GAUSSIAN OPTIMIZERS IN QUANTUM INFORMATION

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Outline

- Gaussian channels
- Minimum output entropy conjecture
- Optimal inputs: passive states
- Attenuator
- Amplifier: p→q norms
- All one-mode Gaussian channels
- Conclusions

Gaussian systems

- Bosonic mode $[\hat{a},\ \hat{a}^\dagger] = \hat{\mathbb{I}}$
- Hamiltonian $\hat{H} = \hat{a}^\dagger \hat{a}$
- Fock basis

$$\hat{H}|n\rangle = n|n\rangle \qquad \langle m|n\rangle = \delta_{mn} \qquad m,\ n \in \mathbb{N}$$

- Gaussian thermal state

$$\hat{\rho}_G = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\ e^{-\beta \hat{H}}}$$

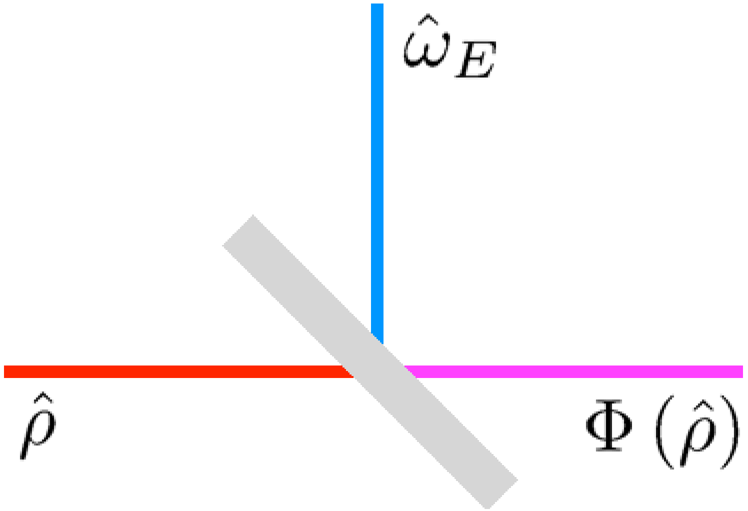
Gaussian channels

$$\hat{U}^\dagger \, \hat{a} \, \hat{U} = \sqrt{\lambda} \, \hat{a} + \sqrt{1-\lambda} \, \hat{e} \quad 0 \leq \lambda \leq 1$$ attenuator

$$\hat{U}^\dagger \, \hat{a} \, \hat{U} = \sqrt{\kappa} \, \hat{a} + \sqrt{\kappa - 1} \, \hat{e}^\dagger \quad \kappa \geq 1$$ amplifier

$$\Phi\left(\hat{\rho}\right) = \mathrm{Tr}_E \left[\hat{U} \, \hat{\rho} \otimes \hat{\omega}_E \, \hat{U}^\dagger \right]$$

$$\Phi\left(\hat{\rho}_G\right) = \hat{\rho}'_G$$



- Quantum limited

$$\hat{\omega}_E = |0\rangle\langle 0|$$

Minimum output entropy conjecture

- Minimum $S\left(\Phi\left(\hat{\rho}\right)\right)$ given $S\left(\hat{\rho}\right)$?

$$S\left(\hat{\rho}\right) = -\mathrm{Tr}\left[\hat{\rho}\ln\hat{\rho}\right]$$

$$S\left(\hat{\rho}\right) = S\left(\hat{\rho}_G\right) \quad \overset{?}{\Longrightarrow} \quad S\left(\Phi\left(\hat{\rho}\right)\right) \geq S\left(\Phi\left(\hat{\rho}_G\right)\right)$$

- Triple trade-off region of quantum-limited attenuator / amplifier
  - o classical+quantum information + entanglement sharing
  - o private+public information + key distribution
- Capacity region of degraded broadcast channel

Wilde, Hayden, Guha, Phys. Rev. Lett. **108**, 140501; Phys. Rev. A **86**, 062306 (2012)
Guha, Shapiro, Erkmen, Phys. Rev. A **76**, 032303 (2007)
Qi, Wilde, arXiv:1605.04922

Majorization and passive states

- ***Majorization***: order relation induced by convex combinations of unitary operations

$$\hat{\rho} \prec \hat{\sigma} \qquad \text{iff} \qquad \hat{\rho} = \sum_i p_i \, \hat{U}_i \, \hat{\sigma} \, \hat{U}_i^\dagger$$

$$\hat{\rho} \prec \hat{\sigma} \qquad \Longrightarrow \qquad S\left(\hat{\rho}\right) \geq S\left(\hat{\sigma}\right)$$

- ***Passive rearrangement***:

minimum average energy with given spectrum

$$\hat{\rho} = \sum_{i=0}^{\infty} p_i \, |\psi_i\rangle\langle\psi_i| \qquad \langle\psi_i|\psi_j\rangle = \delta_{ij} \qquad p_0 \geq p_1 \geq \ldots \geq 0$$

$$\Downarrow$$

$$\hat{\rho}^{\downarrow} = \sum_{i=0}^{\infty} p_i \, |i\rangle\langle i| \qquad \mathrm{Tr}\left[\hat{H}\,\hat{\rho}\right] \geq \mathrm{Tr}\left[\hat{H}\,\hat{\rho}^{\downarrow}\right]$$

Passive input states majorize output

- Optimal state with given spectrum?
- Minimize energy! (one mode)

$$\Phi\left(\hat{\rho}\right) \prec \Phi\left(\hat{\rho}^{\downarrow}\right) \qquad S\left(\Phi\left(\hat{\rho}\right)\right) \geq S\left(\Phi\left(\hat{\rho}^{\downarrow}\right)\right)$$

- Reduces minimum output entropy problem to Fock-diagonal states
- Fails for two modes

GdP, Trevisan, Giovannetti, IEEE Trans. Inf. Theory **62**(5), 2895 (2016)
GdP, Mari, Lloyd, Giovannetti, Phys. Rev. A **93**, 062328 (2016)

Minimum output entropy of attenuator

- $\mathcal{E}_\lambda \circ \mathcal{E}_{\lambda'} = \mathcal{E}_{\lambda\lambda'} \;\Rightarrow$ infinitesimal attenuator
- Isoperimetric inequality

$$S\left(\hat{\rho}\right) = S\left(\hat{\rho}_G\right) \quad \overset{?}{\Longrightarrow} \quad \left.\frac{d}{d\lambda} S\left(\mathcal{E}_\lambda\left(\hat{\rho}\right)\right)\right|_{\lambda=1} \leq \left.\frac{d}{d\lambda} S\left(\mathcal{E}_\lambda\left(\hat{\rho}_G\right)\right)\right|_{\lambda=1}$$

Proof

- Restrict to finite-rank Fock-diagonal input states
- Lagrange multipliers

GdP, Trevisan, Giovannetti, IEEE Trans. Inf. Theory **63**(1), 728 (2017)

Gaussian states achieve p→q norms

- Rényi entropy and Schatten norm

$$\|\hat\rho\|_p = \left(\mathrm{Tr}\;\hat\rho^p\right)^{\frac{1}{p}} \qquad S_p\left(\hat\rho\right) = \frac{p}{1-p}\ln\|\hat\rho\|_p \le S\left(\hat\rho\right) \qquad p>1$$

- Given ϱ_G, for any $q>1$ there exists $1<p<q$ such that for any ϱ

$$\frac{\left\|\mathcal{A}_\kappa\left(\hat\rho\right)\right\|_q}{\|\hat\rho\|_p} \le \frac{\left\|\mathcal{A}_\kappa\left(\hat\rho_G\right)\right\|_q}{\|\hat\rho_G\|_p} = \|\mathcal{A}_\kappa\|_{p\to q}$$

- Prove for attenuator, then

$$\|\mathcal{A}_\kappa\|_{p\to q} = \frac{1}{\kappa}\left\|\mathcal{E}^\dagger_{\frac{1}{\kappa}}\right\|_{p\to q} = \frac{1}{\kappa}\left\|\mathcal{E}_{\frac{1}{\kappa}}\right\|_{\frac{q}{q-1}\to\frac{p}{p-1}}$$

$$\mathrm{Tr}\left[\hat X\;\mathcal{A}_\kappa\left(\hat Y\right)\right] = \frac{1}{\kappa}\mathrm{Tr}\left[\mathcal{E}_{\frac{1}{\kappa}}\left(\hat X\right)\;\hat Y\right]$$

GdP, Trevisan, Giovannetti, arXiv:1610.09967

MOE of one-mode channels

- Given ϱ choose ϱ_G with $S(\hat{\rho}) = S(\hat{\rho}_G)$

$$S_q\left(\mathcal{A}_\kappa\left(\hat{\rho}\right)\right) \geq S_q\left(\mathcal{A}_\kappa\left(\hat{\rho}_G\right)\right) + \frac{p-1}{q-1}\frac{q}{p}\left(S_p\left(\hat{\rho}\right) - S_p\left(\hat{\rho}_G\right)\right)$$

- Limit $q \to 1$

$$\boxed{S\left(\mathcal{A}_\kappa\left(\hat{\rho}\right)\right) \geq S\left(\mathcal{A}_\kappa\left(\hat{\rho}_G\right)\right)}$$

- Attenuator + amplifier generate Gaussian channels

$$\Phi = \mathcal{A}_\kappa \circ \mathcal{E}_\lambda \quad \Longrightarrow \quad S\left(\Phi\left(\hat{\rho}\right)\right) \geq S\left(\Phi\left(\hat{\rho}_G\right)\right)$$

GdP, Trevisan, Giovannetti, arXiv:1610.09970

Conclusions

- Gaussian input states minimize output entropy of one-mode Gaussian channels for fixed input entropy
- Multimode?
- EPnI?

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