Controlled amplification

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Based on joint work with

Cătălin Dohotaru & Mojtaba Komeili

[DH16] [HK16] arXiv:1612.08958

5 algorithms

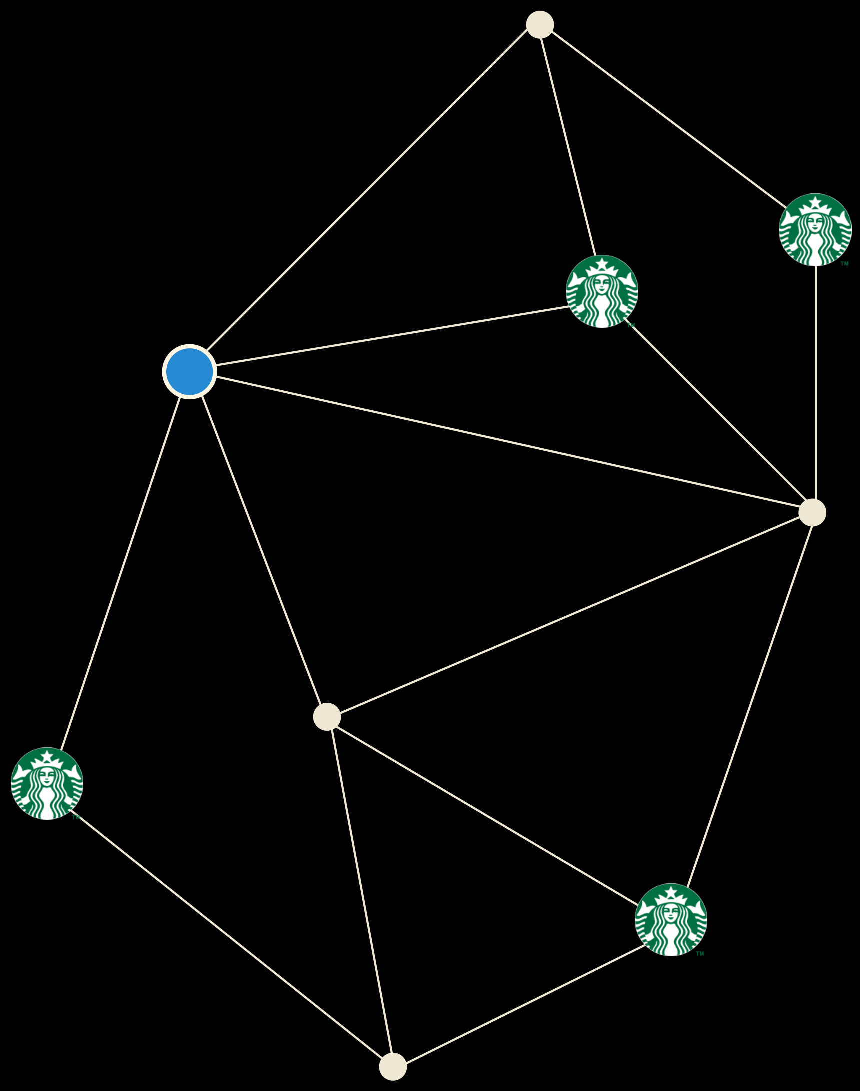
Searching

Quantum search
Amplitude amplification
Quantum walks

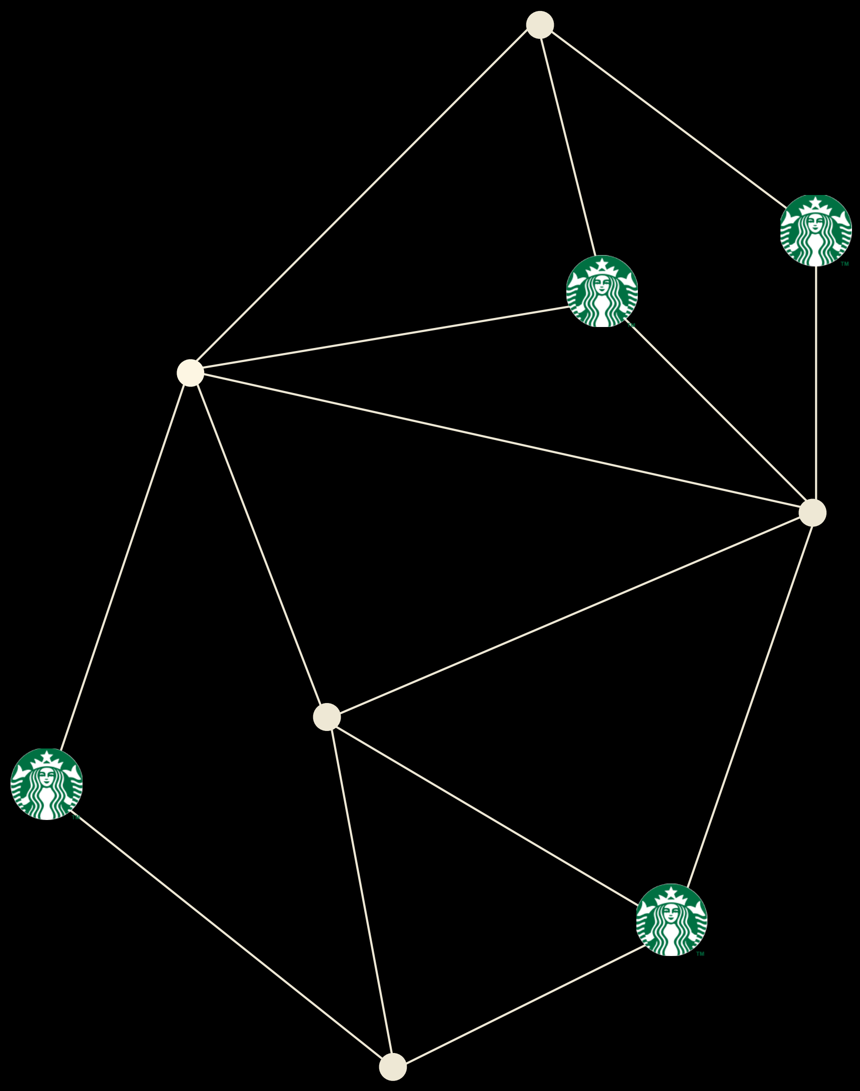
Controlled amplification

Quantum search
Amplitude amplification
Quantum walks

Find Starbucks



Find a marked vertex

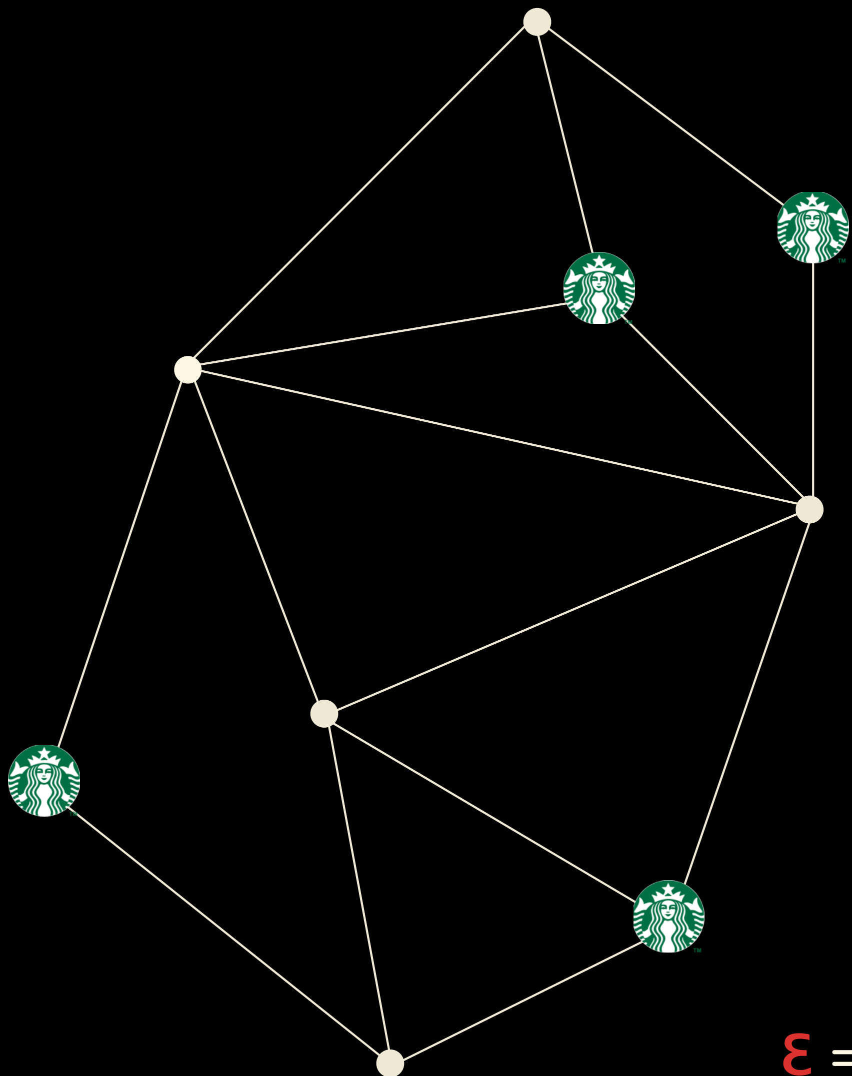


Setup walk at a random vertex

Repeat T times

- Check if vertex marked. If so, halt.
- Update by walking to a random neighbor

Multiple marked M



Setup walk at a random vertex

Repeat **T** times

- **Check** if vertex marked. If so, halt.
- **Update** by walking to a random neighbor

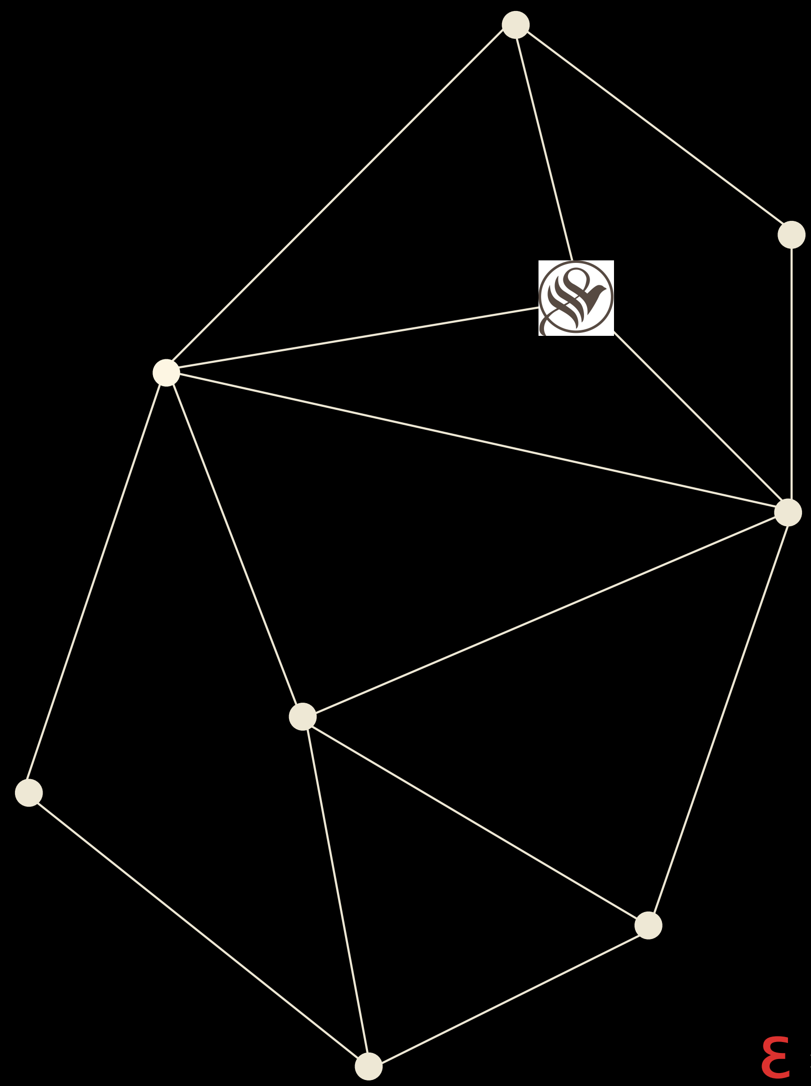
$$T = HT(M)$$

$$\text{Cost} = \mathbf{S} + HT(M)\,(\mathbf{C} + \mathbf{U})$$

$$\varepsilon = \text{Prob [random vertex is marked]}$$

$$= \frac{\text{number of marked}}{\text{number of vertices}} = \frac{|M|}{|V|}$$

Unique marked g



Setup walk at a random vertex

Repeat **T** times

- Check if vertex marked. If so, halt.
- Update by walking to a random neighbor

$T = HT(g)$

$\text{Cost} = \mathbf{S} + HT(g)\,(\mathbf{C} + \mathbf{U})$

$\varepsilon = \text{Prob [random vertex is marked]}$

$= \dfrac{\text{number of marked}}{\text{number of vertices}} = \dfrac{1}{|V|}$

Quantum Walk

Setup walk at a random vertex

Repeat **T** times

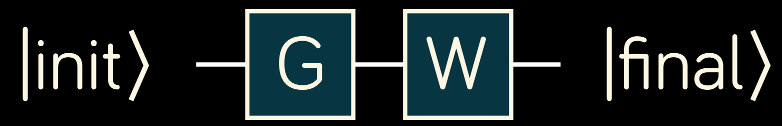
- Check if vertex marked. If so, halt.
- Update by walking to a random neighbor

Quantum Walk

Setup $|\text{init}\rangle$

Repeat T_q times

- Check G
- Update W



$$G|v\rangle = \begin{cases} -|v\rangle & \text{if } v \text{ is marked} \\ |v\rangle & \text{otherwise} \end{cases}$$

$$\text{Cost} = S + T_{quantum}(C + U)$$

Want: $T_{quantum} = \sqrt{T_{classical}}$

[Sze04, Amb04] - Szegedy; Ambainis

Unique marked

	Cost	
Szegedy (Detect)	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[Sze04, Amb04]
Ampl. Ampl.	$\sqrt{1/\varepsilon}\,S + \sqrt{1/\varepsilon}\,C$	[BH97, BHMT02]
Phase Est.	$S + \sqrt{1/(\varepsilon\delta)}\,U + \sqrt{1/\varepsilon}\,C$	[MNRoS11]
Interpolated	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[KMOR16]
Grid/St.trans.	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[Tul08, MNRiS12]

[BH97, BHMT02] - Brassard, Høyer; Brassard, Høyer, Mosca, Tapp
[MNRoS11] - Magniez, Nayak, Roland, Santha
[KMOR16] - Krovi, Magniez, Ozols, Roland
[Tul08] - Tulsi
[MNRiS12] - Magniez, Nayak, Richter, Santha

$$1/\varepsilon \le H \le 1/(\varepsilon\delta)$$

Unique marked

Ampl. Ampl.	$\sqrt{1/\varepsilon}$ **S**			+	$\sqrt{1/\varepsilon}$	**C**	[BH97, BHMT02]
Phase Est.	**S** +	$\sqrt{1/(\varepsilon\delta)}$	**U**	+	$\sqrt{1/\varepsilon}$	**C**	[MNRoS11]
Interpolated	**S** +	$\sqrt{\mathbf{H}}$	**U**	+	$\sqrt{\mathbf{H}}$	**C**	[KMOR16]
Controlled Ampl.	**S** +	$\sqrt{\mathbf{H}}$	**U**	+	$\sqrt{\mathbf{H}}$	**C**	[DH16]
Rec. Ampl. Ampl.	**S** +	$\sqrt{\mathbf{H}}$	**U**	+	$\sqrt{1/\varepsilon}$	**C**	[DH16]
Classical	**S** +	$\mathbf{H}$	**U**	+	$1/\varepsilon$	**C**	[DH16]

$$1/\varepsilon \leq \mathbf{H} \leq 1/(\varepsilon\delta)$$

Multiple marked

Szegedy (Detect)		**S** +	$\sqrt{\mathbf{H}}$ **U** +	$\sqrt{\mathbf{H}}$ **C**	[Sze04, Amb04]
Ampl. Ampl.	$\sqrt{1/\varepsilon}$	**S**	+	$\sqrt{1/\varepsilon}$ **C**	[BH97, BHMT02]
Phase Est.		**S** +	$\sqrt{1/(\varepsilon\delta)}$ **U** +	$\sqrt{1/\varepsilon}$ **C**	[MNRoS11]
Interpolated		**S** +	$\sqrt{\mathbf{H}^+}$ **U** +	$\sqrt{\mathbf{H}^+}$ **C**	[KMOR16]
Controlled Ampl.		**S** +	$\sqrt{\mathbf{H}^+}$ **U** +	$\sqrt{\mathbf{H}^+}$ **C**	[DH16]
Grids		**S** +	$\sqrt{\mathbf{H}}$ **U** +	$\sqrt{\mathbf{H}}$ **C**	[HK16]

$$1/\varepsilon \leq \mathbf{H} \leq 1/(\varepsilon\delta)$$

$$\mathbf{H} \leq \mathbf{H}^+ \leq \mathbf{H}/\delta$$ [AK15]

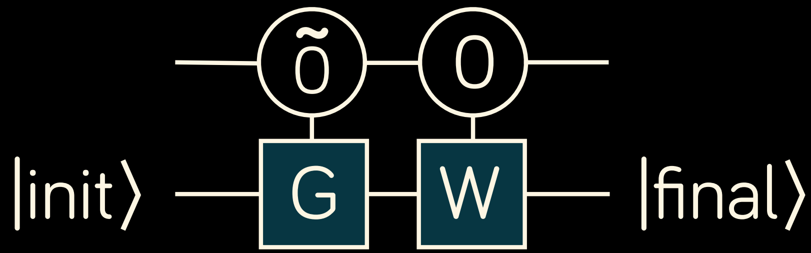
5 Algorithms

Controlled Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[DH16]	
Rec. Ampl. Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{1/\varepsilon}$ C	[DH16]	Unique
Classical	S +	H U +	$1/\varepsilon$ C	[DH16]	
Controlled Ampl.	S +	$\sqrt{H^{+}}$ U +	$\sqrt{H^{+}}$ C	[DH16]	Multiple
Grids	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[HK16]	

$$1/\varepsilon \leq H \leq 1/(\varepsilon\delta)$$

$$H \leq H^{+} \leq H/\delta$$

Controlled amplification



$$|\tilde{0}\rangle\langle\tilde{0}| \otimes G \quad + \quad |\tilde{1}\rangle\langle\tilde{1}| \otimes \mathrm{Id}$$

$\{|\tilde{0}\rangle, |\tilde{1}\rangle\}$ is a basis

Controlled amplification

Comparison

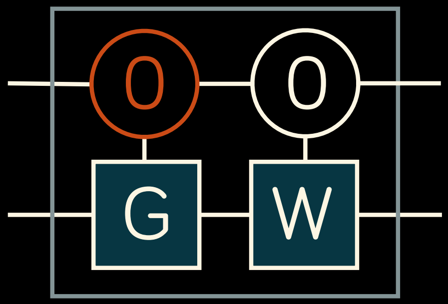
Szegedy (Detect)

Ampl. Ampl.

Phase Est.

Interpolated

Grid/St.trans.



Comparison

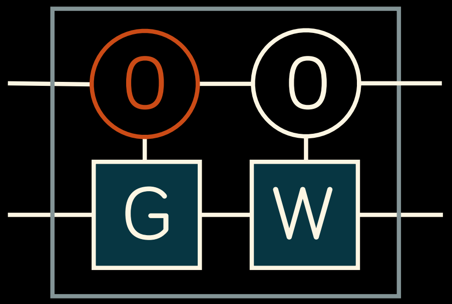
Szegedy (Detect) ✔

Ampl. Ampl.

Phase Est.

Interpolated

Grid/St.trans.



$W = \text{Reflection}(|\text{init}\rangle)$

Comparison

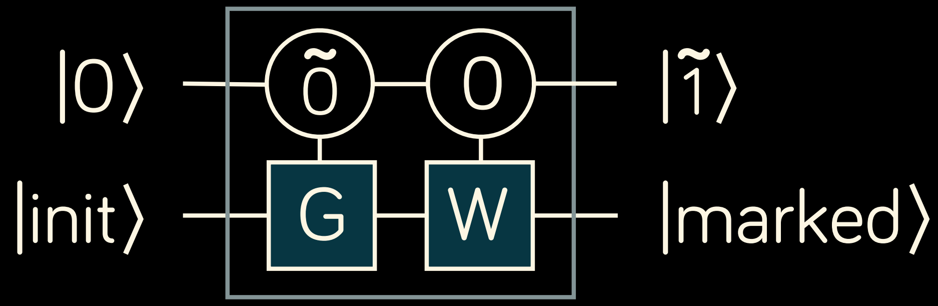
Szegedy (Detect) ✔

Ampl. Ampl. ✔✔

Phase Est.

Interpolated

Grid/St.trans.



$W = \text{Reflection}(|\text{init}\rangle)$

$$|0\rangle|\text{init}\rangle = \sqrt{\tfrac{1}{2}}\left(|\text{stays}\rangle + |\text{rotates}\rangle \right)$$

$$|\tilde{1}\rangle|\text{marked}\rangle = \sqrt{\tfrac{1}{2}}\left(|\text{stays}\rangle - |\text{rotates}\rangle \right)$$

Comparison

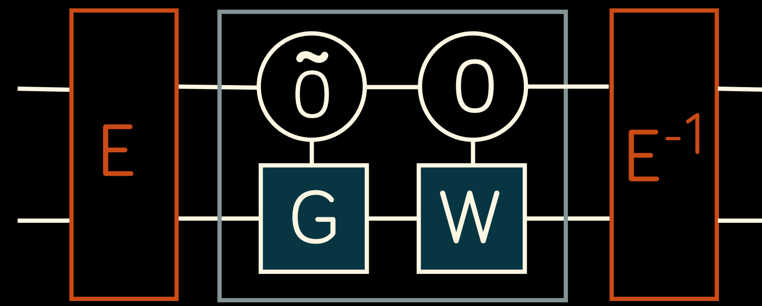
Szegedy (Detect) ✔

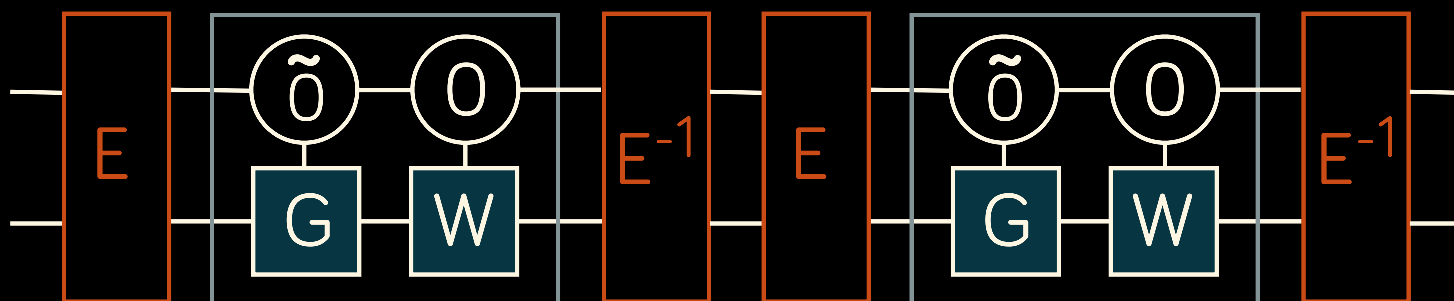
Ampl. Ampl. ✔✔

Phase Est.

Interpolated ✔

Grid/St.trans.





Comparison

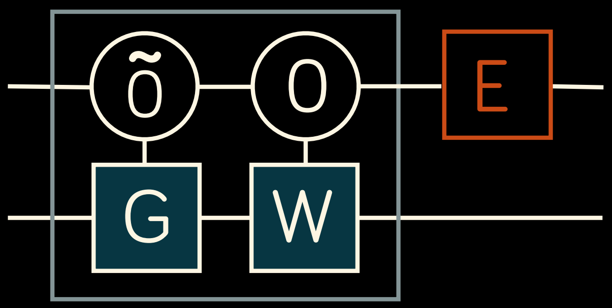
Szegedy (Detect) ✔

Ampl. Ampl. ✔✔

Phase Est.

Interpolated ✔

Grid/St.trans. ✔



Comparison

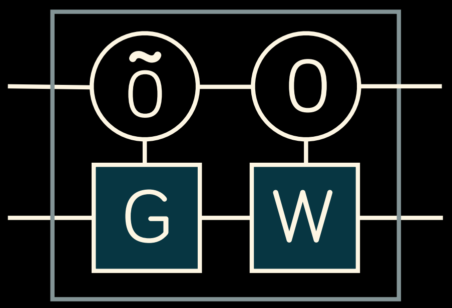
Szegedy (Detect) ✔

Ampl. Ampl. ✔✔

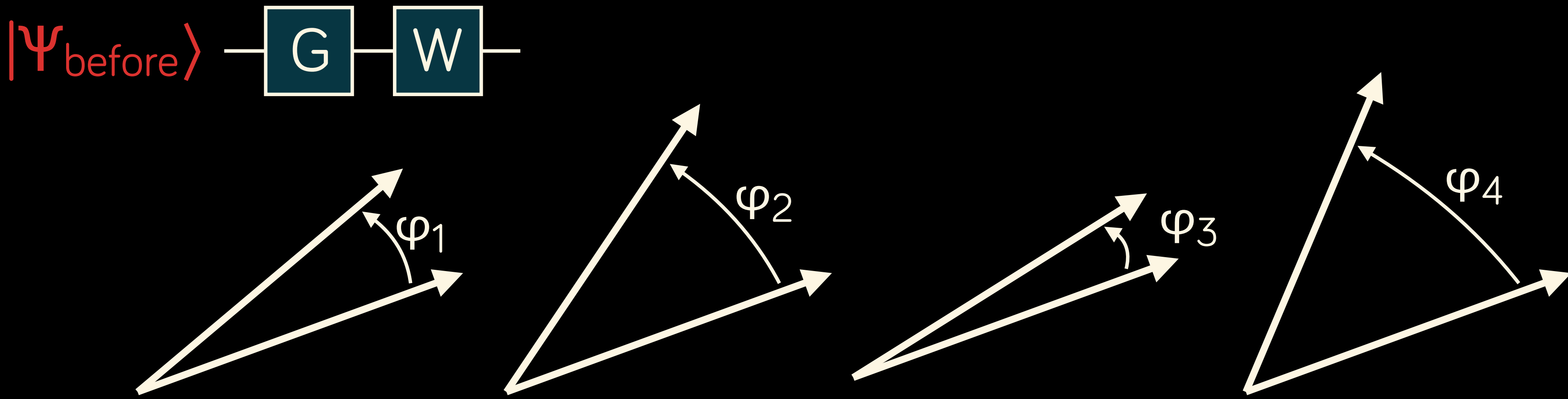
Phase Est.

Interpolated ✔

Grid/St.trans. ✔



Measure of cost

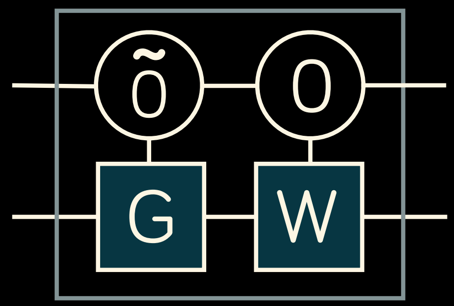


$\alpha_1 \frac{1}{\varphi_1}$ $\alpha_2 \frac{1}{\varphi_2}$ $\alpha_3 \frac{1}{\varphi_3}$ $\alpha_4 \frac{1}{\varphi_4}$

Quantum hitting time $= H_q =$ $\text{Cost} \approx \sqrt{\sum_i \frac{|\alpha_i|^2}{\varphi_i^2}}$

Comparison on cost

U



A

Theorems (Unique marked):

$$H_q^2(U) = H_q^2(A)$$

Final state of $U \approx |\text{marked}\rangle$

$$H_q^2(U) = \frac{1}{\varepsilon} E(W,g)$$

Def (Escape time): $E(W, g) = H_q^2(W)$

5 Algorithms

✔ Controlled Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[DH16]		
Rec. Ampl. Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{1/\varepsilon}$ C	[DH16]	Unique
Classical	S +	H U +	$1/\varepsilon$ C	[DH16]		
Controlled Ampl.	S +	$\sqrt{H^+}$ U +	$\sqrt{H^+}$ C	[DH16]	Multiple
Grids	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[HK16]		

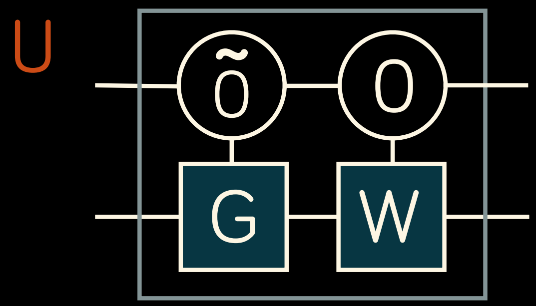
$1/\varepsilon \leq H \leq 1/(\varepsilon\delta)$ $\quad$ $H \leq H^+ \leq H/\delta$

5 Algorithms

✔ Controlled Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[DH16]	
Rec. Ampl. Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{1/\varepsilon}$ C	[DH16]	Unique
Classical	S +	H U +	$1/\varepsilon$ C	[DH16]	
Controlled Ampl.	S +	$\sqrt{H^+}$ U +	$\sqrt{H^+}$ C	[DH16]	Multiple
Grids	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[HK16]	

$1/\varepsilon \leq H \leq 1/(\varepsilon\delta)$ $H \leq H^+ \leq H/\delta$

Algorithm - multiple marked



Theorems (Multiple marked for quantum walks):

Final state of U $\approx$ $|\text{marked}\rangle$

$$H_q^2(U,M) = \frac{1}{\varepsilon} E(W,M)$$

$$E(W,M) = \varepsilon H^+$$

5 Algorithms

	Algorithm				Reference	
✔	Controlled Ampl.	**S** +	$\sqrt{H}$ **U** +	$\sqrt{H}$ **C**	[DH16]	Unique
	Rec. Ampl. Ampl.	**S** +	$\sqrt{H}$ **U** +	$\sqrt{1/\varepsilon}$ **C**	[DH16]	
	Classical	**S** +	H **U** +	$1/\varepsilon$ **C**	[DH16]	
✔	Controlled Ampl.	**S** +	$\sqrt{H^+}$ **U** +	$\sqrt{H^+}$ **C**	[DH16]	Multiple
	Grids	**S** +	$\sqrt{H}$ **U** +	$\sqrt{H}$ **C**	[HK16]	

$1/\varepsilon \leq H \leq 1/(\varepsilon\delta)$ $\quad$ $H \leq H^+ \leq H/\delta$

Lemma: $E(W,M) \leq 1/\delta$ [DH16]

Theorem: $\frac{1}{\varepsilon} \leq H \leq H^+ = \frac{1}{\varepsilon} E(W,M) \leq \frac{1}{\varepsilon}\frac{1}{\delta}$ [DH16]

5 Algorithms

✔ Controlled Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[DH16]	
Rec. Ampl. Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{1/\varepsilon}$ C	[DH16]	Unique
Classical	S +	H U +	$1/\varepsilon$ C	[DH16]	
✔ Controlled Ampl.	S +	$\sqrt{H^+}$ U +	$\sqrt{H^+}$ C	[DH16]	Multiple
Grids	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C	[HK16]	

Unique marked: $H = \frac{1}{\varepsilon} E(W,g)$

Szegedy: P random walk ⟷ W(P) quantum walk

Theorem: If $E(W(P),M) = D$ then $E(W(P^D),M) = 1$ [DH16]

5 Algorithms

	Algorithm	Complexity	Reference	
✔	Controlled Ampl.	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[DH16]	Unique
	Rec. Ampl. Ampl.	$S + \sqrt{H}\,U + \sqrt{1/\varepsilon}\,C$	[DH16]	
✔	Classical	$S + H\,U + 1/\varepsilon\,C$	[DH16]	
✔	Controlled Ampl.	$S + \sqrt{H^+}\,U + \sqrt{H^+}\,C$	[DH16]	Multiple
	Grids	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[HK16]	

Unique marked: $H = \frac{1}{\varepsilon} E(W,g)$

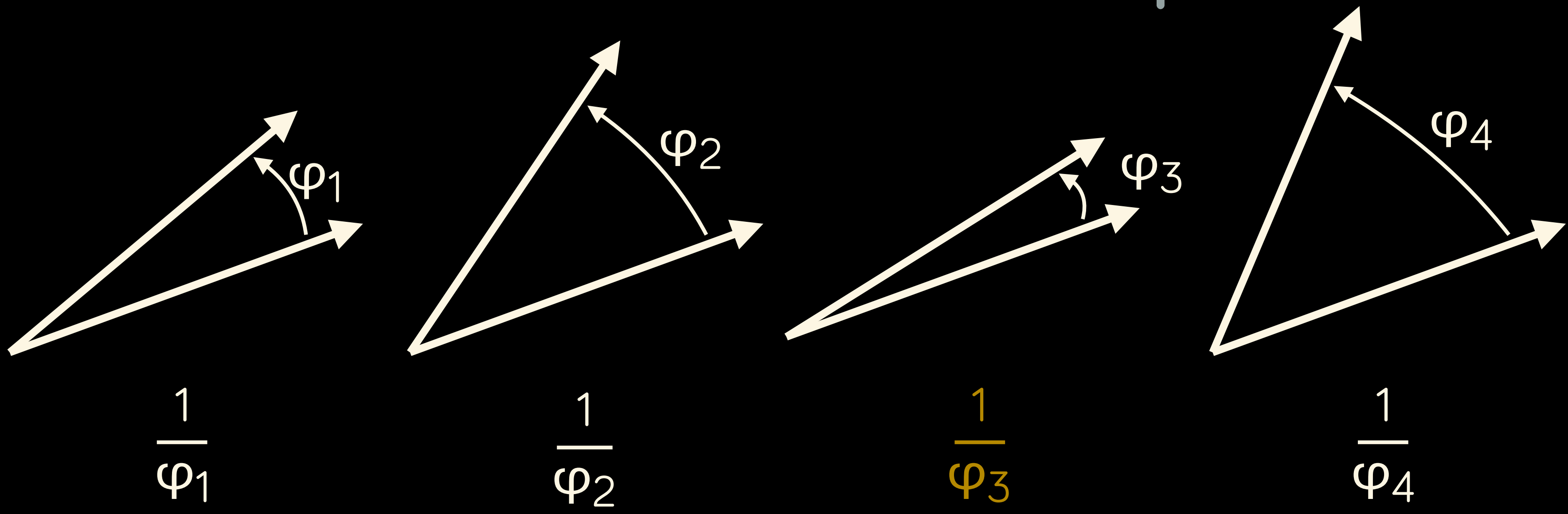
5 Algorithms

	Algorithm					Reference	
✔	Controlled Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C		[DH16]	
	Rec. Ampl. Ampl.	S +	$\sqrt{H}$ U +	$\sqrt{1/\varepsilon}$ C		[DH16]	Unique
✔	Classical	S +	H U +	$1/\varepsilon$ C		[DH16]	
✔	Controlled Ampl.	S +	$\sqrt{H^+}$ U +	$\sqrt{H^+}$ C		[DH16]	Multiple
	Grids	S +	$\sqrt{H}$ U +	$\sqrt{H}$ C		[HK16]	

Unique marked: $H = \frac{1}{\varepsilon} E(W, g)$

Idea: Phase Estimation in the Escape time

Phase Est. in the Escape time

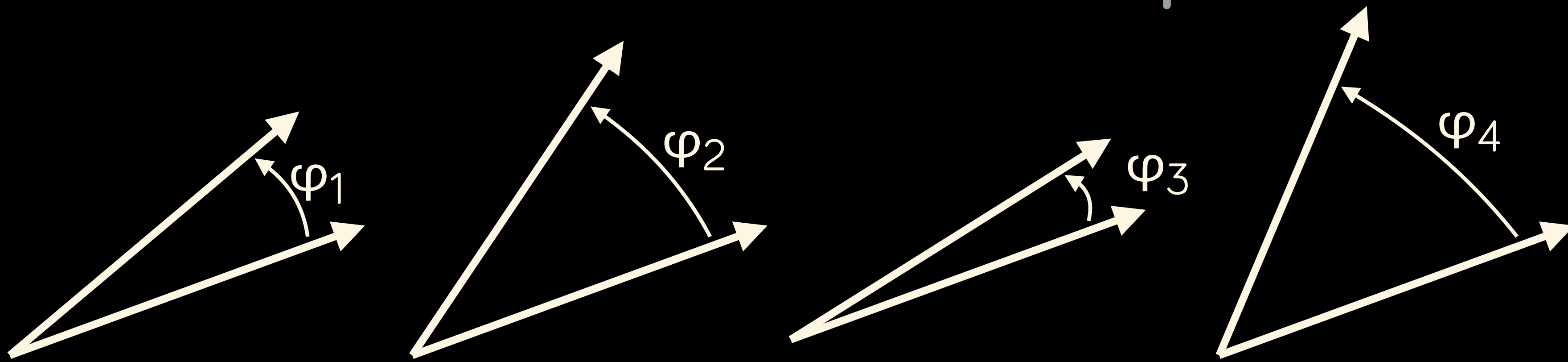


Phase Estimation in $\frac{1}{\text{smallest phase}}$

yields $\mathbf{S} + \sqrt{1/(\varepsilon\delta)}\,\mathbf{U} + \sqrt{1/\varepsilon}\,\mathbf{C}$ algorithm [MNRoS11]

$\delta = (\text{smallest phase})^2$

Phase Est. in the Escape time



$\frac{1}{\varphi_1}$ $\frac{1}{\varphi_2}$ $\frac{1}{\varphi_3}$ $\frac{1}{\varphi_4}$

Phase Estimation in sqrt(Escape time)

yields **S** + $\sqrt{\mathbf{H}}$ **U** + $\sqrt{1/\varepsilon}$ **C** algorithm [MNRoS11]

δ = (smallest phase)^2

5 Algorithms

✔ Controlled Ampl.	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[DH16]	Unique
✔ Rec. Ampl. Ampl.	$S + \sqrt{H}\,U + \sqrt{1/\varepsilon}\,C$	[DH16]	
✔ Classical	$S + H\,U + 1/\varepsilon\,C$	[DH16]	
✔ Controlled Ampl.	$S + \sqrt{H^+}\,U + \sqrt{H^+}\,C$	[DH16]	Multiple
Grids	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[HK16]	

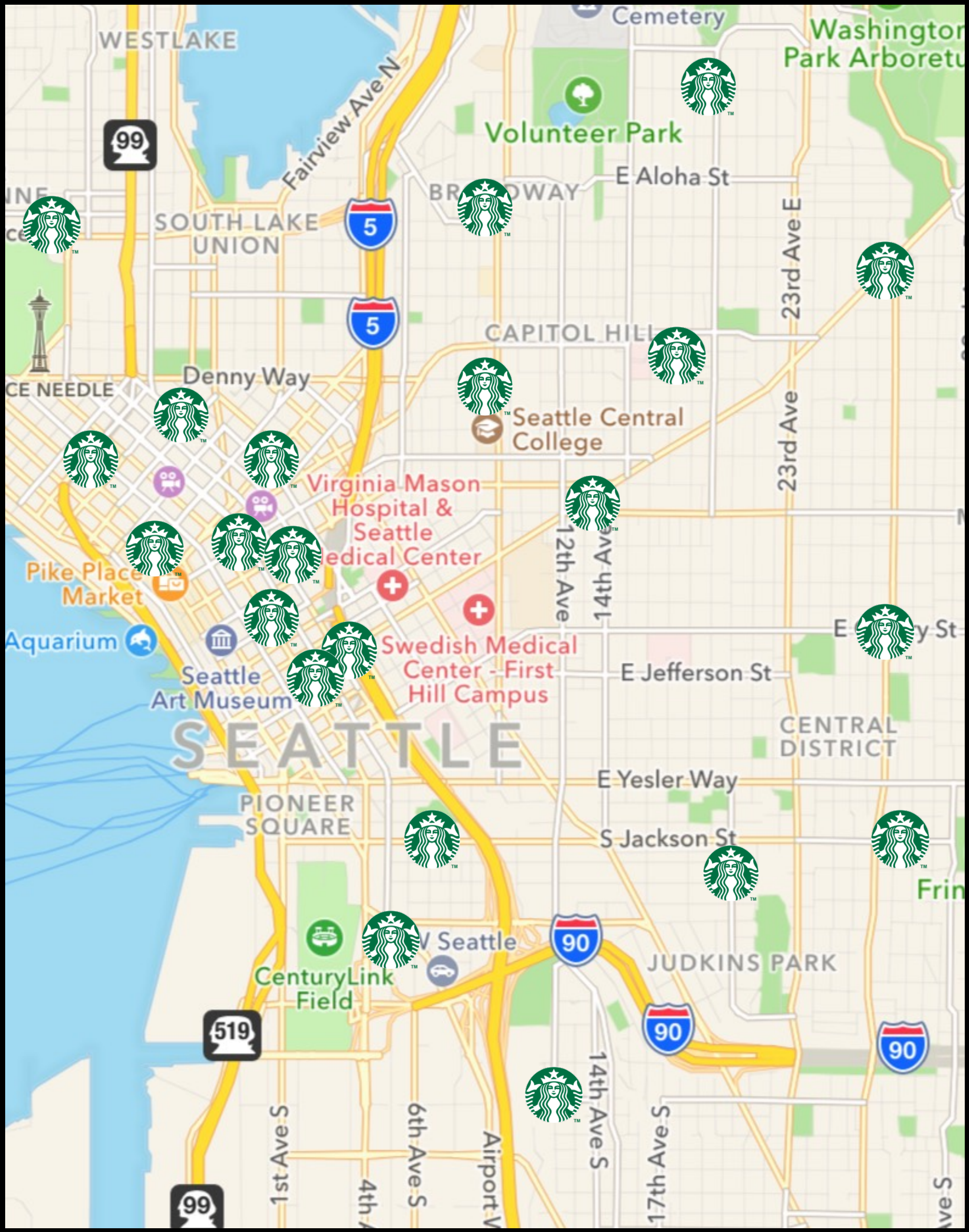
Theorem: $\frac{1}{\varepsilon} \leq H = H^+ = \frac{1}{\varepsilon}E(W,M) \leq \frac{1}{\varepsilon}\frac{1}{\delta}$

5 Algorithms

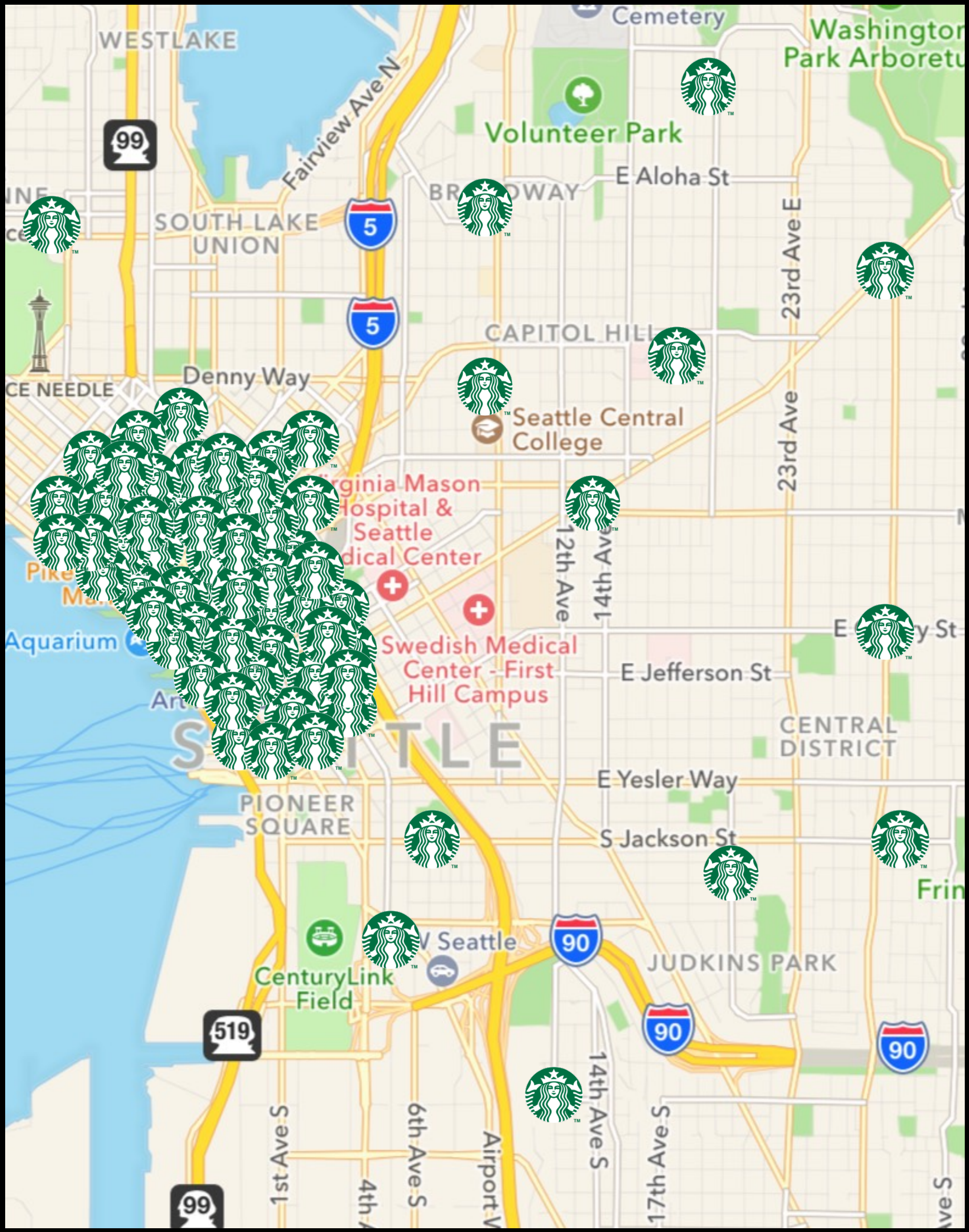
✔	Controlled Ampl.	$S + \sqrt{H}\, U + \sqrt{H}\, C$	[DH16]		
✔	Rec. Ampl. Ampl.	$S + \sqrt{H}\, U + \sqrt{1/\varepsilon}\, C$	[DH16]	Unique	
✔	Classical	$S + H\, U + 1/\varepsilon\, C$	[DH16]		
✔	Controlled Ampl.	$S + \sqrt{H^+}\, U + \sqrt{H^+}\, C$	[DH16]	Multiple	
	Grids	$S + \sqrt{H}\, U + \sqrt{H}\, C$	[HK16]		

Theorem: $\frac{1}{\varepsilon} \leq H = H^+ = \frac{1}{\varepsilon} E(W,M) \leq \frac{1}{\varepsilon}\frac{1}{\delta}$

Finding a Starbucks



Finding a Starbucks



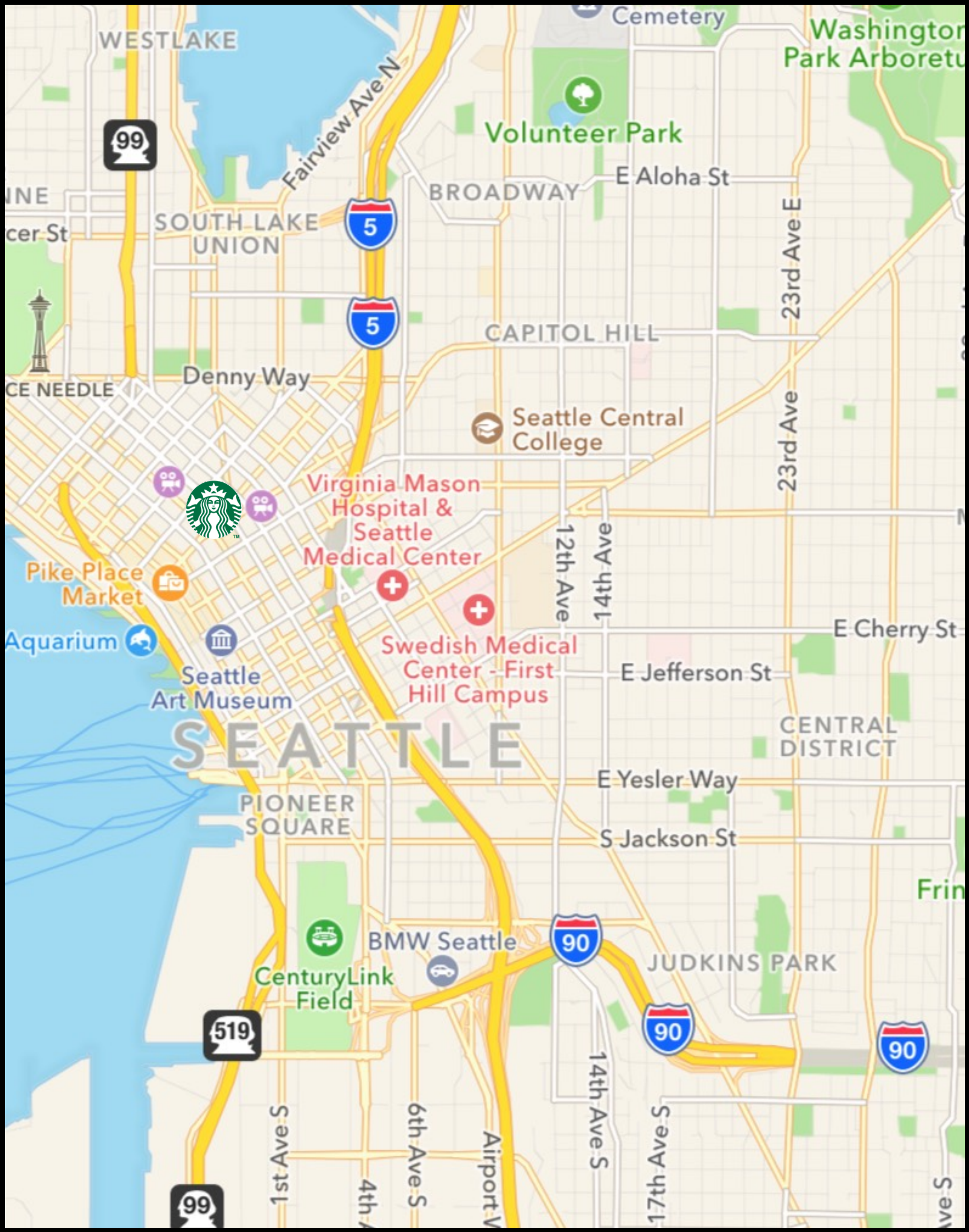
Finding a Starbucks

Grid with cluster of :

Random walk: $\Theta(1)$

~~Quantum walk: $\Theta(N)$~~

~~Quantum walk': $\Theta(N \log N)$~~



Finding a Starbucks

Grid with cluster of :

Random walk: $\Theta(1)$

~~Quantum walk: $\Theta(N)$~~

~~Quantum walk': $\Theta(N \log N)$~~

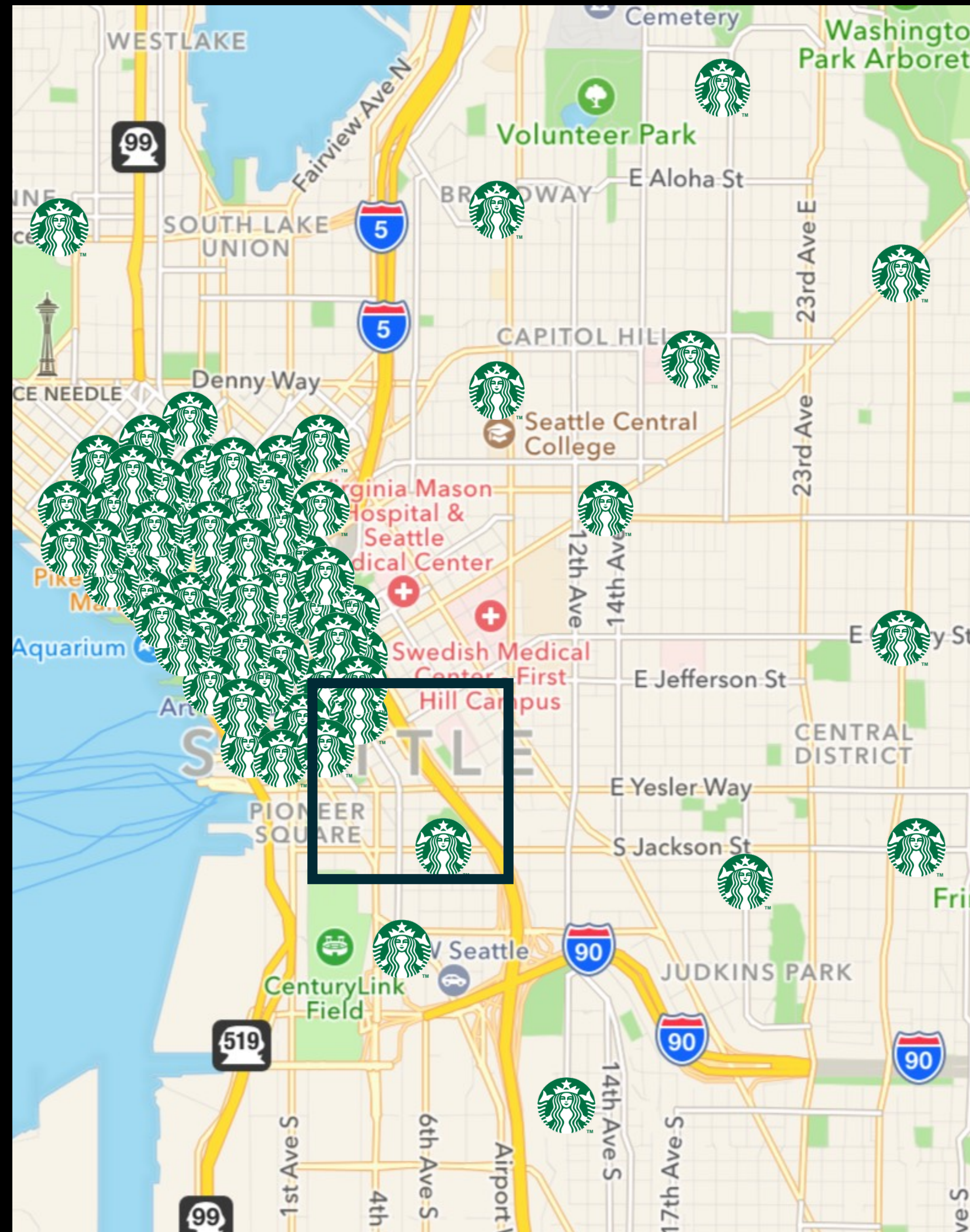
Lemma:
Quantumly search locally

Lemma:
$H^+(W,M) \le H(W,g)$
for worst-case $g \in M$

Theorem: Up to logs,

$$S + \sqrt{H}\, U + \sqrt{H}\, C$$

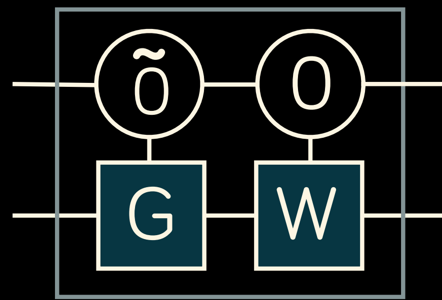
for grids



5 Algorithms

✔ Controlled Ampl.	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[DH16]	Unique
✔ Rec. Ampl. Ampl.	$S + \sqrt{H}\,U + \sqrt{1/\varepsilon}\,C$	[DH16]	
✔ Classical	$S + H\,U + 1/\varepsilon\,C$	[DH16]	
✔ Controlled Ampl.	$S + \sqrt{H^{+}}\,U + \sqrt{H^{+}}\,C$	[DH16]	Multiple
✔ Grids	$S + \sqrt{H}\,U + \sqrt{H}\,C$	[HK16]	

Controlled amplification



✔ Controlled Ampl.	S + $\sqrt{H}$ U + $\sqrt{H}$ C	[DH16]	Unique
✔ Rec. Ampl. Ampl.	S + $\sqrt{H}$ U + $\sqrt{1/\varepsilon}$ C	[DH16]	
✔ Classical	S + H U + $1/\varepsilon$ C	[DH16]	
✔ Controlled Ampl.	S + $\sqrt{H^+}$ U + $\sqrt{H^+}$ C	[DH16]	Multiple
✔ Grids	S + $\sqrt{H}$ U + $\sqrt{H}$ C	[HK16]	

Theorem: $\frac{1}{\varepsilon} \leq H \leq H^+ = \frac{1}{\varepsilon} E(W,M) \leq H(g) \leq \frac{1}{\varepsilon}\frac{1}{\delta}$

Thank you