Fault-tolerant error correction for non-abelian anyons[1]

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QIP
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[1] arXiv:1607.02159

Outline

What are anyons[1] ?

- Localized gapped excitations living on a 2-dimensional surface

[1] A. Kitaev, Annals Phys. **321**, 2-111 (2006)

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- Each excitation is described by a unique label, called its *topological charge* from a finite set $\{a, b, c, \dots\}$
- We can imagine bringing 2 excitation together (a and b), and ask what is their total charge c.
- The possible outcomes are given by the fusion rules:

$$a \times b = \sum_c N_{ab}^c c$$

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Abelian vs non-abelian anyons

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- A Hilbert space is associated to each fusion/splitting process.
- Fusing two anyons a_1 and a_2 collapses the wavefunction into a definite super-selection sector, with probability given by Born's rule:

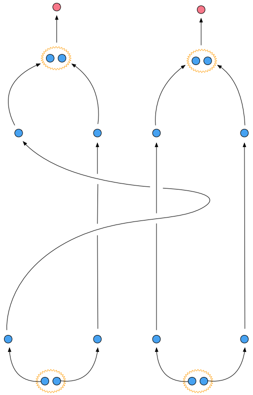
$$P(c) = \langle \psi | \Pi_c^{a_1 a_2} | \psi \rangle. \tag{1}$$

Quantum computation with non-abelian anyons[1]

Measurement

Applying gates

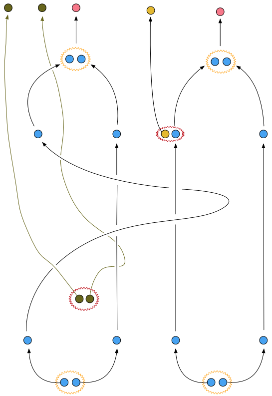
Initialization



[1] M. H. Freedman *et al.*, Commun. Math. Phys. **227**, 605-622 (2002)

Thermal processes can corrupt the information[1]

- At $T > 0$, thermal excitations are present in finite density.
- Thermal excitations can diffuse at no energy cost.
- It really is a scalibility issue: for large systems, such processes are bound to happen.



[1]F. L. Pedrocchi *et al.*, arXiv:1505.03712

Fault-tolerant error correction for non-abelian anyons

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Fault-tolerant error correction for non-abelian anyons

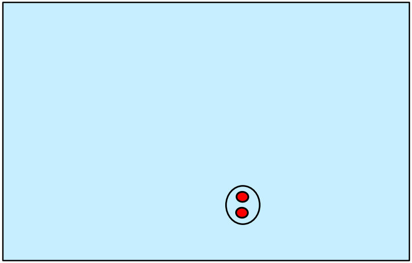
- Our goal is to find an error correction procedure for systems of non-abelian anyons
- We want to include measurement errors
- Fault-tolerant error correction for topologically ordered systems giving rise to abelian anyons have been studied extensively.[1]

[1] Dennis *et al.*, J. Math. Phys. **43**, 4452 (2002)

Anyons and topological order

Anyons appear as excitations in topologically ordered systems[1]. The ground space is degenerate and quantum information can be encoded in such states.

- Logical operations consist of creating a pair of excitations, performing non-trivial loop, and fuse the excitations back to the vacuum.
- World lines with the same topology have the same effect on the ground space.



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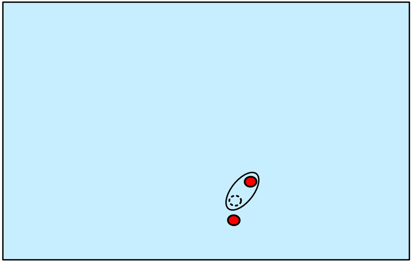

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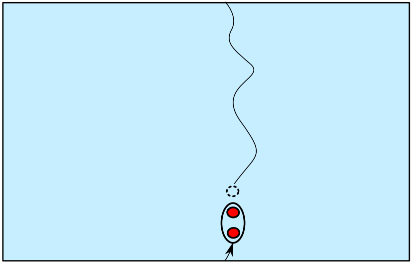

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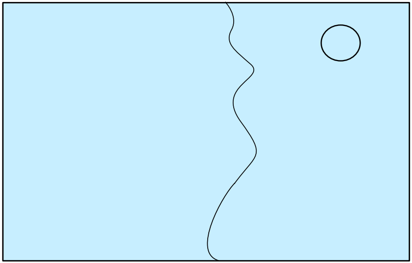

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Topological quantum error correction for abelian anyons have been extensively studied (*i.e.* the toric code)

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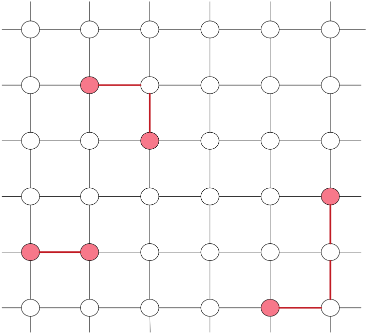

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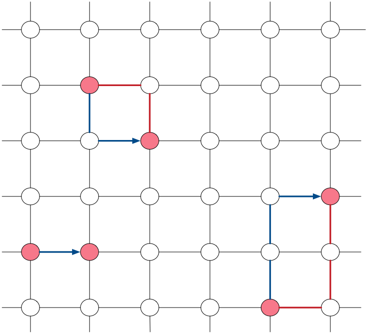

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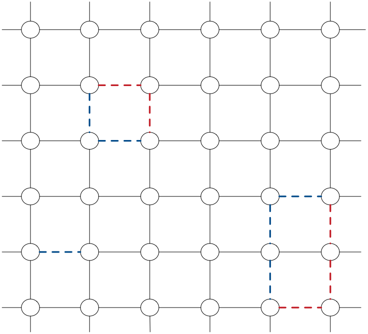
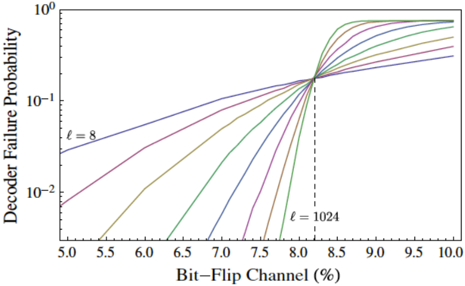
- **The correction operations are performed.**



Various families of decoding algorithms

- Perfect matching
- Mapping to statistical physics problems
- Clustering methods
- **Cellular automaton**
- **Renormalization methods**

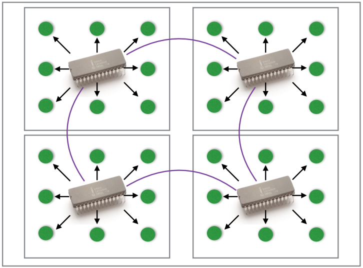


[a]

[a] G. Duclos-Cianci *et al.*, PRL **104**, 050504 (2010)

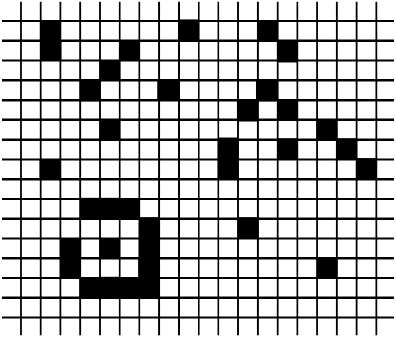
Cellular Automata

- Classical device acting on a small neighborhood
- Apply predetermined local operations depending on the state of the sites in the neighborhood
- Can communicate with neighboring automata
- Can have a memory and instruction of a programs



Emerging structure of the noise[1]

- Each actual error is characterize by a level n.
- If fits in a box of size $Q^n \times Q^n \times U^n$ and is separated by at least aQ^n sites (bU^n time steps) from other actual errors.
- The notion of *actual error* is recursively defined over the level.



[1] J. H. Harrington, Ph. D. thesis, Caltech (2004)

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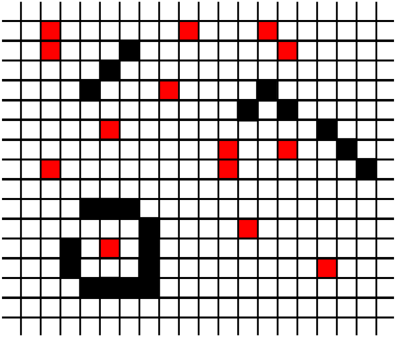

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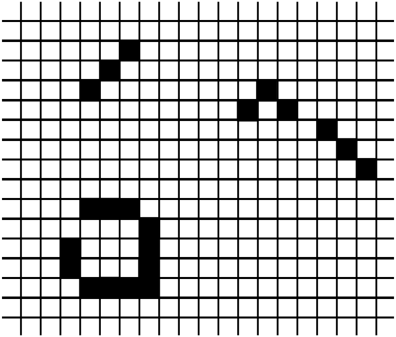

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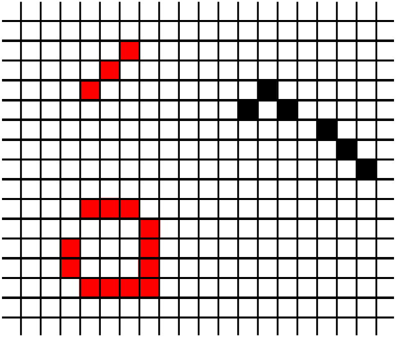

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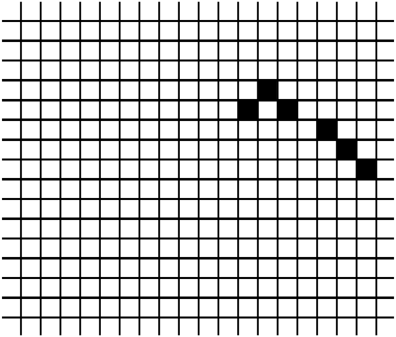

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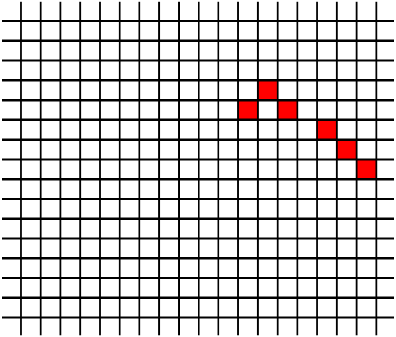

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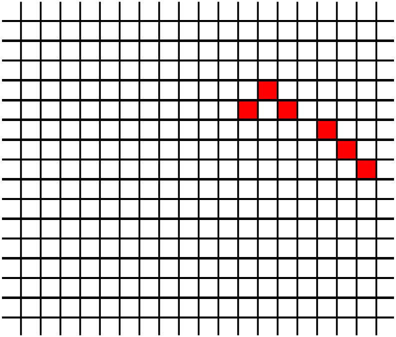
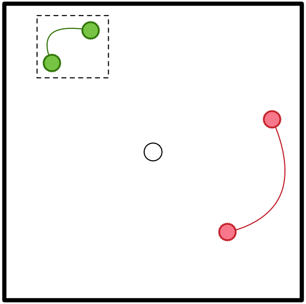


The rate of apperance of a level-n actual errors goes as $\epsilon_n \sim e^{-2^n}$

The idea behind Harrington's algorithm

- Cellular automata periodically measure topological charges.



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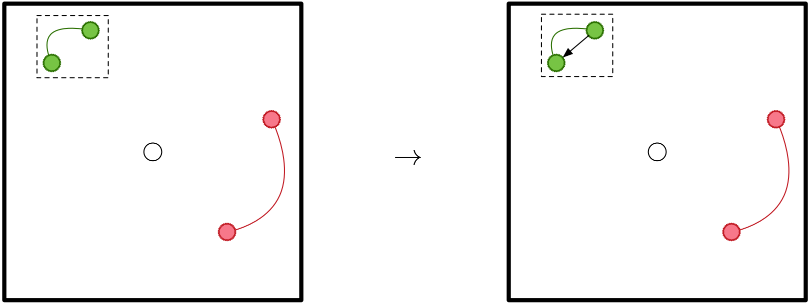

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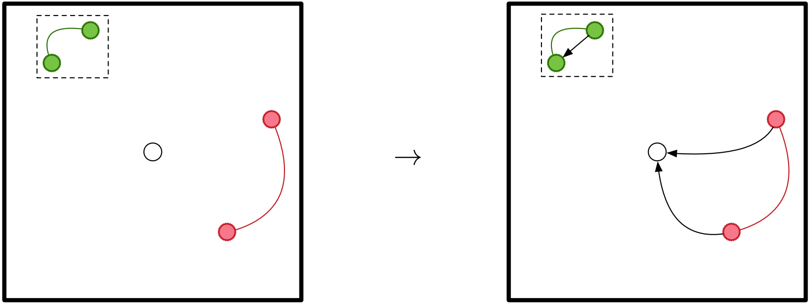
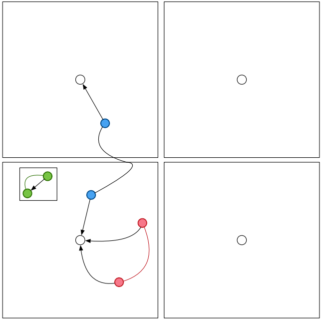
- If an excitation is isolated, it is displaced to the colony center.



The need for renormalization

- An error chain extending over 2 or more colonies cannot get corrected using such simple local rules.



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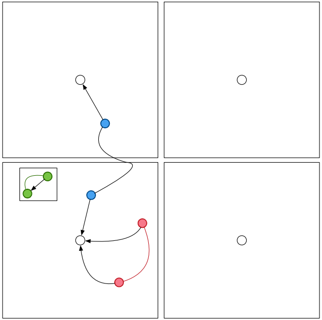

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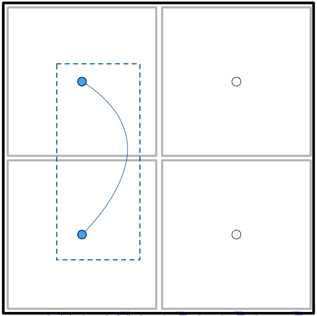

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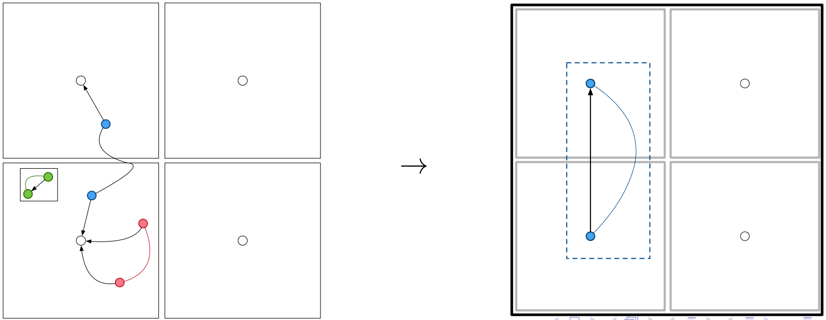
- Colonies are periodically grouped into renormalized colonies.
- Renormalized transition rules are periodically applied.



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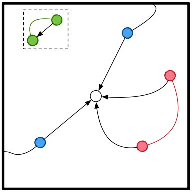
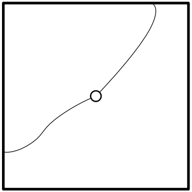
The properties above combined with the fact that $\epsilon_n \sim e^{-2^n}$ leads to the existence of a threshold.

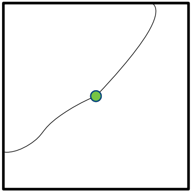
Complications for non-abelian anyons: probabilistic evolution

The fusion channel of 2 or more anyons is in general not deterministic:



$$= \alpha \qquad + \beta$$





We introduce the notion of a *trajectory domain* of an error. It roughly corresponds to the set of sites having a probability of becoming charged because of a given error.

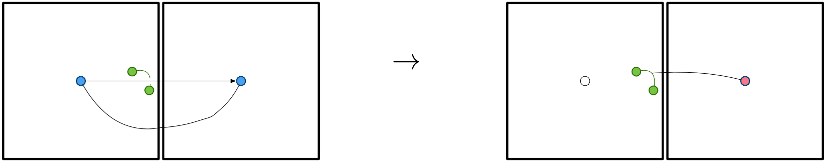
Complications for non-abelian anyons: renormalized charge

The total charge present in a colony becomes path-dependent and subject to rapid fluctuations.

The notion of *renormalized charge* needs to be carefully defined, and must include the interactions of the errors with the transition rules

Complications for non-abelian anyons: interactions between renormalization levels

The hierarchic classification of errors does not capture the 'topological interaction' between anyons caused by different actual errors.



We introduce the notion of *causally-linked clusters* of errors, sets of actual errors which can potentially interact with each others through the application of transition rules.

Key properties for non-cyclic anyons

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Non-cyclic anyons are anyons such that for any sequence of labels $\{x_0, x_1, \ldots, x_n\}$ such that $x_0 = x_n$ (and not the vacuum), then $\prod_{i=0}^{n} N_{x_i \bar{x}_i}^{x_{i+1}} = 0$.

A threshold for non-cyclic anyons

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Threshold theorem

If $\mathcal{A}$ is non-cyclic, there exists a critical value $p_c > 0$ such that if $p + q < p_c$, for any number of time steps T and any $\epsilon > 0$, there exists a linear system size $L = Q^n \in \mathcal{O}(\log \frac{1}{\epsilon})$ such that with probability of at least $1 - \epsilon$, the encoded quantum state can in principle be recovered after T time steps.

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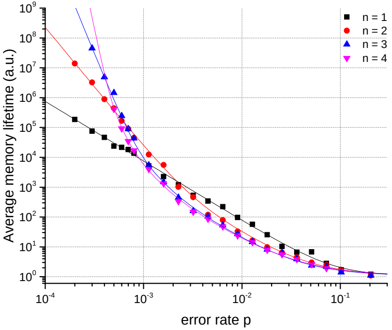
The theorem provides an upper bound on the numerical value of $p_c < 2,7 \times 10^{-20} \times (3D + 1)^{-4}$.

Numerical simulations

We performed numerical simulations for Ising anyons. They suggest a threshold in the range of $10^{-4} \sim 10^{-3}$.



Future directions

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- How do we modify the algorithm to the case where we have computational anyons ?
- How about braiding in a fault-tolerant manner ?

Thank you for your attention !