \frac{2}{3} \sqrt{\frac{T \cdot \ln k}{2} \left(1 - \frac{1}{k^2}\right)} \geq \frac{2}{3} \sqrt{1 - \frac{1}{k^2}} R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{\text{univ}})$.

Proof: Define $\ell = T^{3/4}$ and $R = \left\lceil \frac{T-\ell-1}{2} \right\rceil$. We use the following adversary for even number of experts k :

1. Choose $j \in [R]$ uniformly at random. With probability 0.5 don't advance any expert in the first step.
2. Divide all experts into two equal parties, numbered A and B . For the next j rounds, advance all the experts in party A in even numbered rounds, and all experts in party B in odd numbered rounds.

3. For next ℓ steps, pick any expert i and keep advancing just expert i . Do nothing in remaining steps.

The regret of the algorithm is

$$\begin{aligned} & \frac{1}{R} \sum_{j=0}^{R-1} \left[\frac{1}{2} \left(\sum_{t=1}^j \left[\frac{1}{2} - \frac{1}{e^{\eta(2t)} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{k-1}{e^{d \cdot \eta(2j+d+1)} + k-1} \right) + \right. \\ & \left. \frac{1}{2} \left(\sum_{t=1}^j \left[\frac{1}{2} - \frac{1}{e^{\eta(2t+1)} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{k-1}{e^{d \cdot \eta(2j+d+2)} + k-1} \right) \right]. \end{aligned} \quad (7)$$

Since $\eta(t)$ can be arbitrary nonnegative number, we break (7) into terms with the same $\eta(t)$ (we also drop a few terms to simplify the expression). In the following, we will also assume that $e^{\eta(t)} = 1 + \frac{\alpha(t)}{\sqrt{T}}$, where $\alpha(t) = \Theta(1)$ for every $t \in [T]$. Later we will explain why this assumption is without loss of generality.

$$\begin{aligned} (7) & \geq \frac{1}{2R} \sum_{t=\ell}^{T-\ell-1} \left([R - \lceil t/2 \rceil] \cdot \left[\frac{1}{2} - \frac{1}{e^{\eta(t)} + 1} \right] + \sum_{d=0}^{\ell-1} \frac{k-1}{e^{\eta(t)d} + k-1} \right) \\ & \simeq \sum_{t=\ell}^{T-\ell-1} \left(\left[\frac{R - \lceil t/2 \rceil}{R} \right] \frac{\alpha(t)}{8\sqrt{T}} + \frac{\sqrt{T} \ln(k)}{2R \cdot \alpha(t)} \right) \geq \sum_{t=\ell}^{T-\ell-1} 2 \sqrt{\left[\frac{R - \lceil t/2 \rceil}{R} \right] \frac{\alpha(t)}{8\sqrt{T}} \frac{\sqrt{T} \ln(k)}{2R \cdot \alpha(t)}} \\ & = \sqrt{\frac{\ln(k)}{2 \cdot 2R}} \sum_{t=\ell}^{T-\ell-1} \sqrt{\left[\frac{R - \lceil t/2 \rceil}{R} \right]} \simeq \sqrt{\frac{\ln(k)}{2 \cdot T}} \int_0^1 \sqrt{1-x} \, dx = \sqrt{\frac{\ln(k)}{2T}} \cdot \frac{2}{3}. \end{aligned} \quad (8)$$

In the above derivation we obtain the first approximation \simeq by using approximations from (4) and (6).

We now argue that the assumption $\alpha(t) = \Theta(1)$ is without loss of generality for every $t \in [T]$. We apply a similar argument as in Theorem 1, but now for each individual term with a particular $\eta(t)$. The term $\sum_{d=0}^{\ell-1} \frac{k-1}{e^{\eta(t)d} + k-1}$ in (8) is already large enough for the estimate when $\alpha(t) = o(1)$. The term $[R - \lceil t/2 \rceil] \cdot \left[\frac{1}{2} - \frac{1}{e^{\eta(t)} + 1} \right]$ also places a strong bound of $O(1)$ on $\alpha(t)$, when $[R - \lceil t/2 \rceil]$ is constant fraction of T . To argue about t close to the threshold T , we can slightly modify the adversary by playing with a small constant probability ε entirely “looping” strategy (without “straight line” part). This would make the coefficient in front of $\left[\frac{1}{2} - \frac{1}{e^{\eta(t)} + 1} \right]$ to be sufficiently large, and at the same time would decrease the lower bound by at most $1 - \varepsilon$ factor. Taking ε arbitrary small we obtain the bound in (8). This concludes the proof for the even number of experts

For the odd number of experts k . We slightly modify the adversary analogous to the case of odd number of experts in Theorem 1. This gives us an additional factor of $1 - \frac{1}{k^2}$ for each of the looping terms. ■

Remark 3 One can slightly improve the lower bound in Theorem 2 and get a better factor than $\frac{2}{3}$. To this end we employ a more complicated adversary by playing with some probability $p > 0$ the same strategy as in Theorem 2 and with the remaining $1 - p$ probability playing purely looping strategy (see Appendix A.2).

3 Geometric horizon

We prove two main results in this Section. We derive the structure of the optimal adversary for 2 experts and show that the optimal regret for 2 experts is exactly $\frac{0.391}{\sqrt{\delta}}$ as $\delta \rightarrow 0$ (see Appendix B.1). For an arbitrary number of experts k , we derive a regret lower bound of $\frac{1}{2} \sqrt{\frac{\ln(k)}{2\delta}}$ (see Appendix B.2).

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A Finite horizon

A.1 Theorem 1 for odd number of experts k

Theorem 1 For $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{single}, \mathcal{A}_{dec}$: i) For even k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{lsdet}) = \sqrt{\frac{T \cdot \ln k}{2}} = R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{univ})$.

ii) For odd k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{lsdet}) \geq \sqrt{\frac{T \cdot \ln k}{2} \left(1 - \frac{1}{k^2}\right)} \geq \sqrt{1 - \frac{1}{k^2}} R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{univ})$.

Proof: We have already proven this theorem for even k . So let $k = 2 \cdot m + 1$ and let $\eta(t)$ be as before the update rate (non increasing in t) of the optimal MWA. We employ almost the same adversary as for even k , although, since k now is odd, we split experts into two parties of *almost* equal sizes. As in the case of even k we choose $\ell = T^{3/4}$ and let $\eta' = \eta(T - \ell)$.

1. divide all experts into two parties of sizes m and $m + 1$ respectively, advance all experts within the party of size m in one step then advance all experts within the other party of size $m + 1$ in the next step, repeat these cycles of two steps $\frac{T - \ell}{2}$ times.
2. fix on one expert and keep advancing just that expert for the remaining ℓ steps.

This adversary obtains the regret of at least

$$\frac{T - \ell}{2} \left[\frac{m + 1}{2m + 1} - \frac{m + 1}{m \cdot e^{\eta'} + m + 1} \right] + \sum_{d=0}^{\ell-1} \frac{k - 1}{e^{d\eta'} + k - 1}. \quad (9)$$

We use the estimate (6) for the second part of (9). For the first part of (9) we closely follow the derivation in (4) and obtain the following estimate.

$$\frac{T - \ell}{2} \left[\frac{m + 1}{2m + 1} - \frac{m + 1}{m \cdot e^{\eta'} + m + 1} \right] \sim \frac{T}{2} \cdot \frac{(m + 1) \cdot m \cdot \alpha / \sqrt{T}}{(2m + 1)^2} = \frac{\alpha \sqrt{T} (k - 1)(k + 1)}{8k^2}. \quad (10)$$

Therefore, the regret from (9) is at least

$$\frac{\alpha \sqrt{T}}{8} \cdot \left(1 - \frac{1}{k^2}\right) + \frac{\sqrt{T} \ln(k)}{\alpha} \geq 2 \cdot \sqrt{\frac{\alpha \sqrt{T}}{8} \left(1 - \frac{1}{k^2}\right)} \cdot \frac{\sqrt{T} \ln(k)}{\alpha} = \sqrt{\frac{T \ln(k)}{2} \left(1 - \frac{1}{k^2}\right)},$$

which concludes the proof of the theorem. ■

A.2 Improved lower bound for \mathcal{A}_{rand} .

We use a slightly more complicated adversary than the one used in Theorem 2 to get a better regret. Call this adversary (which we describe in the proof of the theorem below) $D_{lsrand++}$. We show:

Theorem 3 For $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{arb}, \mathcal{A}_{rand}$: i) For even k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{lsrand++}) \geq 0.68 \sqrt{\frac{T \cdot \ln k}{2}} \geq 0.68 R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{univ})$.

ii) For odd k : $R_k(\mathcal{A}_{\mathcal{F}}, D_{lsrand++}) \geq 0.68 \sqrt{\frac{T \cdot \ln k}{2} \left(1 - \frac{1}{k^2}\right)} \geq 0.68 \sqrt{1 - \frac{1}{k^2}} R_k(\mathcal{A}_{\mathcal{F}}, \mathcal{D}_{univ})$.

Proof: We closely follow the proof of Theorem 2, although now we employ a more complicated adversary by following with a probability $p > 0$ the same strategy as in Theorem 2 and with the remaining $1 - p$ probability playing purely looping strategy (without “straight line” part) as follows.

1. With probability 0.5 don’t advance any expert in the first step.
2. Divide all experts into two equal (almost equal, when k is odd) parties, numbered A and B . For the next T rounds, advance all the experts in party A in even numbered rounds, and all experts in party B in odd numbered rounds.

The regret of the MWA for even k with respect to this adversary is

$$p \cdot (7) + (1 - p) \cdot \left(\frac{1}{2} \sum_{t=1}^{\lfloor \frac{T}{2} \rfloor} \left[\frac{1}{2} - \frac{1}{e^{\eta(2t)} + 1} \right] + \frac{1}{2} \sum_{t=1}^{\lfloor \frac{T-1}{2} \rfloor} \left[\frac{1}{2} - \frac{1}{e^{\eta(2t+1)} + 1} \right] \right) \quad (11)$$

We closely follow derivation (8) and assume that $e^{\eta(t)} = 1 + \frac{\alpha(t)}{\sqrt{T}}$, where $\alpha(t) = \Theta(1)$ for every $t \in [T]$. We also recall that $R = \lfloor \frac{T-\ell-1}{2} \rfloor$.

$$\begin{aligned} (11) &\gtrsim \sum_{t=\ell}^{T-\ell-1} \left(\left[p \cdot \frac{R - \lfloor t/2 \rfloor}{R} + 1 - p \right] \frac{\alpha(t)}{8\sqrt{T}} + p \cdot \frac{\sqrt{T} \ln(k)}{2R \cdot \alpha(t)} \right) \\ &\geq \sum_{t=\ell}^{T-\ell-1} 2 \sqrt{\left[p \cdot \frac{R - \lfloor t/2 \rfloor}{R} + 1 - p \right] \frac{\alpha(t)}{8\sqrt{T}} \cdot p \cdot \frac{\sqrt{T} \ln(k)}{2R \cdot \alpha(t)}} \\ &\simeq \sqrt{\frac{p \ln(k)}{2 \cdot T}} \int_0^1 \sqrt{p(1-x) + 1 - p} \, dx = \sqrt{\frac{\ln(k)}{2T}} \cdot \frac{2}{3\sqrt{p}} \left(1 - (1-p)^{3/2} \right). \end{aligned} \quad (12)$$

The right hand side of expression (12) is maximized for $p \sim 0.866$ at a value slightly larger than 0.68. Note that we could assume that $\alpha(t) = \Theta(1)$, as otherwise either the “looping” term, or the “straight-line” respective term for the particular $\eta(t)$ would be greater than the bound we used in (12).

The derivation for the odd number of experts k is almost the same with an additional factor of $1 - \frac{1}{k^2}$ for each of the looping terms. ■

A.3 Optimal adversary for two experts for a broader family than MWA.

Our goal in this section is to identify the structure of the optimal adversary for a broader family of algorithms than MWA. We consider a simple case of $k = 2$ experts. We assume that in general any algorithm is parametrized by the distance d between lagging and leading experts at time t and picks the lagging expert with probability $p(d, t)$ and the leading expert with probability $1 - p(d, t)$. Thus, when the adversary increases d by 1, i.e., increases the gain of the leading expert by 1, the regret benchmark (namely, the gains of the leading expert) increases by 1, where as the algorithm is correct only with probability $1 - p(d, t)$, and this therefore inflicts a regret of $p(d, t)$ on the algorithm. On the other hand, if the adversary decreases d by 1, then the benchmark doesn’t change, whereas the algorithm succeeds with probability $p(d, t)$, and this therefore inflicts a regret of $-p(d, t)$. When the adversary doesn’t change d , the regret inflicted is 0. For $k = 2$ experts we consider a family of algorithms $\mathcal{A}_{\text{conv}}$ given by the following two properties:

- (i) $p(d, t)$ does not decrease with t for a fixed d ;
- (ii) $p(d, t)$ is convex and decreasing in variable d for a fixed t , i.e., $p(d + 1, t) + p(d - 1, t) \geq 2p(d, t)$ and $p(d, t) \leq p(d - 1, t)$ for any $d \geq 1$.

Note that family $\mathcal{A}_{\text{conv}}$ contains \mathcal{A}_{dec} for $k = 2$ experts.

Against a specific algorithm, an optimal adversary can always be found in the class of deterministic adversaries. The actions of the optimal adversary (against a specific algorithm) depend only on the distance d between leading and lagging experts and time step t . At each time step, the adversary may either increase or decrease the gap d by 1, or leave d unchanged. We denote these actions of the adversary by $d \xrightarrow{t} d + 1$, $d \xrightarrow{t} d - 1$, and $d \xrightarrow{t} d$.

1. **Pushing $d \rightarrow d$ to $d = 0$:** we observe that at any time t two consecutive actions $d \xrightarrow{t} d + 1$ and $d + 1 \xrightarrow{t+1} d + 1$ give at most as much regret as $d \xrightarrow{t} d$ and $d \xrightarrow{t+1} d + 1$, since $p(d, t) + 0 \leq 0 + p(d, t + 1)$. This means that the optimal adversary can always use the latter pair of actions instead of the former pair of actions (the rest of the actions are unaffected) and inflict at least as much regret on the algorithm.
2. **Pushing loops $d \rightarrow d + 1 \rightarrow d$ to $0 \rightarrow 1 \rightarrow 0$:** Let $t_0 + 1$ be the first time the optimal adversary has decreased distance d between lagging and leading experts, i.e., the adversary played $d + 1 \xrightarrow{t_0+1} d$. We may assume by the previous observation that the adversary plays the incremental actions only at the end of the $[t_0]$ interval, i.e., he first plays $0 \xrightarrow{t} 0$ for $1 \leq t < t_0 - d$ times and then plays $0 \xrightarrow{t_0-d} 1 \xrightarrow{t_0-d+1} 2 \xrightarrow{t_0-d+2} \dots \xrightarrow{t_0} d + 1 \xrightarrow{t_0+1} d$. Suppose that $d > 0$, then the adversary would not decrease the regret by substituting actions $d \xrightarrow{t_0} d + 1 \xrightarrow{t_0+1} d$ with $d \xrightarrow{t_0} d - 1 \xrightarrow{t_0+1} d$. Indeed, using properties (ii) and (i) for the algorithm we get

$$p(d - 1, t_0 + 1) + p(d + 1, t_0 + 1) \geq 2p(d, t_0 + 1) \geq 2p(d, t_0).$$

Equivalently, $\text{Regret}(d \xrightarrow{t_0} d + 1 \xrightarrow{t_0+1} d) = p(d, t_0) - p(d + 1, t_0 + 1) \leq -p(d, t_0) + p(d - 1, t_0 + 1) = \text{Regret}(d \xrightarrow{t_0} d - 1 \xrightarrow{t_0+1} d)$. Furthermore, the play of adversary is dominated by $0 \xrightarrow{t_0-d} 1 \xrightarrow{t_0-d+1} 2 \xrightarrow{t_0-d+2} \dots \xrightarrow{t_0} d$. The same argument applied to the time t after $t_0 - d + 1$ allows us to say that all actions $d \rightarrow d - 1$ (together with some previous $d - 1 \rightarrow d$ action) are dominated by the loops $0 \rightarrow 1 \rightarrow 0$. To this end, we can conclude that adversary play is dominated by a sequence: $0 \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow d$.

3. **$0 \rightarrow 0$ is dominated:** We observe that at any time t the sequence $0 \xrightarrow{t} 1 \xrightarrow{t+1} 0 \xrightarrow{t+2} 0$ dominates $0 \xrightarrow{t} 0 \xrightarrow{t+1} 1 \xrightarrow{t+2} 0$. Indeed,

$$p(0, t) - p(1, t + 1) = 0.5 - p(1, t + 1) \geq 0.5 - p(1, t + 2) = p(0, t + 1) - p(1, t + 2).$$

Thus, adversary plays all sequences $0 \rightarrow 1 \rightarrow 0$ before $0 \rightarrow 0$. Finally, two consecutive actions $0 \xrightarrow{t} 0$, $0 \xrightarrow{t+1} 0$ are dominated by $0 \xrightarrow{t} 1 \xrightarrow{t+1} 0$, so the optimal adversary would use at most one action of the form $0 \rightarrow 0$.

Without loss of generality, the optimal adversary can be assumed to be looping for $\frac{T-\ell}{2}$ steps at 0 ($0 \rightarrow 1 \rightarrow 0$) and then monotonically increasing d for ℓ steps at which point the game ends⁷.

⁷The optimal adversary could also use one extra single step loop $0 \rightarrow 0$, which is not important for our asymptotic analysis as T goes to infinity.

B Geometric horizon

B.1 Asymptotic regret of the optimal adversary for 2 experts.

We begin our analysis of MWA with the case of 2 experts and first identify the structure of the optimal adversary. This adversary turns out to be in some sense the opposite to the optimal adversary in the finite horizon model: it keeps increasing the gap between lagging and leading experts for the first few rounds and only then enters the looping phase until the process stops — we denote this by \mathcal{D}_{sl} .

Structure of the optimal adversary. In the geometric horizon model the actions of the optimal adversary against any specific algorithm depend only on the distance d between leading and lagging experts and don't depend on the time step. Thus to describe the optimal adversary, one just needs to specify what will the lag in the next step be, given that it is d in this step. Further, the optimal adversary at no value of d will decide to maintain the same d in next step, i.e., it advances exactly one expert at a time, thereby increasing d by 1 or decreasing d by 1 in each step. Without loss of generality we assume that the optimal adversary advances the leading expert for the first $\ell + 1$ steps (ℓ may be infinite) and at step $\ell + 1$ advances the lagging expert. At that point $d = \ell$, and since the optimal adversary advanced the leading expert the first time d was equal to ℓ , it will do the same now too. Extending this reasoning to all remaining steps we conclude that the optimal adversary follows straight line strategy for the first ℓ steps and at that point when $d = \ell$, switches to the looping strategy for the remaining time with d looping between ℓ and $\ell + 1$.

Remark 4 *Note that the above structural reasoning is applicable not just for the family \mathcal{A}_{single} , but for the entire universe of algorithms \mathcal{A}_{univ} . While the following precise calculations on regret are for the family \mathcal{A}_{single} of MWA, the very simple adversary structure is fully general and applies to \mathcal{A}_{univ} also, i.e., $R_2(\mathcal{A}_{univ}, \mathcal{D}_{sl}) = R_2(\mathcal{A}_{univ}, \mathcal{D}_{univ})$.*

For a particular MWA with a parameter η the optimal adversary achieves the regret of

$$\sum_{d=0}^{\ell-1} \frac{(1-\delta)^{d+1}}{e^{\eta d} + 1} + (1-\delta)^{\ell+1} \sum_{k=0}^{\infty} \left[\frac{(1-\delta)^{2k}}{e^{\eta \ell} + 1} - \frac{(1-\delta)^{2k+1}}{e^{\eta(\ell+1)} + 1} \right]. \quad (13)$$

We analyze the asymptotic regret of the optimal adversary given in the above expression and compute the regret of the optimal algorithm in case of $k = 2$ experts for the family \mathcal{A}_{single} in Theorem 4 below.

Remark 5 *In contrast to the finite horizon model, the analysis of looping and straight-line strategies of the optimal adversary in geometric horizon are related and cannot be decomposed into two independent quantities. This is because the looping phase comes after the straight-line phase in the geometric horizon model, and therefore the regret it inflicts is strongly influenced by the number of straight-line steps that have passed. On the other hand, in the finite horizon model, the straight-line phase follows the looping phase, and the regret inflicted by the straight-line phase is almost independent of the number of loops that have passed (the only dependence is via the number of rounds in the game that remain and it does not matter for asymptotics). This fundamental difference in structure manifests in how the optimal regret values compare in these two settings: where as in finite horizon we have a regret value of $\sqrt{\frac{T \ln 2}{2}}$, in the geometric horizon setting instead of having the equivalent $\sqrt{\frac{\ln 2}{2\delta}}$, we get the optimal regret to be $\frac{0.391}{\sqrt{\delta}}$ — the former is 50.5% larger than the latter. In other words, it is more difficult for the adversary to inflict regret in the geometric horizon setting.*

Theorem 4 In geometric horizon, $R_2(\mathcal{A}_{single}, \mathcal{D}_{sl}) = \frac{0.391}{\sqrt{\delta}} = R_2(\mathcal{A}_{single}, \mathcal{D}_{univ})$.

Proof: We denote $e^\eta = \tau = 1 + \alpha\sqrt{\delta}$. We also can estimate $\eta \sim e^\eta - 1 = \alpha\sqrt{\delta}$, where $\alpha = \Omega(1)$ (if $\alpha \neq \Omega(1)$, then, similar to the finite horizon model, straight or looping adversary alone already achieves regret of $\omega(\frac{1}{\sqrt{\delta}})$). Without loss of generality we assume that $\ell = \beta \cdot \frac{1}{\sqrt{\delta}} = o(\frac{1}{\delta})$. Then $(1 - \delta)^\ell \sim 1$. We further analyze separately each part of (13). First, we have

$$\begin{aligned} \sum_{d=0}^{\ell-1} \frac{(1-\delta)^{d+1}}{\tau^d + 1} &\sim \sum_{d=0}^{\ell-1} \frac{1}{\tau^d + 1} \sim \int_0^\ell \frac{dx}{e^{\eta x} + 1} \\ &= \frac{1}{\eta} \ln \left(2 \frac{e^{\ell\eta}}{e^{\ell\eta} + 1} \right) \sim \frac{\ln(2)}{\alpha\sqrt{\delta}} - \frac{\ln(1 + e^{-\alpha\beta})}{\alpha\sqrt{\delta}}. \end{aligned} \quad (14)$$

The second part we can estimate as follows

$$\begin{aligned} (1-\delta)^{\ell+1} \sum_{k=0}^{\infty} \left[\frac{(1-\delta)^{2k}}{\tau^\ell + 1} - \frac{(1-\delta)^{2k+1}}{\tau^{\ell+1} + 1} \right] &\sim \left[\frac{1}{\tau^\ell + 1} - \frac{1-\delta}{\tau^{\ell+1} + 1} \right] \cdot \sum_{k=0}^{\infty} (1-\delta)^{2k} \\ &= \left[\frac{1}{\tau^\ell + 1} - \frac{1}{\tau^{\ell+1} + 1} + \frac{\delta}{\tau^{\ell+1} + 1} \right] \frac{1}{2\delta - \delta^2} \sim O(1) + \left[\frac{\tau^\ell(\tau - 1)}{(\tau^\ell + 1)(\tau^{\ell+1} + 1)} \right] \frac{1}{2\delta} \\ &\sim \frac{1}{2\delta} \cdot \frac{e^{\alpha\beta} \cdot \alpha\sqrt{\delta}}{(1 + e^{\alpha\beta})^2}. \end{aligned}$$

Combining these two estimates we get that (13) is asymptotically equal to

$$\frac{\ln(2)}{\alpha\sqrt{\delta}} - \frac{\ln(1 + e^{-\alpha\beta})}{\alpha\sqrt{\delta}} + \frac{1}{2\delta} \cdot \frac{e^{\alpha\beta} \cdot \alpha\sqrt{\delta}}{(1 + e^{\alpha\beta})^2} = \frac{\ln(2)}{\sqrt{\delta}\alpha} - \frac{\ln(1 + e^{-\alpha\beta})}{\sqrt{\delta}\alpha} + \frac{\alpha}{2\sqrt{\delta}} \frac{e^{-\alpha\beta}}{(1 + e^{-\alpha\beta})^2}. \quad (15)$$

With a new notation $\gamma = \frac{1}{1 + e^{-\alpha\beta}}$ the above expression becomes

$$\frac{\ln(2)}{\alpha\sqrt{\delta}} + \frac{\ln(\gamma)}{\alpha\sqrt{\delta}} + \frac{\alpha}{2\sqrt{\delta}} \gamma(1 - \gamma) = \frac{\ln(2\gamma)}{\alpha\sqrt{\delta}} + \frac{\alpha}{2\sqrt{\delta}} \gamma(1 - \gamma) = \frac{1}{\sqrt{\delta}} \cdot h(\alpha, \gamma).$$

We recall that the player first picks parameter $\alpha \in (0, \infty)$ for the algorithm and then the adversary decides on the optimal $\gamma(\alpha) = \frac{1}{1 + e^{-\alpha\beta}}$ which can be anything in $[\frac{1}{2}, 1]$. In other words we are looking for

$$\inf_{\alpha \in (0, \infty)} \max_{\gamma(\alpha) \in [\frac{1}{2}, 1]} h(\alpha, \gamma). \quad (16)$$

For each fixed α we conclude that $\gamma(\alpha)$ is either $\frac{1}{2}$, or 1, or such that $\frac{d}{d\gamma} h(\gamma) = \frac{1}{\alpha\gamma} + \frac{\alpha}{2} - \alpha\gamma = 0$. The latter expression gives us a quadratic equation on γ , which has only one positive root:

$$\gamma(\alpha) = \frac{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4}{\alpha^2}}}{2}.$$

We note that $h_\gamma(\alpha, \gamma) > 0$, when $\gamma = \frac{1}{2}$. Therefore, $h(\alpha, \gamma)$ does not attain its maximum at $\gamma = \frac{1}{2}$. For $\alpha < \sqrt{2}$ the above expression is greater than 1 and, hence, $\gamma(\alpha) = 1$. We further note that $h_\gamma(\alpha, \gamma) = \frac{1}{\alpha} - \frac{\alpha}{2} < 0$ at $\gamma = 1$. Thus $\gamma(\alpha) = \frac{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4}{\alpha^2}}}{2}$ is a unique maximum of $h(\alpha, \gamma)$ for any $\alpha > \sqrt{2}$.

Now we need to find optimal α in

$$\inf_{\alpha \in (0, \infty)} h(\alpha, \gamma(\alpha)).$$

When $\alpha \in [0, \frac{1}{2}]$ we have $\gamma(\alpha) = 1$ and $h(\alpha, \gamma) = \frac{\ln(2)}{\alpha}$, which attains its minimum of $\frac{\ln(2)}{\sqrt{2}}$ at $\alpha = \sqrt{2}$.

We know that the optimal $\alpha = \Theta(1)$. Thus $h(\alpha, \gamma(\alpha))$ attains its minimum for some finite α . For $\alpha \in [\sqrt{2}, \infty)$ this minimum could be attained either at $\alpha = \sqrt{2}$, or at such α that $\frac{d}{d\alpha} h(\alpha, \gamma(\alpha)) = 0$. Note that $\frac{d}{d\alpha} h(\alpha, \gamma(\alpha)) = h_\alpha(\alpha, \gamma(\alpha)) + \gamma'(\alpha) \cdot h_\gamma(\alpha, \gamma(\alpha))$, whereas $h_\gamma(\alpha, \gamma(\alpha)) = 0$. Thus we get $h_\alpha(\alpha, \gamma(\alpha)) = 0$. Now we can write the following system of equations:

$$\begin{cases} h_\gamma(\alpha, \gamma(\alpha)) = 0 \\ h_\alpha(\alpha, \gamma(\alpha)) = 0 \end{cases} \Leftrightarrow \begin{cases} -\frac{\ln(2\gamma)}{\alpha^2} + \frac{1}{2}\gamma(1-\gamma) = 0 \\ \frac{1}{\alpha\gamma} - \alpha\gamma + \frac{\alpha}{2} = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha^2 = \frac{2\ln(2\gamma)}{\gamma(1-\gamma)} \\ \alpha^2 = \frac{1}{\gamma(\gamma-0.5)} \end{cases}$$

Solving numerically this system of equations we find a unique solution: $\alpha = 2.200$ and $\gamma(\alpha) = 0.769$ with $h(\alpha, \gamma) = 0.391$. This number is smaller than $h(\sqrt{2}, 1) = \frac{\ln(2)}{\sqrt{2}}$. Therefore, $\alpha = 2.2$, $\gamma = 0.769$ is the optimal solution to (16). The resulting optimal regret of MWA is $\frac{0.391}{\sqrt{\delta}}$ with the optimal parameter $\eta \sim 2.2 \times \sqrt{\delta}$. ■

Remark 6 The regret $\frac{0.391}{\sqrt{\delta}}$ of $\mathcal{A}_{\text{single}}$ in the geometric horizon model is by 10.6% larger than the regret of $\frac{1}{2\sqrt{2\delta}}$ of the optimal algorithm ([10]) for $k = 2$.

B.2 Improved regret lower bound for k experts.

In this section, we derive new regret lower bound for $\mathcal{A}_{\text{single}}$ in the geometric horizon model. We show that for any number of experts k , as $\delta \rightarrow 0$, $\mathcal{A}_{\text{single}}$ cannot obtain a regret smaller than $\frac{1}{2} \sqrt{\frac{\ln(k)}{2\delta}}$. This is an improvement over the previously best known lower bound of $\sqrt{\frac{T \log_2(k)}{16}}$ in the finite horizon model and its corresponding implication (it's not even clear if there is any) in the geometric horizon model.

One obstacle in directly generalizing the adversary for $k = 2$ experts is that there is no immediate generalization of the looping phase after ℓ steps of straight-line phase have passed, when $k \geq 3$. On the other hand, we still manage to get a lower bound which is only factor 2 away from the upper bound by employing the looping and straight strategies separately.

Theorem 5 In geometric horizon, $R_k(\mathcal{A}_{\text{single}}, \{D_\ell, D_s\}) \geq \frac{1}{2} \sqrt{\frac{\ln k}{2\delta}} \geq \frac{1}{2} R_k(\mathcal{A}_{\text{single}}, \mathcal{D}_{\text{univ}})$.

Proof: We begin with the case of 2 experts and then generalize the proof. We show that the looping and straight-line primitives in combination yield the desired lower bound. In the description below, η refers to the parameter in the exponent, as described in the paragraph close to the beginning of this section. The main idea is to show that different regimes of η are rendered ineffective by different primitives.

Straight-line and looping primitives. A quick reminder of the straight-line and looping primitives:

1. The *straight-line primitive* picks an arbitrary expert and always keeps advancing that expert by 1 in each step. Thus the lag between the leading and all lagging experts keeps monotonically increasing by one in each step.

2. The *looping primitive* gives labels A and B to the two experts, and advances expert number 1 in odd-numbered steps and the other expert in even-numbered steps. In effect, the difference between these two experts loops between 0 and 1.

Remarkably, these two simple primitives generate regrets of $\frac{\ln(k)}{\eta}$ and $\frac{\eta}{8\delta}$ in geometric model and that match exactly with the standard upper bound analysis for MWA ($\mathcal{A}_{\text{single}}$). The only difference between the upper and lower bounds is that in the former, one gets the *sum* of these two regret terms, but in our lower bound, we get only the *max* of these two regret terms. Thus, our lower bound is exactly a factor 2 away from the known upper bound of $\sqrt{\frac{\ln(k)}{2\delta}}$.

For convenience, we write $e^\eta = \tau = 1 + \alpha\sqrt{\delta}$. The two primitives/adversaries together place strong bounds on what α should be: they imply that $\alpha = \Theta(1)$. We show this in 2 steps: first we show that $\alpha = O(1)$, and then show that $\alpha = \Theta(1)$.

1. The looping adversary forces α to be $O(1)$. The regret of MWA on a looping adversary is given by

$$\sum_{d=0}^{\infty} (1-\delta)^{2d+1} \left[\frac{1}{2} - \frac{1-\delta}{1+\tau} \right] = \frac{1-\delta}{1-(1-\delta)^2} \left[\frac{1}{2} - \frac{1-\delta}{1+\tau} \right] \sim \frac{1}{2\delta} \left[\frac{1}{2} - \frac{1-\delta}{1+\tau} \right].$$

Since MWA's regret upper bound in the geometric horizon model is $\Theta\left(\sqrt{\frac{1}{\delta}}\right)$, our lower bound on the regret $\frac{1}{2\delta} \left[\frac{1}{2} - \frac{1-\delta}{1+\tau} \right] = \frac{1}{2\delta} \left[\frac{\alpha\sqrt{\delta}+2\delta}{2(2+\alpha\sqrt{\delta})} \right]$ is at most $\Theta\left(\sqrt{\frac{1}{\delta}}\right)$. This implies that $\alpha = O(1)$.

2. We use straight-line adversary to show that $\alpha = \Theta(1)$. We argue that when $\alpha = o(1)$, the straight-line adversary will result in the regret of $\omega\left(\sqrt{\frac{1}{\delta}}\right)$. In fact, we don't even need to consider $\alpha = o(1)$. Just assume that α is a very small constant c independent of δ . Then $\tau = 1 + \alpha\sqrt{\delta} = 1 + c\sqrt{\delta}$. Consider running the straight-line adversary for just $\frac{1}{c\sqrt{\delta}}$ steps. For any $d = o\left(\frac{1}{\delta}\right)$, we have $(1-\delta)^{d+1} \sim 1$. Also, for all $d \leq \frac{1}{c\sqrt{\delta}}$, we have $\tau^d = (1 + c\sqrt{\delta})^d \leq e$. Thus the straight-line adversary running till $\frac{1}{c\sqrt{\delta}}$ has a regret of

$$\sum_{d=0}^{\frac{1}{c\sqrt{\delta}}} (1-\delta)^{d+1} \frac{1}{\tau^d + 1} \sim \sum_{d=0}^{\frac{1}{c\sqrt{\delta}}} \frac{1}{\tau^d + 1} \geq \sum_{d=0}^{\frac{1}{c\sqrt{\delta}}} \frac{1}{e+1} = \frac{1}{e+1} \left(1 + \frac{1}{c\sqrt{\delta}} \right).$$

As c gets smaller, the regret becomes very large, and hence there is a lower bound on c . This shows that $\alpha = \Theta(1)$.

Regret calculation for k experts. We are now ready to show the lower bound on the regret for arbitrary number k of experts. While one could do something more sophisticated for k experts, we consider just the looping and straight line adversaries similar to the case of $k = 2$ experts. The straight-line adversary for arbitrary k fixes on an arbitrary expert and advances just that expert by 1 in each step till the game dies. The looping adversary splits the set of k experts into two teams A and B of equal size (if k is odd, the sets are of size $(k-1)/2$ each and it won't matter for asymptotics), and all experts in team A are advanced in odd numbered rounds, and those in B are advanced in even numbered rounds. It is immediate to see that the looping adversary gets a regret of exactly what it obtained for the case of 2 experts, namely, $\frac{1}{2\delta} \left[\frac{1}{2} - \frac{1-\delta}{1+\tau} \right] \sim \frac{\alpha}{8\sqrt{\delta}}$ (the regret is only higher when k is odd). The straight line adversary generates a higher regret, and it amounts to $\frac{\ln(k)}{\alpha\sqrt{\delta}}$. To see this, note that with a lead of d , MWA follows the leading expert with probability

$\frac{\tau^d}{\tau^d + k - 1}$, and each of the $k - 1$ other experts with probability $\frac{1}{\tau^d + k - 1}$. This generates a regret of $\sum_{d=0}^{\infty} (1 - \delta)^{d+1} \frac{k-1}{\tau^d + k - 1}$. As in the case of 2 experts, we lower bound this regret by stopping the straight line adversary after ℓ steps where $\ell = \omega\left(\sqrt{\frac{1}{\delta}}\right)$, but $\ell = o\left(\frac{1}{\delta}\right)$. Thus, the straight-line adversary generates a regret of at least

$$\begin{aligned}
\sum_{d=0}^{\ell-1} (1 - \delta)^{d+1} \frac{k-1}{\tau^d + k - 1} &\sim \sum_{d=0}^{\ell-1} \frac{k-1}{e^{\eta d} + k - 1} \sim \int_0^{\ell} \frac{k-1}{e^{\eta x} + k - 1} dx \\
&= \frac{1}{\eta} \int_0^{\ell\eta} \left[1 - \frac{e^y}{e^y + k - 1} \right] dy = \frac{1}{\eta} [y - \ln(e^y + k - 1)]_{y=0}^{y=\ell\eta} \\
&= \frac{1}{\eta} \ln \left(k \frac{e^{\ell\eta}}{e^{\ell\eta} + k - 1} \right) \sim \frac{\ln(k)}{\eta}. \tag{17}
\end{aligned}$$

The looping adversary, as already mentioned, obtains a regret of

$$\frac{1}{2\delta} \left[\frac{1}{2} - \frac{1 - \delta}{1 + \tau} \right] = \frac{1}{2\delta} \left[\frac{\alpha\sqrt{\delta} + 2\delta}{2(2 + \alpha\sqrt{\delta})} \right] \sim \frac{1}{2\delta} \frac{\alpha\sqrt{\delta}}{4} = \frac{\alpha}{8\sqrt{\delta}}.$$

Thus we get a regret lower bound of $\max\left(\frac{\ln(k)}{\alpha\sqrt{\delta}}, \frac{\alpha}{8\sqrt{\delta}}\right)$, which is minimized when $\alpha = \sqrt{8\ln(k)}$ giving a lower bound of $\frac{1}{2}\sqrt{\frac{\ln(k)}{2\delta}}$. ■