# Resource Theory of Work and Heat


C. Sparaciari

*and*

J. Oppenheim

T. Fritz



ArXiv

**1607.01302**

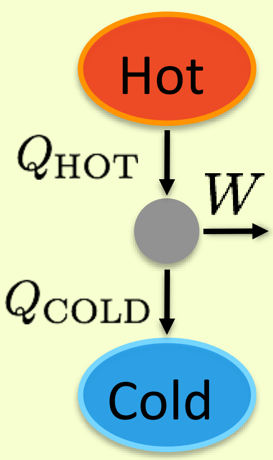


University College London - Dep. Physics and Astrophysics

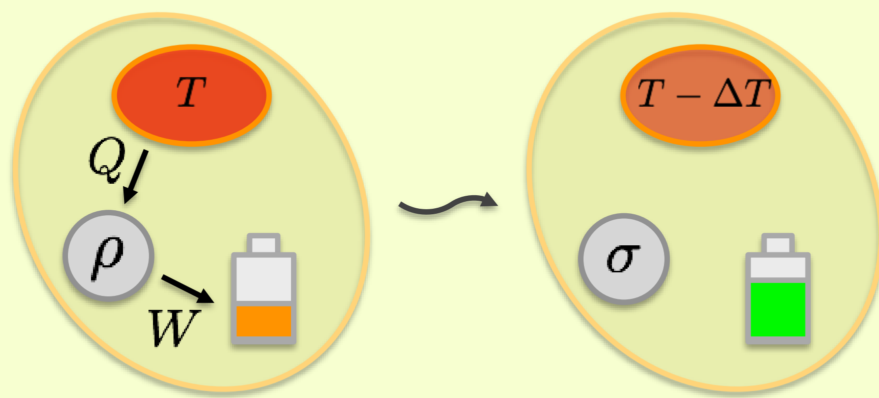
# Motivations

Work and heat
as resources


Finite size engines[1]




Work and heat for
finite size baths

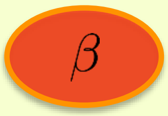




[1] H. Tajima and M. Hayashi, *arXiv:1405.6457* (2014)

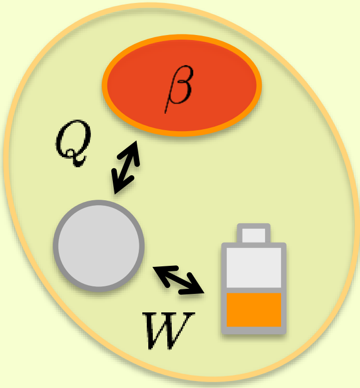
[2] D. Reeb and M. Wolf, *New J. Phys.* **16**, 103011 (2014)

# Work and heat



: thermal reservoir


: battery[3]




**Work** : ordered energy transfer between system and battery

- Work is the energy stored in the battery
- Classical case $W = \Delta F_\beta$, where $F_\beta = E - \beta^{-1} S$

**Heat** : disordered energy transfer between system and reservoir

- Heat is the energy exchanged with the bath
- Classical case $Q = \beta^{-1} \Delta S$


[3] M. Horodecki and J. Oppenheim, *Nat. Comm.* **4**, 2059 (2013).

# Resource theory

e.g. **Thermal Operations**[4-5]

- State space
  - All states describing the quantum system S
- Allowed operations
  - Energy-preserving unitaries
  - Addition of thermal states
  - Forgetting part of the system
- Free states
  - Thermal states at a given $\beta$

$$\tau_\beta = e^{-\beta H}/Z$$

[4] D. Janzing et al., *Int. J. Theor. Phys.* **39**, 2717–2753 (2000)
[5] F. G. S. L. Brandão et al., *Phys. Rev. Lett.* **111**, 250404 (2013).

# Goals

- Formulation of a theory without free thermal states
  - **Asymptotic equivalence** of quantum states
  - Microscopic to macroscopic description
  - State space as an energy-entropy diagram
- Definition of **work and heat** in the theory
  - Corrections for finite size bath
  - Limitations to **efficiency** of microscopic engines

# Index

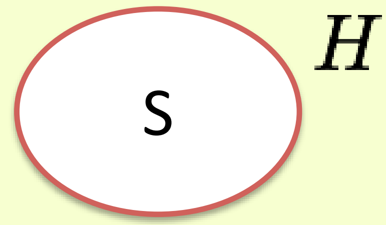
- Framework and asymptotic equivalence
- Energy-entropy diagram
- Work and heat with finite size bath
- Conclusions and outlooks

# Index

- Framework and asymptotic equivalence
- Energy-entropy diagram
- Work and heat with finite size bath
- Conclusions and outlooks

# Framework and allowed operations

- Closed *qudit* system
- Fixed Hamiltonian $H$




**OPERATIONS**

- Energy-preserving unitary operations $U$

$$[H,U]=0$$

- Tracing out of any subsystems

**Asymptotic Limit**

$$\rho \to \rho^{\otimes n}$$

$$H \to \sum_{i=1}^{n} H_i$$

Non-interacting systems

# Asymptotic equivalence

Consider a quantum system with Hamiltonian $H$, and two states $\rho$ and $\sigma$.

1. The states have same **energy** and **entropy**

$$S(\rho) = S(\sigma) \qquad\qquad \langle H \rangle_\rho = \langle H \rangle_\sigma$$

2. There exists:
   - energy-preserving unitary operator $U$
   - sub-linear ancilla $\eta$

$$\left\| \mathrm{Tr}_A \left[ U \left( \rho^{\otimes n} \otimes \eta \right) U^\dagger \right] - \sigma^{\otimes n} \right\|_1 \to 0 \quad \text{as} \quad n \to \infty$$

UCL

# Remarks on asymptotic equivalence

$$\left\| \mathrm{Tr}_A \left[ U \left( \rho^{\otimes n} \otimes \eta \right) U^\dagger \right] - \sigma^{\otimes n} \right\|_1 \to 0 \quad \text{as} \quad n \to \infty$$

- The **ancilla** has:
  i. sub-linear size: $\dim\left(\mathcal{H}_{\mathrm{anc}}\right) = d^{O(\sqrt{n \log n})}$
  ii. bounded Hamiltonian: $\|H_{\mathrm{anc}}\| \leq O(n^{\frac{2}{3}})$
- Equivalence classes $\Longrightarrow$ **energy-entropy** diagram
- Apply to different conserved quantities

# Composition of the ancillary system

1. Randomness Source : $\frac{\mathbb{I}}{d^{n_1}}$ , with $n_1 = O(\sqrt{n \log n})$

   Modify the **probability** distribution of $\rho^{\otimes n}$ into $\sigma^{\otimes n}$

2. Register : $|0\rangle^{\otimes n_2}$, with $n_2 = O(\sqrt{n \log n})$

   Make the operation which maps $\rho^{\otimes n}$ into $\sigma^{\otimes n}$ **reversible**

3. Energy/Coherence Storage :

   Make the unitary operation **energy-preserving** $[H, U] = 0$

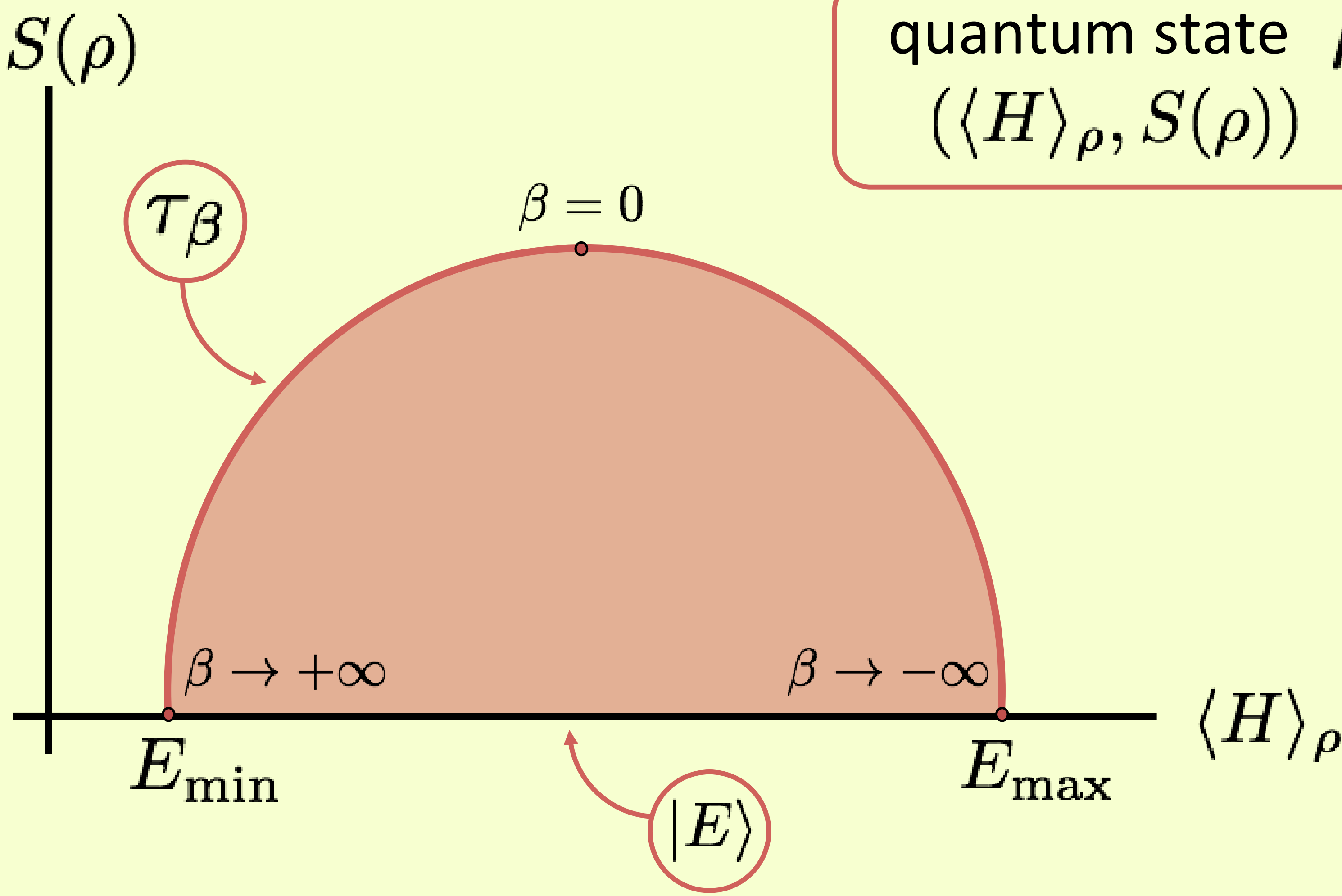
# Index

- Framework and asymptotic equivalence
- Energy-entropy diagram
- Work and heat with finite size bath
- Conclusions and outlooks

# The energy-entropy diagram

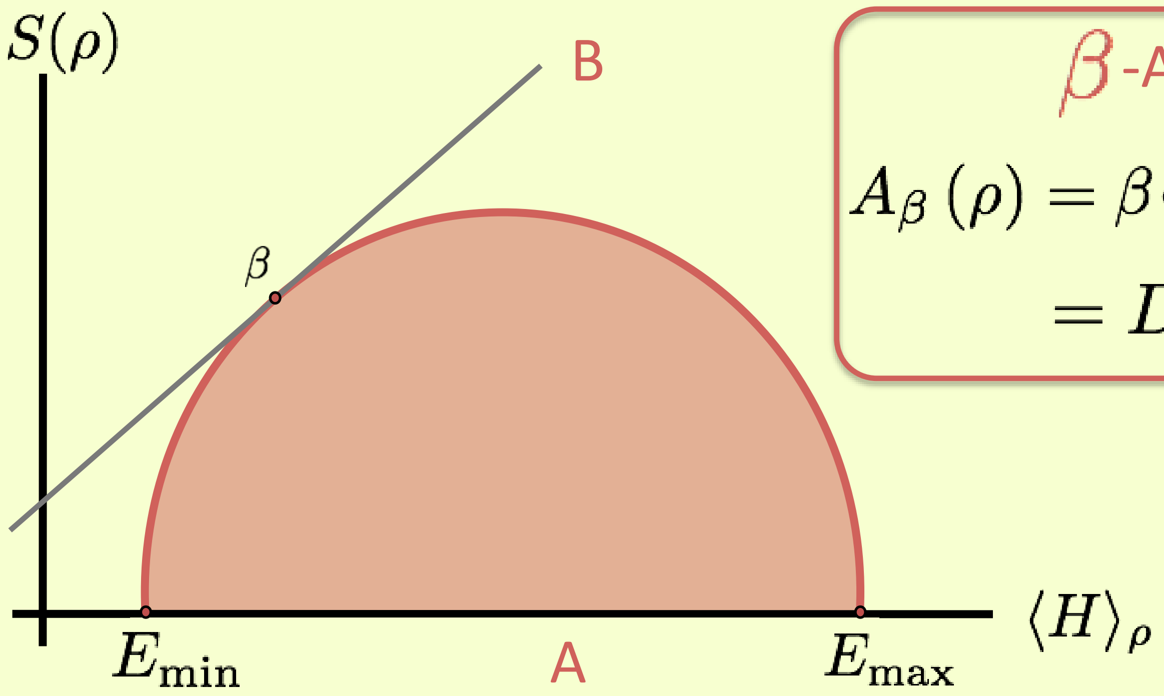
quantum state $\rho$

$(\langle H \rangle_\rho, S(\rho))$




[6] T. Fritz, *Mathematical Structures in Computer Science*, 1–89 (2015)

# Linear ineq. and energy-entropy diagram



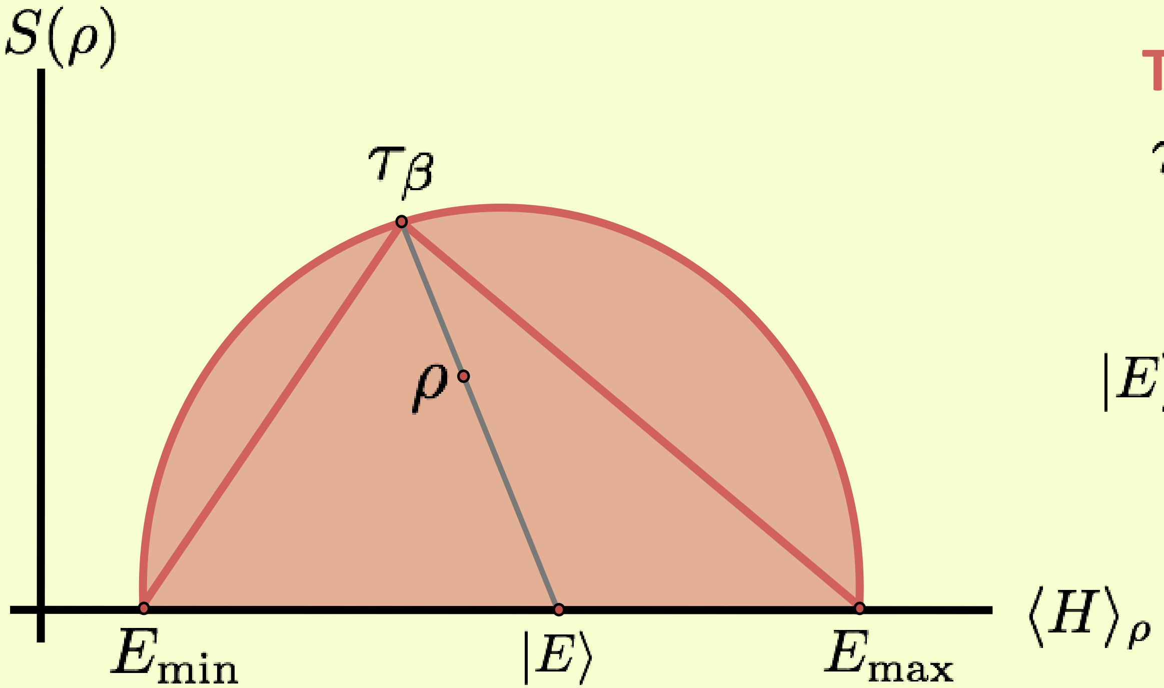

$\beta$-ATHERMALITY

$$A_\beta(\rho) = \beta \langle H \rangle_\rho - S(\rho) + \log Z_\beta$$
$$= D(\rho \| \tau_\beta)$$

A. Entropy inequality: $S(\rho) \geq 0$

B. $\beta$-athermality inequality: $A_\beta(\rho) \geq 0 \ , \quad \forall \beta$

# Excited and thermal state conversion




**Thermal State**

$$\tau_\beta = e^{-\beta H}/Z$$

**Pure State**

$$|E\rangle \; : \; \langle E|H|E\rangle = E$$

$$\rho^{\otimes n} \leftrightarrow \tau_\beta^{\otimes k} \otimes |E\rangle\langle E|^{\otimes n-k}$$

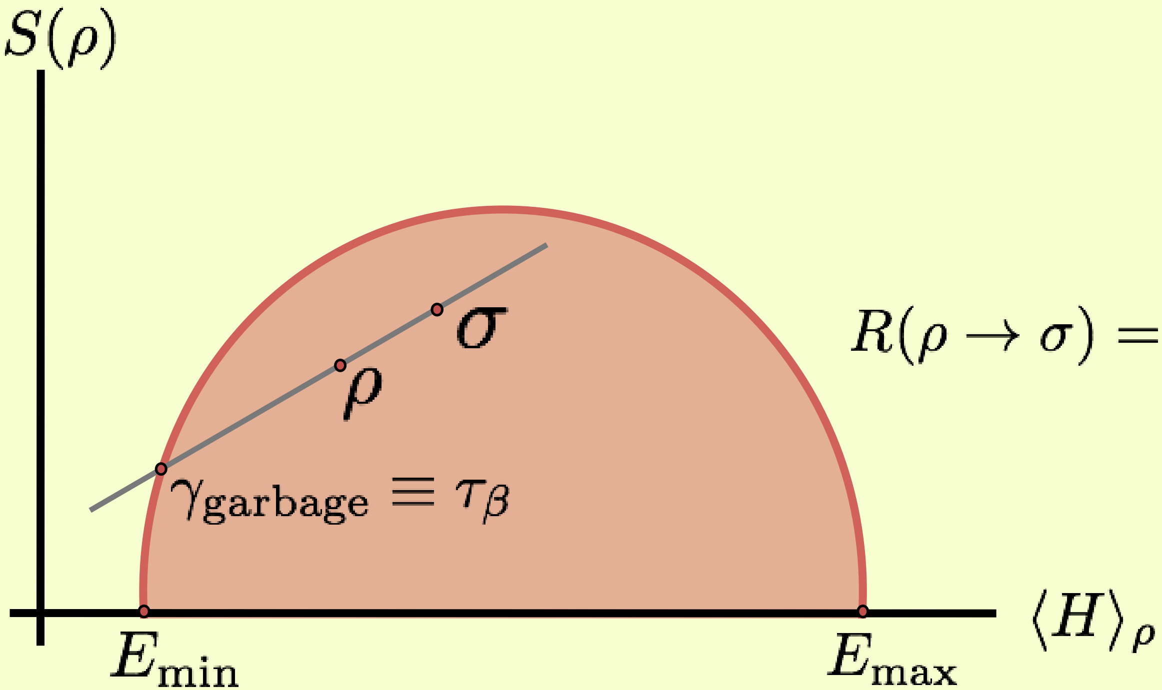
# Rate of conversion

Given $n$ copies of $\rho$, how many copies of $\sigma$ do we get?

$$\rho^{\otimes n} \leftrightarrow \sigma^{\otimes m} \otimes \gamma_{\text{garbage}}$$

**RATE** $\rho \rightarrow \sigma$

$$R(\rho \rightarrow \sigma) = \frac{m}{n}$$

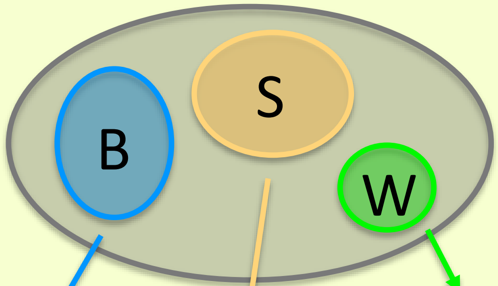



$$R(\rho \rightarrow \sigma) = \min \left\{ \frac{S(\rho)}{S(\sigma)}, \inf_{\beta \in \mathbb{R}} \frac{A_\beta(\rho)}{A_\beta(\sigma)} \right\}$$

# Index

- Framework and asymptotic equivalence
- Energy-entropy diagram
- Work and heat with finite size bath
- Conclusions and outlooks

# From closed systems to finite reservoirs




**Finite-sized thermal bath**

$$\frac{m}{n} = O(1)\ ;\ n, m \to \infty$$

$$\omega_{\text{in}}^{\otimes N} = \tau_{\beta_1}^{\otimes m} \otimes \rho^{\otimes n} \otimes \left|E_{\text{min}}\right\rangle\left\langle E_{\text{min}}\right|^{\otimes l}$$

$$\omega_{\text{out}}^{\otimes N} = \tau_{\beta_2}^{\otimes m} \otimes \sigma^{\otimes n} \otimes \left|E_{\text{max}}\right\rangle\left\langle E_{\text{max}}\right|^{\otimes l}$$

For the asymptotic equivalence theorem

$$\omega_{\text{in}}^{\otimes N} \leftrightarrow \omega_{\text{out}}^{\otimes N}\ ,\ N \to \infty \quad \Leftrightarrow \quad \begin{array}{l} S(\omega_{\text{in}}) = S(\omega_{\text{out}}) \\ E(\omega_{\text{in}}) = E(\omega_{\text{out}}) \end{array}$$

[7] P. Ehrenfest and T. Ehrenfest, *The Conceptual Foundations of the Statistical Approach in Mechanics* (2015).

# Work and heat with finite size bath

$$\tau_{\beta_1}^{\otimes m} \otimes \rho^{\otimes n} \otimes |E_{\min}\rangle\langle E_{\min}|^{\otimes l} \to \tau_{\beta_2}^{\otimes m} \otimes \sigma^{\otimes n} \otimes |E_{\max}\rangle\langle E_{\max}|^{\otimes l}$$

**WORK EXTRACTED**

$$W = \frac{l}{n}\left(E_{\max} - E_{\min}\right)$$

**HEAT PROVIDED**

$$Q = \frac{m}{n}\left(\langle H\rangle_{\tau_{\beta_1}} - \langle H\rangle_{\tau_{\beta_2}}\right)$$

We get:

$$W = F_{\beta_{\text{eff}}}(\rho) - F_{\beta_{\text{eff}}}(\sigma)$$

$$Q = \beta_{\text{eff}}^{-1}\left(S(\sigma) - S(\rho)\right)$$

where:

$$F_\beta(\rho) = \langle H\rangle_\rho - \beta^{-1}S(\rho)$$

and:

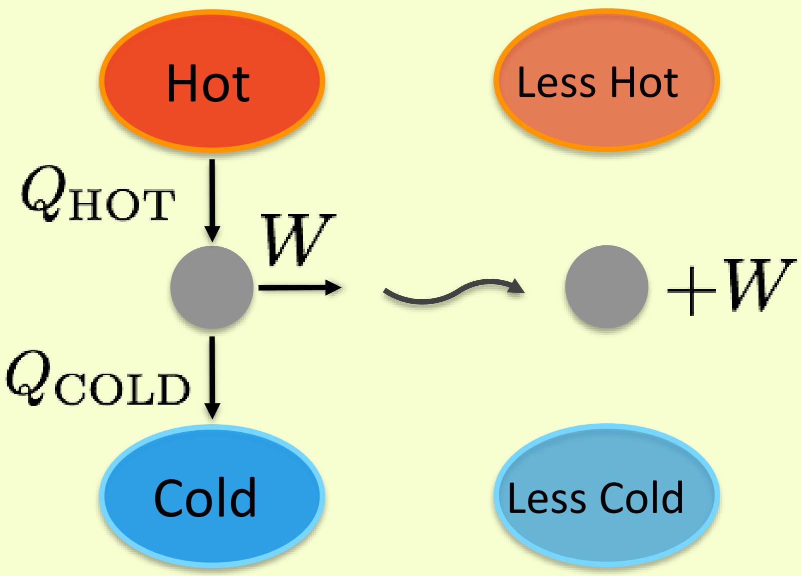
$$\beta_{\text{eff}} = \frac{S(\tau_{\beta_1}) - S(\tau_{\beta_2})}{\langle H\rangle_{\tau_{\beta_1}} - \langle H\rangle_{\tau_{\beta_2}}}$$

# Heat engines and finite thermal reservoirs

RESERVOIR : $\tau_{\beta_1} \rightarrow \tau_{\beta_{\text{HOT}}}$ $\tau_{\beta_2} \rightarrow \tau_{\beta_{\text{LESS HOT}}}$

SYSTEM : $\rho \rightarrow \tau_{\beta_{\text{COLD}}}$ $\sigma \rightarrow \tau_{\beta_{\text{LESS COLD}}}$

$$\tau_{\beta_{\text{HOT}}}^{\otimes m} \otimes \tau_{\beta_{\text{COLD}}}^{\otimes n} \otimes |E_{\min}\rangle\langle E_{\min}|^{\otimes l} \leftrightarrow \tau_{\beta_{\text{LESS HOT}}}^{\otimes m} \otimes \tau_{\beta_{\text{LESS COLD}}}^{\otimes n} \otimes |E_{\max}\rangle\langle E_{\max}|^{\otimes l}$$




## Efficiency

$$\eta^{\text{HE}} = \frac{W}{Q_{\text{HOT}}}$$

$$\eta^{\text{HE}} < 1 - \frac{T_{\text{COLD}}}{T_{\text{HOT}}} = \eta^{\text{HE}}_{\text{Carnot}}$$

# Index

- Framework and asymptotic equivalence
- Energy-entropy diagram
- Work and heat with finite size bath
- Conclusions and outlooks

# Conclusions

- Introduced a theory **without infinite thermal bath**
  - Classification of states in terms of $\langle H \rangle_\rho$ and $S(\rho)$
  - Resource space as **energy-entropy diagram**

- Thermodynamics with **finite size thermal bath**
  - Work and heat exchanged for $\rho \longrightarrow \sigma$:

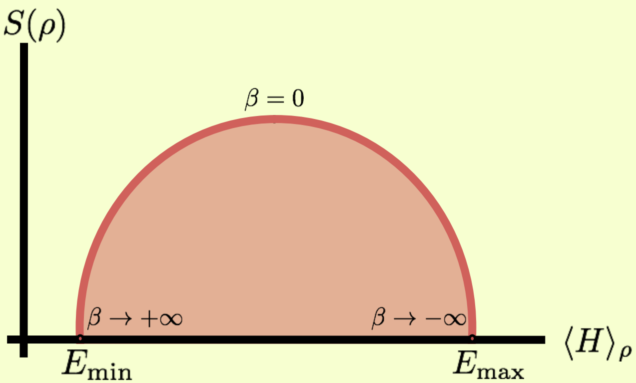
$$W = F_{\beta_{\text{eff}}}(\rho) - F_{\beta_{\text{eff}}}(\sigma) \qquad Q = \beta_{\text{eff}}^{-1}\left(S(\sigma) - S(\rho)\right)$$

  - **Efficiency** of heat engines is Sub-Carnot

# Outlook

- Improve our idealized theory
  - Many copies limit, non-interacting particles
  - Finite-grained operations are allowed[8]

- Using the energy-entropy diagram to describe
  - work and heat
  - efficiency of engines




[8] C. Perry et al., *arXiv:1511.06553* (2015).

FIN

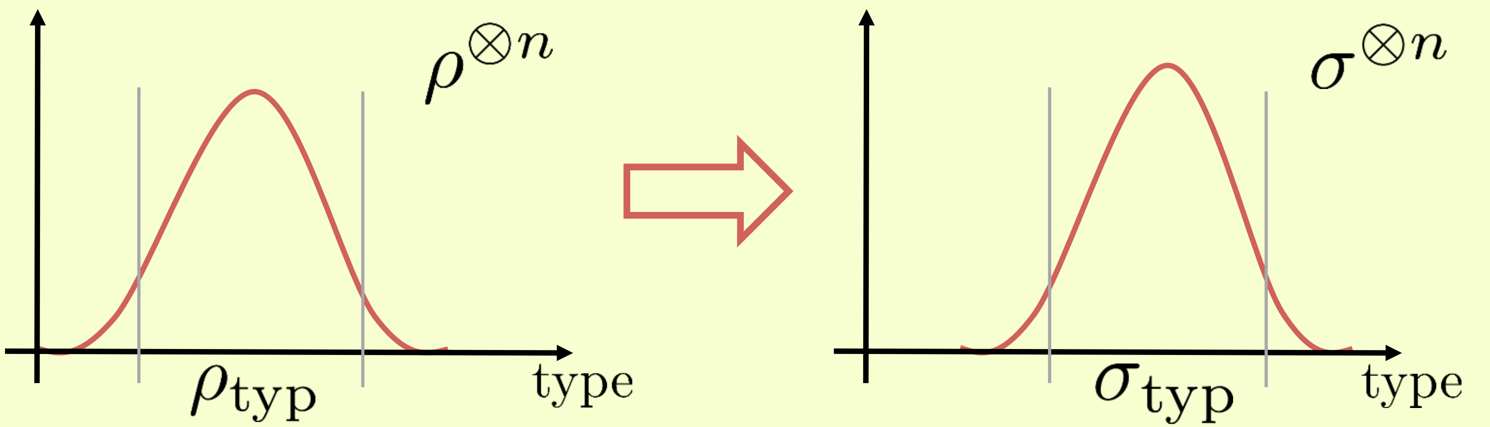
## Sketch of the proof asymptotic equivalence

- If $\rho^{\otimes n} \rightarrow \sigma^{\otimes n}$ then $S(\rho) = S(\sigma)$ and $\langle H \rangle_\rho = \langle H \rangle_\sigma$

  Follow from **asymptotic continuity** of $S$ and $\langle H \rangle$

- If $S(\rho) = S(\sigma)$ and $\langle H \rangle_\rho = \langle H \rangle_\sigma$ then $\rho^{\otimes n} \rightarrow \sigma^{\otimes n}$

  Protocol based on the **central limit theorem**



# Work and heat: classical case

$$\tau_{\beta_1}^{\otimes m} \otimes \rho^{\otimes n} \otimes |E_{\min}\rangle \langle E_{\min}|^{\otimes l} \rightarrow \tau_{\beta_2}^{\otimes m} \otimes \sigma^{\otimes n} \otimes |E_{\max}\rangle \langle E_{\max}|^{\otimes l}$$

The **bath temperature** changes infinitesimally:

$$\beta_2 = \beta_1 + \varepsilon$$

**Work extracted**

$$W = F_{\beta_1}(\rho) - F_{\beta_1}(\sigma) + O(\varepsilon)$$

**Heat provided**

$$Q = \frac{1}{\beta_1}\left(S(\sigma) - S(\rho)\right) + O(\varepsilon)$$

**Bath size**

$$\frac{m}{n} \propto \left| \frac{S(\sigma) - S(\rho)}{\beta \langle \Delta^2 H \rangle_{\tau_\beta}} \right| \frac{1}{\varepsilon}$$

# Work and heat: an example

## Hamiltonian

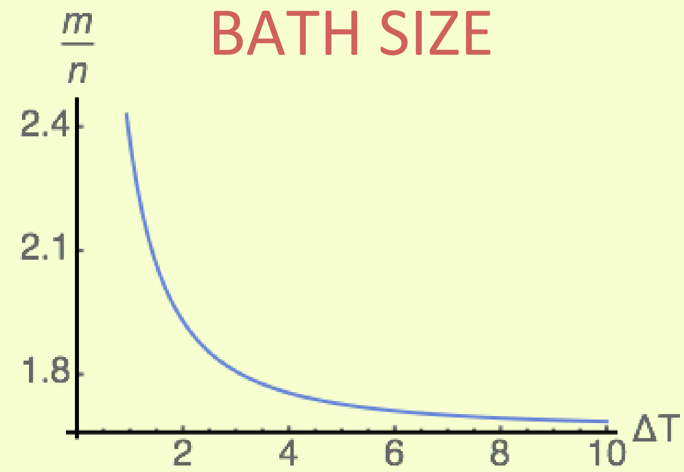
$$H = 1\,|1\rangle\langle 1| + 2\,|2\rangle\langle 2|$$

## Quantum states

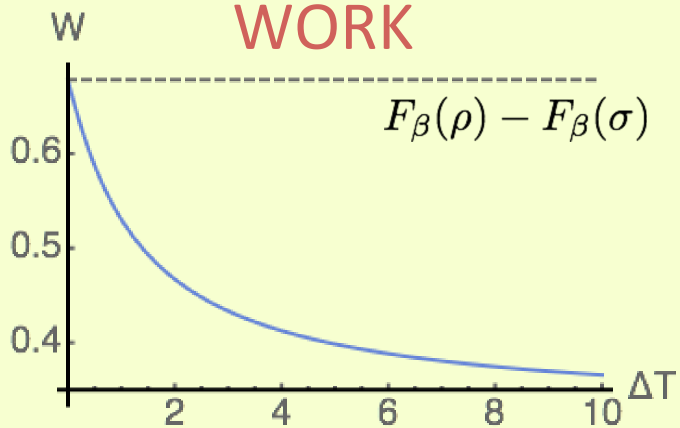
$$\rho = 0.2\,|0\rangle\langle 0| + 0.4\,|1\rangle\langle 1| + 0.4\,|2\rangle\langle 2|$$

$$\sigma = 0.7\,|0\rangle\langle 0| + 0.3\,|1\rangle\langle 1|$$

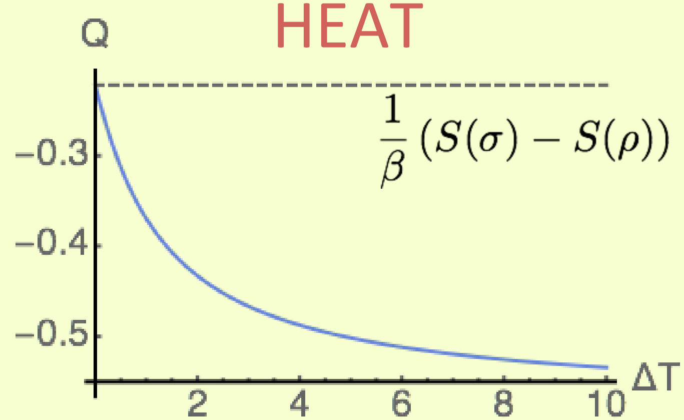
BATH SIZE




WORK



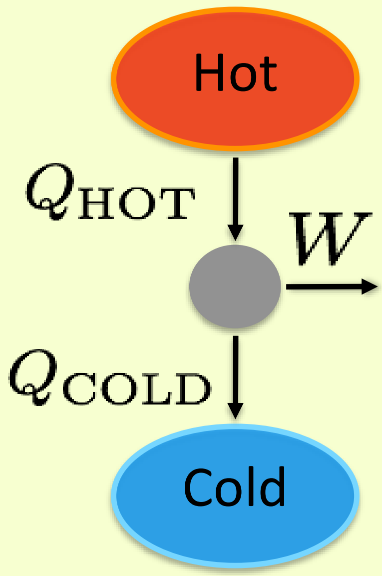

HEAT



# Index

- Framework and asymptotic equivalence
- Energy-entropy diagram
- Work and heat with finite size bath
- Limitation to finite size engines
- Conclusions and outlooks

# Efficiency of finite size engines




$$\eta^{\text{HE}} = 1 - \beta_{\text{eff}}^{\text{HOT}} / \beta_{\text{eff}}^{\text{COLD}}$$

$$\eta^{\text{HE}} < 1 - \frac{T_{\text{COLD}}}{T_{\text{HOT}}} = \eta_{\text{Carnot}}^{\text{HE}}$$

Efficiency is lower than Carnot

Where:

$$\beta_{\text{eff}}^{\text{X}} = \frac{S(\tau_{\beta_{\text{X}}}) - S(\tau_{\beta_{\text{LESS X}}})}{\langle H \rangle_{\tau_{\beta_{\text{X}}}} - \langle H \rangle_{\tau_{\beta_{\text{LESS X}}}}} \qquad \text{X} = \text{HOT, COLD}$$

# Quantum Information Group in UCL

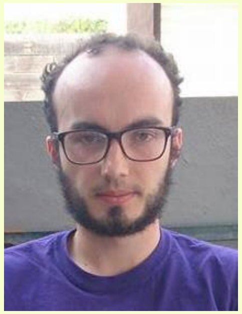

← Jonathan Oppenheim


← Lluis Masanes


← Mischa Woods


← Joan Camps


← Alvaro Alhambra


← Thomas Galley


← Jon Richens