Asymptotic entanglement manipulation under PPT operations: new SDP bounds and irreversibility

Xin Wang

UTS: Centre for Quantum Software and Information

Joint work with **Runyao Duan** (UTS:QSI)

Background

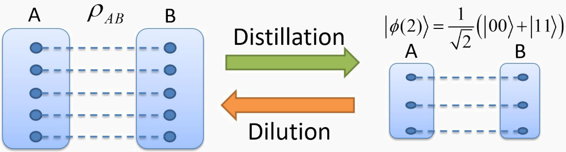
- Entanglement
  - Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$
  - Non-entangled (separable) state: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$

Background

- Entanglement
  - Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$
  - Non-entangled (separable) state: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$
- Entanglement theory studies the detection, quantification, manipulation and applications of entanglement.

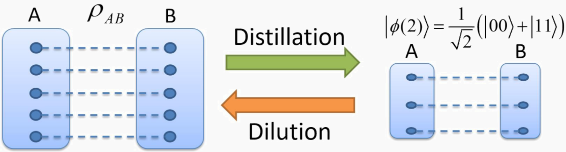
Background

- Entanglement
  - Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$
  - Non-entangled (separable) state: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$
- Entanglement theory studies the detection, quantification, manipulation and applications of entanglement.
- Two fundamental processes in entanglement manipulations
  - **Entanglement distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, 1999, 2001): To extract standard 2⊗2 maximally entangled states (EPR pairs) from a given state ρ by LOCC



Background

- Entanglement
  - Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$
  - Non-entangled (separable) state: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$
- Entanglement theory studies the detection, quantification, manipulation and applications of entanglement.
- Two fundamental processes in entanglement manipulations
  - **Entanglement distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, 1999, 2001): To extract standard 2⊗2 maximally entangled states (EPR pairs) from a given state ρ by LOCC



  - **Entanglement dilution**: To prepare a given state ρ with the standard EPR pairs by LOCC

Distillable entanglement and entanglement cost

- **Distillable entanglement**: The optimal (maximal) number of EPR pairs we can extract from ρ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.

Distillable entanglement and entanglement cost

- **Distillable entanglement**: The optimal (maximal) number of EPR pairs we can extract from ρ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.

- **Entanglement cost**: The optimal (minimal) number of EPR pairs we need to prepare ρ in an asymptotic setting,

$$E_C(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{LOCC}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}.$$

It is equal to the regurlized entanglement of formation (Hayden, Horodecki, Terhal 2001).

Distillable entanglement and entanglement cost

- **Distillable entanglement**: The optimal (maximal) number of EPR pairs we can extract from ρ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.

- **Entanglement cost**: The optimal (minimal) number of EPR pairs we need to prepare ρ in an asymptotic setting,

$$E_C(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{LOCC}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}.$$

It is equal to the regurlized entanglement of formation (Hayden, Horodecki, Terhal 2001).

- It is natural to ask whether $E_C \stackrel{?}{=} E_D$.

Entanglement manipulations and irreversibility

- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\operatorname{Tr}_B |\psi\rangle\langle\psi|).$$

Entanglement manipulations and irreversibility

- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B\,|\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac 2001).

Entanglement manipulations and irreversibility

- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B\,|\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac 2001).
  - In particular, $0 = E_D < E_C$ for any bound entangled states (Yang, Horodecki, Horodecki, Synak-Radtke 2005).

Entanglement manipulations and irreversibility

- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B\,|\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac 2001).
  - In particular, $0 = E_D < E_C$ for any bound entangled states (Yang, Horodecki, Horodecki, Synak-Radtke 2005).
- Enlarge the set of operations?

Entanglement manipulations and irreversibility

- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B\, |\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac 2001).
  - In particular, $0 = E_D < E_C$ for any bound entangled states (Yang, Horodecki, Horodecki, Synak-Radtke 2005).
- Enlarge the set of operations?
- One candidate is the set of **PPT operations** (quantum operations completely preserving positivity of partial transpose). Note that $LOCC \subsetneq SEP \subsetneq PPT$.

Entanglement manipulations under PPT operations

- PPT distillable entanglement (Rains 1999, 2001)

$$E_{D,PPT}(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

Entanglement manipulations under PPT operations

- PPT distillable entanglement (Rains 1999, 2001)

$$E_{D,PPT}(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

PPT entanglement cost (Audenaert, Plenio, Eisert 2003)

$$E_{C,PPT}(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{PPT}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}$$

Entanglement manipulations under PPT operations

- PPT distillable entanglement (Rains 1999, 2001)

$$E_{D,PPT}(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

PPT entanglement cost (Audenaert, Plenio, Eisert 2003)

$$E_{C,PPT}(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{PPT}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}$$

Clearly,

$$E_D \leq E_{D,PPT} \leq E_{C,PPT} \leq E_C$$

Reversibility under PPT operations?

- The class of antisymmetric states is an example of reversibility under PPT operations (Audenaert, Plenio, Eisert 2003).

Reversibility under PPT operations?

- The class of antisymmetric states is an example of reversibility under PPT operations (Audenaert, Plenio, Eisert 2003).
- Any state with a nonpositive partial transpose is distillable under PPT operations (Eggeling, Vollbrecht, Werner, Wolf 2001).

Reversibility under PPT operations?

- The class of antisymmetric states is an example of reversibility under PPT operations (Audenaert, Plenio, Eisert 2003).
- Any state with a nonpositive partial transpose is distillable under PPT operations (Eggeling, Vollbrecht, Werner, Wolf 2001).
- **An old open problem** (Audenaert, Plenio, Eisert 2003):

$$E_{D,PPT}(\rho) = E_{C,PPT}(\rho)?$$

(The 20^{th} problem listed at the website of Werner's group.)

Reversibility under PPT operations?

- The class of antisymmetric states is an example of reversibility under PPT operations (Audenaert, Plenio, Eisert 2003).
- Any state with a nonpositive partial transpose is distillable under PPT operations (Eggeling, Vollbrecht, Werner, Wolf 2001).
- **An old open problem** (Audenaert, Plenio, Eisert 2003):

$$E_{D,PPT}(\rho) = E_{C,PPT}(\rho)?$$

(The 20^{th} problem listed at the website of Werner's group.)

- (Brandão and Plenio 2008) Entanglement can be reversibly interconverted under asymptotically non-entangling operations.

Main question and outline

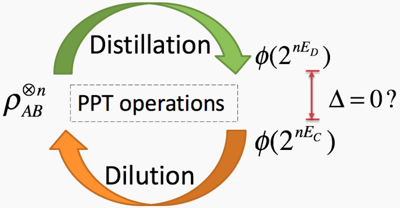
This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?

Main question and outline

This talk is about

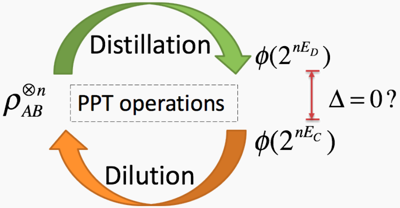
- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



Main question and outline

This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



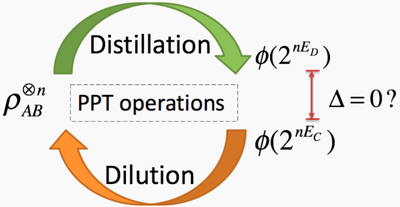
We will show

- Improved upper bounds for $E_{D,PPT}$

Main question and outline

This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



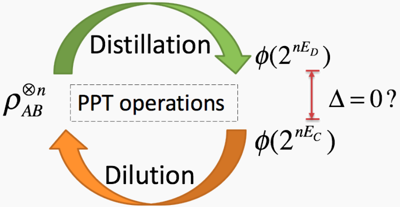
We will show

- Improved upper bounds for $E_{D,PPT}$
- Efficiently computable lower bound for $E_{C,PPT}$

Main question and outline

This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



We will show

- Improved upper bounds for $E_{D,PPT}$
- Efficiently computable lower bound for $E_{C,PPT}$
- The irreversibility under PPT operations:

$$\exists \rho, \text{ s.t. } E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$

An Upper bound of E_D: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, or PPT) is formidable. Only known for very limited cases.

An Upper bound of E_D: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, or PPT) is formidable. Only known for very limited cases.

- **Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005):

$$E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_B}\|_1,$$

where T_B means the partial transpose over the system B and $\|\cdot\|_1$ is the trace norm.

An Upper bound of E_D: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, or PPT) is formidable. Only known for very limited cases.

- **Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005):

$$E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_B}\|_1,$$

where T_B means the partial transpose over the system B and $\|\cdot\|_1$ is the trace norm.

  - **Negativity** $N(\rho_{AB}) = (\|\rho_{AB}^{T_B}\|_1 - 1)/2$ (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998)
  - (Rains, 2001; Vidal and Werner 2002):

$$E_D(\rho_{AB}) \leq E_{D,PPT}(\rho_{AB}) \leq E_N(\rho_{AB}).$$

  - E_N has many nice properties (see later).

A better SDP upper bound of E_D

- Primal SDP:

$$E_W(\rho) = \max \ \log_2 \operatorname{Tr} \rho R, \ \text{s.t.} \ \ |R^{T_B}| \le \mathbb{1}, R \ge 0. \tag{1}$$

- Dual SDP:

$$E_W(\rho) = \min \log_2 \|X^{T_B}\|_1, \ \ \text{s.t.} \ \ X \ge \rho. \tag{2}$$

A better SDP upper bound of E_D

- Primal SDP:

$$E_W(\rho) = \max \ \log_2 \mathrm{Tr}\rho R, \ \text{s.t.} \ \ |R^{T_B}| \leq \mathbb{1}, R \geq 0. \tag{1}$$

- Dual SDP:

$$E_W(\rho) = \min \log_2 \|X^{T_B}\|_1, \ \ \text{s.t.} \ \ X \geq \rho. \tag{2}$$

- Properties of E_W:
  i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.

A better SDP upper bound of E_D

- Primal SDP:

$$E_W(\rho) = \max \ \log_2 \mathrm{Tr} \rho R, \ \text{s.t.} \ \ |R^{T_B}| \le \mathbb{1}, R \ge 0. \tag{1}$$

- Dual SDP:

$$E_W(\rho) = \min \log_2 \|X^{T_B}\|_1, \ \ \text{s.t.} \ \ X \ge \rho. \tag{2}$$

- Properties of E_W:
  - i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.
  - ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \le E_W(\rho).$$

A better SDP upper bound of E_D

- Primal SDP:

$$E_W(\rho) = \max \ \log_2 \mathrm{Tr}\rho R, \ \text{s.t.} \ \ |R^{T_B}| \le \mathbb{1}, R \ge 0. \quad (1)$$

- Dual SDP:

$$E_W(\rho) = \min \log_2 \|X^{T_B}\|_1, \ \ \text{s.t.} \ \ X \ge \rho. \quad (2)$$

- Properties of E_W:
  - i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.
  - ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \le E_W(\rho).$$

  - iii) Detecting genuine PPT distillable entanglement: $E_W(\rho) > 0$ iff $E_D(\rho) > 0$, i.e., ρ is PPT distillable.
  - iv) **Non-increasing in average under PPT (LOCC) operations**

A better SDP upper bound of E_D

- Primal SDP:

$$E_W(\rho) = \max \ \log_2 \mathrm{Tr}\rho R, \ \text{s.t.} \ \ |R^{T_B}| \leq \mathbb{1}, R \geq 0. \tag{1}$$

- Dual SDP:

$$E_W(\rho) = \min \log_2 \|X^{T_B}\|_1, \ \ \text{s.t.} \ \ X \geq \rho. \tag{2}$$

- Properties of E_W:
  - i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.
  - ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \leq E_W(\rho).$$

  - iii) Detecting genuine PPT distillable entanglement: $E_W(\rho) > 0$ iff $E_D(\rho) > 0$, i.e., ρ is PPT distillable.
  - iv) **Non-increasing in average under PPT (LOCC) operations**
  - v) **Improved over logarithmic negativity**: $E_W(\rho) \leq E_N(\rho)$ and the inequality is strict in general.

A better SDP upper bound of E_D

- Primal SDP:

$$E_W(\rho) = \max \ \log_2 \mathrm{Tr}\rho R, \ \text{s.t.} \ \ |R^{T_B}| \le \mathbb{1}, R \ge 0. \qquad (1)$$

- Dual SDP:

$$E_W(\rho) = \min \log_2 \|X^{T_B}\|_1, \ \ \text{s.t.} \ \ X \ge \rho. \qquad (2)$$

- Properties of E_W:
  - i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.
  - ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \le E_W(\rho).$$

  - iii) Detecting genuine PPT distillable entanglement: $E_W(\rho) > 0$ iff $E_D(\rho) > 0$, i.e., ρ is PPT distillable.
  - iv) **Non-increasing in average under PPT (LOCC) operations**
  - v) **Improved over logarithmic negativity**: $E_W(\rho) \le E_N(\rho)$ and the inequality is strict in general.
- E_N has all above properties except v)!

Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho \log \rho - \rho \log \sigma)$
- Relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997) with respect to PPT states

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \quad \text{s.t.} \quad \sigma, \sigma^{T_B} \geq 0, \operatorname{Tr} \sigma = 1.$$

Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho\log\rho - \rho\log\sigma)$
- Relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997) with respect to PPT states

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \quad \text{s.t.} \quad \sigma, \sigma^{T_B} \geq 0, \operatorname{Tr}\sigma = 1.$$

- Asymptotic relative entropy of entanglement w.r.t. PPT states

$$E^{\infty}_{R,PPT}(\rho) = \lim_{n\to\infty} \frac{1}{n} E_{R,PPT}(\rho^{\otimes n}).$$

Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho \log \rho - \rho \log \sigma)$
- Relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997) with respect to PPT states

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma, \sigma^{T_B} \geq 0, \operatorname{Tr}\sigma = 1.$$

- Asymptotic relative entropy of entanglement w.r.t. PPT states

$$E_{R,PPT}^{\infty}(\rho) = \lim_{n\to\infty} \frac{1}{n} E_{R,PPT}(\rho^{\otimes n}).$$

- (Rains 2001) Rains' bound is the best known upper bound on the PPT distillable entanglement, i.e., $E_{D,PPT}(\rho) \leq R(\rho)$.
- **Rains' bound** (Rains 2001; Audenaert, De Moor, Vollbrecht, Werner'02)

$$R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \operatorname{Tr}|\sigma^{T_B}| \leq 1,$$

Rains bound is not additive

- **A conjecture** (Audenaert, De Moor, Vollbrecht, Werner 2002): Rains' bound is always additive;

Rains bound is not additive

- **A conjecture** (Audenaert, De Moor, Vollbrecht, Werner 2002): Rains' bound is always additive;
- **An open problem** (Plenio and Virmani 2007):

$$E_{R,PPT}^{\infty}(\rho) = R(\rho)?$$

Rains bound is not additive

- **A conjecture** (Audenaert, De Moor, Vollbrecht, Werner 2002): Rains' bound is always additive;
- **An open problem** (Plenio and Virmani 2007):

$$E_{R,PPT}^{\infty}(\rho) = R(\rho)?$$

- Evidence: Rains' bound equals to $E_{R,PPT}^{\infty}$ for Werner states (Audenaert, Eisert, Jane, Plenio, Virmani, De Moor 2001) and orthogonally invariant states (Audenaert, et al. 2002).

Rains bound is not additive

- **A conjecture** (Audenaert, De Moor, Vollbrecht, Werner 2002): Rains' bound is always additive;
- **An open problem** (Plenio and Virmani 2007):

$$E^{\infty}_{R,PPT}(\rho) = R(\rho)?$$

- Evidence: Rains' bound equals to $E^{\infty}_{R,PPT}$ for Werner states (Audenaert, Eisert, Jane, Plenio, Virmani, De Moor 2001) and orthogonally invariant states (Audenaert, et al. 2002).

Theorem

There exists a two-qubit state ρ such that

$$R(\rho^{\otimes 2}) < 2R(\rho).$$

Meanwhile,

$$E^{\infty}_{R,PPT}(\rho) < R(\rho).$$

Rains' bound is not additive: Proof ideas

i) Construct a $2 \otimes 2$ state ρ so that we can explicitly find a PPT state σ such that

$$R(\rho) = E_{R,PPT}(\rho) = S(\rho\|\sigma)$$

via a technique in (Miranowicz, Ishizaka'08, $R = E_{R,PPT}$ for any $2 \otimes 2$ state; see also Gour, Friedland'11 and Girard+'14.)

Rains' bound is not additive: Proof ideas

i) Construct a $2 \otimes 2$ state ρ so that we can explicitly find a PPT state σ such that

$$R(\rho) = E_{R,PPT}(\rho) = S(\rho \| \sigma)$$

via a technique in (Miranowicz, Ishizaka'08, $R = E_{R,PPT}$ for any $2 \otimes 2$ state; see also Gour, Friedland'11 and Girard+'14.)

ii) Finding a PPT state τ via an algorithm developed in (Girard, Zinchenko, Friedland, Gour'15). This gives an upper bound on $E_{R,PPT}(\rho^{\otimes 2})$, i.e.,

$$E_{R,PPT}(\rho^{\otimes 2}) \leq S(\rho^{\otimes 2} \| \tau).$$

Rains' bound is not additive: Proof ideas

i) Construct a $2 \otimes 2$ state ρ so that we can explicitly find a PPT state σ such that

$$R(\rho) = E_{R,PPT}(\rho) = S(\rho\|\sigma)$$

via a technique in (Miranowicz, Ishizaka'08, $R = E_{R,PPT}$ for any $2 \otimes 2$ state; see also Gour, Friedland'11 and Girard+'14.)

ii) Finding a PPT state τ via an algorithm developed in (Girard, Zinchenko, Friedland, Gour'15). This gives an upper bound on $E_{R,PPT}(\rho^{\otimes 2})$, i.e.,

$$E_{R,PPT}(\rho^{\otimes 2}) \leq S(\rho^{\otimes 2}\|\tau).$$

iii) Compare $S(\rho^{\otimes 2}\|\tau)$ and $2E_{R,PPT}(\rho)$, achieve the goal by showing

$$R(\rho^{\otimes 2}) \leq E_{R,PPT}(\rho^{\otimes 2}) \leq S(\rho^{\otimes 2}\|\tau) < 2S(\rho\|\sigma) = 2R(\rho).$$

and $E_{R,PPT}^{\infty}(\rho) \leq E_{R,PPT}(\rho_r^{\otimes 2})/2 < R(\rho)$.

Rains' bound is not additive: Proof ideas

i) Construct a $2 \otimes 2$ state ρ so that we can explicitly find a PPT state σ such that

$$R(\rho) = E_{R,PPT}(\rho) = S(\rho\|\sigma)$$

via a technique in (Miranowicz, Ishizaka'08, $R = E_{R,PPT}$ for any $2 \otimes 2$ state; see also Gour, Friedland'11 and Girard+'14.)

ii) Finding a PPT state τ via an algorithm developed in (Girard, Zinchenko, Friedland, Gour'15). This gives an upper bound on $E_{R,PPT}(\rho^{\otimes 2})$, i.e.,

$$E_{R,PPT}(\rho^{\otimes 2}) \leq S(\rho^{\otimes 2}\|\tau).$$

iii) Compare $S(\rho^{\otimes 2}\|\tau)$ and $2E_{R,PPT}(\rho)$, achieve the goal by showing

$$R(\rho^{\otimes 2}) \leq E_{R,PPT}(\rho^{\otimes 2}) \leq S(\rho^{\otimes 2}\|\tau) < 2S(\rho\|\sigma) = 2R(\rho).$$

and $E^{\infty}_{R,PPT}(\rho) \leq E_{R,PPT}(\rho_r^{\otimes 2})/2 < R(\rho)$.

iv) An example of semi-analytical and semi-numerical proof.

Rains' bound is not additive: Proof ideas (cont.)

We construct ρ_r and σ_r such that $R(\rho_r) = E_{R,PPT}(\rho_r) = S(\rho_r||\sigma_r)$:

$$\rho_r = \frac{1}{8}|00\rangle\langle 00| + x|01\rangle\langle 01| + \frac{7-8x}{8}|10\rangle\langle 10| + \frac{32r^2-(6+32x)r+10x+1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|)$$

$$\sigma_r = \frac{1}{4}|00\rangle\langle 00| + \frac{1}{8}|11\rangle\langle 11| + r|01\rangle\langle 01| + (\frac{5}{8}-r)|10\rangle\langle 10| + \frac{1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|).$$

with x and y are determined by r.

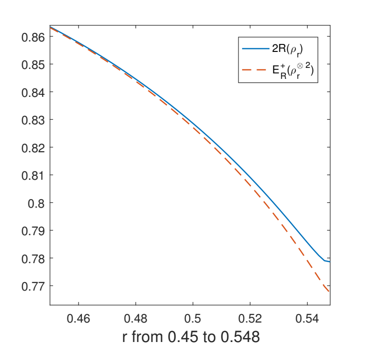
Rains' bound is not additive: Proof ideas (cont.)

We construct ρ_r and σ_r such that $R(\rho_r) = E_{R,PPT}(\rho_r) = S(\rho_r||\sigma_r)$:

$$\rho_r = \frac{1}{8}|00\rangle\langle 00| + x|01\rangle\langle 01| + \frac{7-8x}{8}|10\rangle\langle 10| + \frac{32r^2-(6+32x)r+10x+1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|)$$

$$\sigma_r = \frac{1}{4}|00\rangle\langle 00| + \frac{1}{8}|11\rangle\langle 11| + r|01\rangle\langle 01| + (\frac{5}{8}-r)|10\rangle\langle 10| + \frac{1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|).$$

with x and y are determined by r. When $0.45 \le r \le 0.548$, we show the gap between $2R(\rho_r)$ and $E_R^+(\rho_r^{\otimes 2}) = S(\rho_r^{\otimes 2}||\tau_r)$:



Application & New problem

- Regularization of Rains' bound: $R^\infty(\rho) = \inf_{k\geq 1} \frac{R(\rho^{\otimes k})}{k}$.
- A better upper bound on distillable entanglement:

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq R(\rho),$$

and the second inequality could be strict.

Application & New problem

- Regularization of Rains' bound: $R^{\infty}(\rho) = \inf_{k\geq 1} \frac{R(\rho^{\otimes k})}{k}$.
- A better upper bound on distillable entanglement:

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq R(\rho),$$

  and the second inequality could be strict.
- Remark: Hayashi introduced R^{∞} in his book in 2006.

Application & New problem

- Regularization of Rains' bound: $R^\infty(\rho) = \inf_{k\geq 1} \frac{R(\rho^{\otimes k})}{k}$.
- A better upper bound on distillable entanglement:

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq R(\rho),$$

  and the second inequality could be strict.
- Remark: Hayashi introduced R^∞ in his book in 2006.
- **New problem and an old open problem**
  - $R^\infty(\rho) = E^\infty_{R,PPT}(\rho)$?

Application & New problem

- Regularization of Rains' bound: $R^\infty(\rho) = \inf_{k\geq 1} \frac{R(\rho^{\otimes k})}{k}$.
- A better upper bound on distillable entanglement:

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq R(\rho),$$

  and the second inequality could be strict.
- Remark: Hayashi introduced R^∞ in his book in 2006.
- **New problem and an old open problem**
  - $R^\infty(\rho) = E_{R,PPT}^\infty(\rho)$?
  - Note that

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq E_{R,PPT}^\infty(\rho) \leq E_{C,PPT}(\rho).$$

Application & New problem

- Regularization of Rains' bound: $R^\infty(\rho) = \inf_{k\geq 1} \frac{R(\rho^{\otimes k})}{k}$.
- A better upper bound on distillable entanglement:

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq R(\rho),$$

  and the second inequality could be strict.
- Remark: Hayashi introduced R^∞ in his book in 2006.
- **New problem and an old open problem**
  - $R^\infty(\rho) = E^\infty_{R,PPT}(\rho)$?
  - Note that

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq E^\infty_{R,PPT}(\rho) \leq E_{C,PPT}(\rho).$$

  - Dream: if $R^\infty(\rho) < E^\infty_{R,PPT}(\rho)$, then we will have $E_{D,PPT}(\rho) < E_{C,PPT}(\rho)$!

Application & New problem

- Regularization of Rains' bound: $R^{\infty}(\rho) = \inf_{k\geq 1} \frac{R(\rho^{\otimes k})}{k}$.
- A better upper bound on distillable entanglement:

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq R(\rho),$$

  and the second inequality could be strict.
- Remark: Hayashi introduced R^{∞} in his book in 2006.
- **New problem and an old open problem**
  - $R^{\infty}(\rho) = E^{\infty}_{R,PPT}(\rho)$?
  - Note that

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq E^{\infty}_{R,PPT}(\rho) \leq E_{C,PPT}(\rho).$$

  - Dream: if $R^{\infty}(\rho) < E^{\infty}_{R,PPT}(\rho)$, then we will have $E_{D,PPT}(\rho) < E_{C,PPT}(\rho)$!

  - How to evaluate R^{∞} and $E^{\infty}_{R,PPT}$?

Irreversibility under PPT operations

Theorem (Key result)

There exists entangled state ρ such that $R^\infty(\rho) < E_{R,PPT}^\infty(\rho)$. Thus, the asymptotic entanglement manipulation under PPT operations is irreversible:

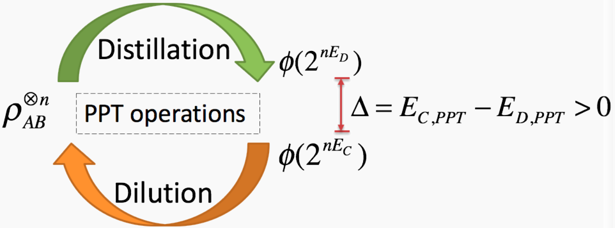
$$\exists \rho, \text{ s.t. } E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$

Irreversibility under PPT operations

Theorem (Key result)

There exists entangled state ρ such that $R^\infty(\rho) < E_{R,PPT}^\infty(\rho)$. Thus, the asymptotic entanglement manipulation under PPT operations is irreversible:

$$\exists \rho, \text{ s.t. } E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$



Lower bound of $E_{R,PPT}^{\infty}$

Our key contribution is an efficiently computable lower bound on the regularized relative entropy of entanglement w.r.t. PPT states.

A lower bound for $E_{R,PPT}^{\infty}$

Let P be the projection over the support of state ρ. Then

$$E_{R,PPT}^{\infty}(\rho) \geq E_{\eta}(\rho) = -\log_2 \eta(P),$$

where

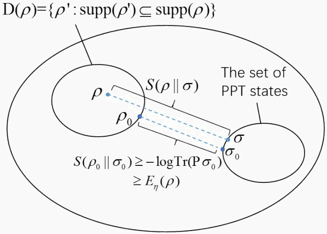
$$\eta(P) = \min t, s.t. -t\mathbb{1} \leq Y^{T_B} \leq t\mathbb{1}, -Y \leq P^{T_B} \leq Y.$$

Lower bound of $E^{\infty}_{R,PPT}$: Sketch of the proof

- Relax the problem to an SDP:

$$\min_{\sigma\in PPT} S(\rho\|\sigma) \geq \min_{\rho_0\in D(\rho),\sigma_0\in PPT} S(\rho_0\|\sigma_0)$$
$$\geq \min_{\sigma_0\in PPT} -\log \operatorname{Tr} P\sigma_0.$$

Also see min-relative entropy (Datta 2009):
$S(\rho\|\sigma) \geq D_{\min}(\rho\|\sigma) = -\log \operatorname{Tr} P\sigma$

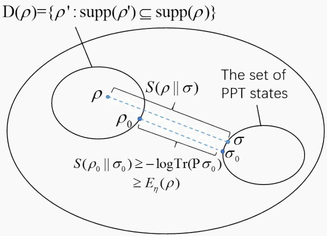


Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof

- Relax the problem to an SDP:

$$\min_{\sigma \in PPT} S(\rho||\sigma) \geq \min_{\rho_0 \in D(\rho), \sigma_0 \in PPT} S(\rho_0||\sigma_0)$$
$$\geq \min_{\sigma_0 \in PPT} -\log \operatorname{Tr} P\sigma_0.$$

Also see min-relative entropy (Datta 2009):
$S(\rho||\sigma) \geq D_{\min}(\rho||\sigma) = -\log \operatorname{Tr} P\sigma$



- Utilizing the weak duality of SDP and did a further relaxtion

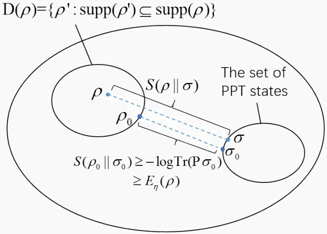
$$E_{R,PPT}(\rho) \geq \min_{\sigma_0 \in \mathrm{PPT}} -\log \operatorname{Tr} P\sigma_0$$
$$\geq \max -\log t \text{ s.t. } Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y \quad (\text{not additive ☹})$$

Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof

- Relax the problem to an SDP:

$$\min_{\sigma \in PPT} S(\rho||\sigma) \geq \min_{\rho_0 \in D(\rho), \sigma_0 \in PPT} S(\rho_0||\sigma_0)$$
$$\geq \min_{\sigma_0 \in PPT} -\log \mathrm{Tr}\, P\sigma_0.$$

Also see min-relative entropy (Datta 2009):
$S(\rho||\sigma) \geq D_{\min}(\rho||\sigma) = -\log \mathrm{Tr}\, P\sigma$



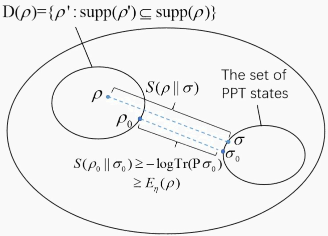
- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq \min_{\sigma_0 \in \mathrm{PPT}} -\log \mathrm{Tr}\, P\sigma_0$$
$$\geq \max -\log t \text{ s.t. } Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y \quad (\text{not additive} ☹)$$
$$\geq \max -\log t \text{ s.t. } \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y = E_\eta.$$

Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof

- Relax the problem to an SDP:

$$\min_{\sigma\in PPT} S(\rho\|\sigma) \geq \min_{\rho_0\in D(\rho),\sigma_0\in PPT} S(\rho_0\|\sigma_0)$$
$$\geq \min_{\sigma_0\in PPT} -\log \operatorname{Tr} P\sigma_0.$$



Also see min-relative entropy (Datta 2009):
$S(\rho\|\sigma) \geq D_{\min}(\rho\|\sigma) = -\log \operatorname{Tr} P\sigma$

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq \min_{\sigma_0\in \mathrm{PPT}} -\log \operatorname{Tr} P\sigma_0$$
$$\geq \max -\log t \text{ s.t. } Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y \quad (\text{not additive} ☹)$$
$$\geq \max -\log t \text{ s.t. } \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y \quad = E_\eta.$$

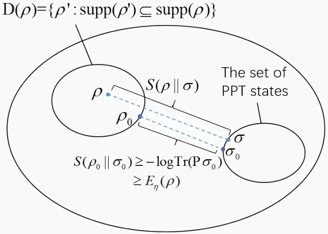
- Utilizing the strong duality of SDP to obtain

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2), ☺$$

Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof

- Relax the problem to an SDP:

$$\min_{\sigma \in PPT} S(\rho\|\sigma) \geq \min_{\rho_0 \in D(\rho), \sigma_0 \in PPT} S(\rho_0\|\sigma_0)$$
$$\geq \min_{\sigma_0 \in PPT} -\log \operatorname{Tr} P\sigma_0.$$



Also see min-relative entropy (Datta 2009):
$S(\rho\|\sigma) \geq D_{\min}(\rho\|\sigma) = -\log \operatorname{Tr} P\sigma$

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq \min_{\sigma_0 \in \mathrm{PPT}} -\log \operatorname{Tr} P\sigma_0$$
$$\geq \max -\log t \text{ s.t. } Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y \quad (\text{not additive} ☹)$$
$$\geq \max -\log t \text{ s.t. } \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y \quad = E_\eta.$$

- Utilizing the strong duality of SDP to obtain

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2), \ ☺$$

thus we have $E_{R,PPT}^{\infty}(\rho) \geq \lim_{n\to\infty} \frac{1}{n} E_\eta(\rho^{\otimes n}) = E_\eta(\rho).$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

We have

$$E_{D,PPT}(\rho) = R^{\infty}(\rho) = \log_2(1+\frac{1}{\sqrt{2}}) < 1 = E^{\infty}_{R,PPT}(\rho) = E_{C,PPT}(\rho).$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

  We have

$$E_{D,PPT}(\rho) = R^{\infty}(\rho) = \log_2(1+\frac{1}{\sqrt{2}}) < 1 = E_{R,PPT}^{\infty}(\rho) = E_{C,PPT}(\rho).$$

- Sufficient condition for the irreversibility: If $E_\eta(\rho) > E_W(\rho) = \min_{X_{AB} \geq \rho} \log_2 \|X_{AB}^{T_B}\|_1$, then

$$E_{D,PPT}(\rho) \leq E_W(\rho) < E_\eta(\rho) \leq E_{C,PPT}(\rho),$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

We have

$$E_{D,PPT}(\rho) = R^{\infty}(\rho) = \log_2(1+\frac{1}{\sqrt{2}}) < 1 = E^{\infty}_{R,PPT}(\rho) = E_{C,PPT}(\rho).$$

- Sufficient condition for the irreversibility: If $E_\eta(\rho) > E_W(\rho) = \min_{X_{AB} \geq \rho} \log_2 \|X_{AB}^{T_B}\|_1$, then

$$E_{D,PPT}(\rho) \leq E_W(\rho) < E_\eta(\rho) \leq E_{C,PPT}(\rho),$$

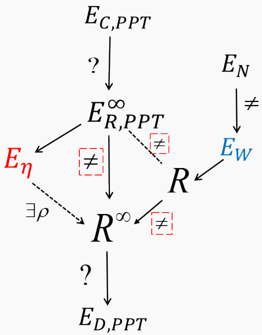
- **Example 2:** The above example can be generalized to any rank-2 state ρ supporting on the $3 \otimes 3$ anti-symmetric subspace: $E_{D,PPT}(\rho) \leq E_W(\rho) < 1 = E_\eta(\rho) = E_{C,PPT}(\rho)$.

Conclusion

Results:

- Better SDP upper bound on E_D
- Non-additivity of Rains' bound
- SDP lower bound for $E_{R,PPT}^{\infty}$
- Irreversibility under PPT operations:

$$E_{D,PPT} \neq E_{C,PPT}.$$

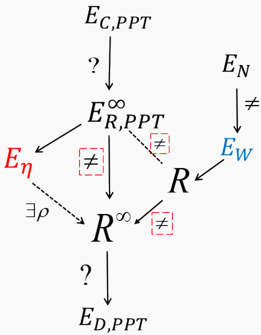


Conclusion

Results:

- Better SDP upper bound on E_D
- Non-additivity of Rains' bound
- SDP lower bound for $E_{R,PPT}^{\infty}$
- Irreversibility under PPT operations:

$$E_{D,PPT} \neq E_{C,PPT}.$$



Discussions:

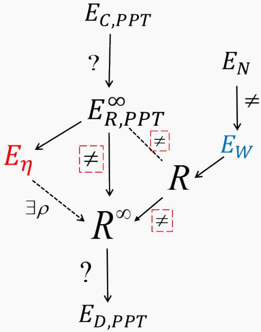
- $E_{D,PPT}(\rho) = R^{\infty}(\rho)$?

Conclusion

Results:

- Better SDP upper bound on E_D
- Non-additivity of Rains' bound
- SDP lower bound for $E^{\infty}_{R,PPT}$
- Irreversibility under PPT operations:

$$E_{D,PPT} \neq E_{C,PPT}.$$



Discussions:

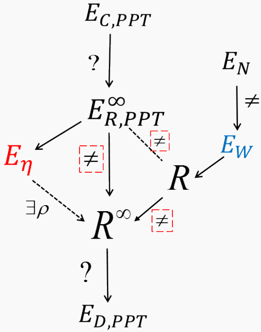
- $E_{D,PPT}(\rho) = R^{\infty}(\rho)$?
- Note that E_η is not tight for the $3 \otimes 3$ anti-symmetric state σ_a, how to improve E_η?

Conclusion

Results:

- Better SDP upper bound on E_D
- Non-additivity of Rains' bound
- SDP lower bound for $E_{R,PPT}^{\infty}$
- Irreversibility under PPT operations:

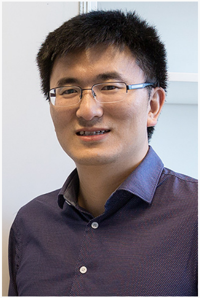
$$E_{D,PPT} \neq E_{C,PPT}.$$



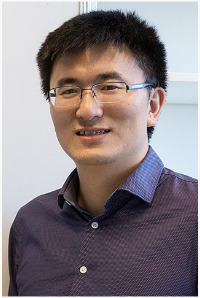
Discussions:

- $E_{D,PPT}(\rho) = R^{\infty}(\rho)$?
- Note that E_η is not tight for the $3 \otimes 3$ anti-symmetric state σ_a, how to improve E_η?
- How to evaluate the distillable entanglement without using PPT operations?

arXiv: 1606.09421, 1605.00348, 1601.07940



arXiv: 1606.09421, 1605.00348, 1601.07940



Thank you for your attention!