The Thermality of Quantum Approximate Markov Chains

with implications to the Locality of Edge States and Entanglement Spectrum

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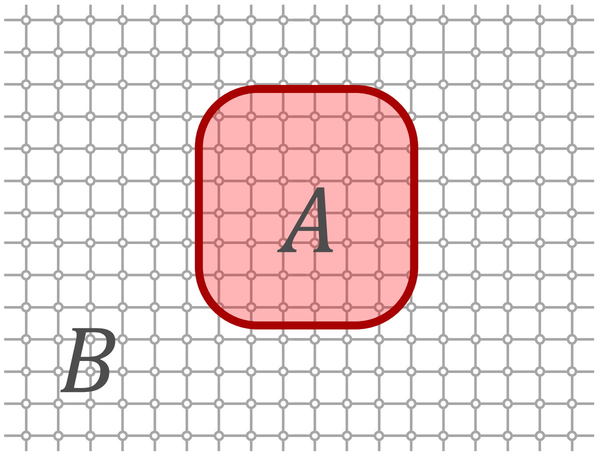
Motivation

When many-body systems are described by **local (short-range) Hamiltonians**, states have special correlation properties.

Area law for gapped ground states: restricts *entanglement* (rigorously proven for 1D systems [Hastings, 07])

Area law for Gibbs (thermal) states: restricts *correlations* (proven for any dim. [Wolf, *et al.*, 07])



⇨ efficient descriptions of many-body states (MPS, PEPS, MPO,...)

A useful consequence of area laws: Q. How to characterize ?

small "conditional mutual information (CMI)" on certain regions

(Applications: [Kim, '12,'13], [Swingle & Kim, 14], [Kastryano & Brandao, '16] ...)

Motivation

When many-body systems are described by **local (short-range) Hamiltonians**, states have special correlation properties.

This talk:

"approximate Markov chains"
↓

1. Characterizing states with small CMI in terms of Gibbs states
(cf. previous talk by Kastoryano)
2. An application to "entanglement spectrum" of 2D gapped systems

Area law for gapped ground states: restricts *entanglement*

Area law for Gibbs (thermal) states: restricts *correlations*
(proven for any dim. [Wolf, *et al.*, 07])

⇨ efficient descriptions of many-body states (MPS, PEPS, MPO,...)

A useful consequence of area laws: Q. How to characterize ?
small "conditional mutual information (CMI)" on certain regions
(Applications: [Kim, '12,'13], [Swingle & Kim, 14], [Kastryano & Brandao, '16] ...)

Outline of this talk

Part I: A characterization of approximate Markov chains

- ◆ Area law for Gibbs States
- ◆ Quantum Markov Chains & Approximate Quantum Markov Chains
- ◆ Equivalence to Gibbs states of short-range Hamiltonians

Part II: An application to entanglement spectrum in 2D systems

- ◆ Topological Entanglement Entropy and Entanglement Spectrum
- ◆ Previous Results on Entanglement Spectrum
- ◆ Locality of Entanglement Hamiltonian and Spectrum

Part I:
A characterization of approximate Markov chains

Area law for Gibbs states

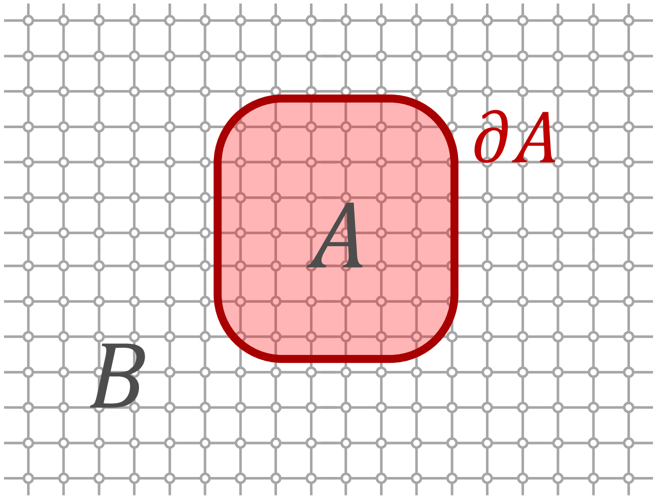
Hamiltonian

WLOG: nearest-neighbor

$$H = \sum_i h_{i,i+1}, \quad \|h_i\| \le J.$$

Gibbs state

$$\rho = \frac{1}{Z} e^{-\beta H}, \qquad Z = \mathrm{tr} e^{-\beta H}.$$



[Wolf, et al., '07]

$$I(A{:}B)_\rho := S(A)_\rho + S(B)_\rho - S(AB)_\rho \le 2\beta J |\partial A|$$

➢ $S(A)_\rho := -\mathrm{tr}\rho_A \log_2 \rho_A$

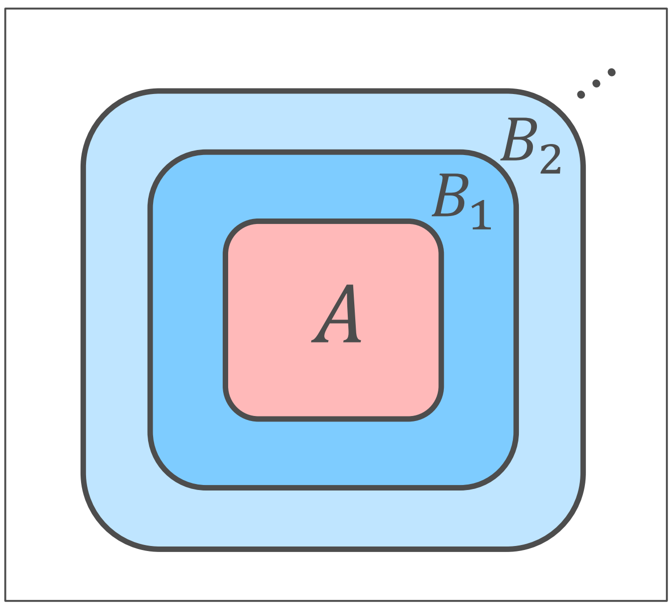
Conditional Mutual Information of Gibbs States

The conditional mutual information:

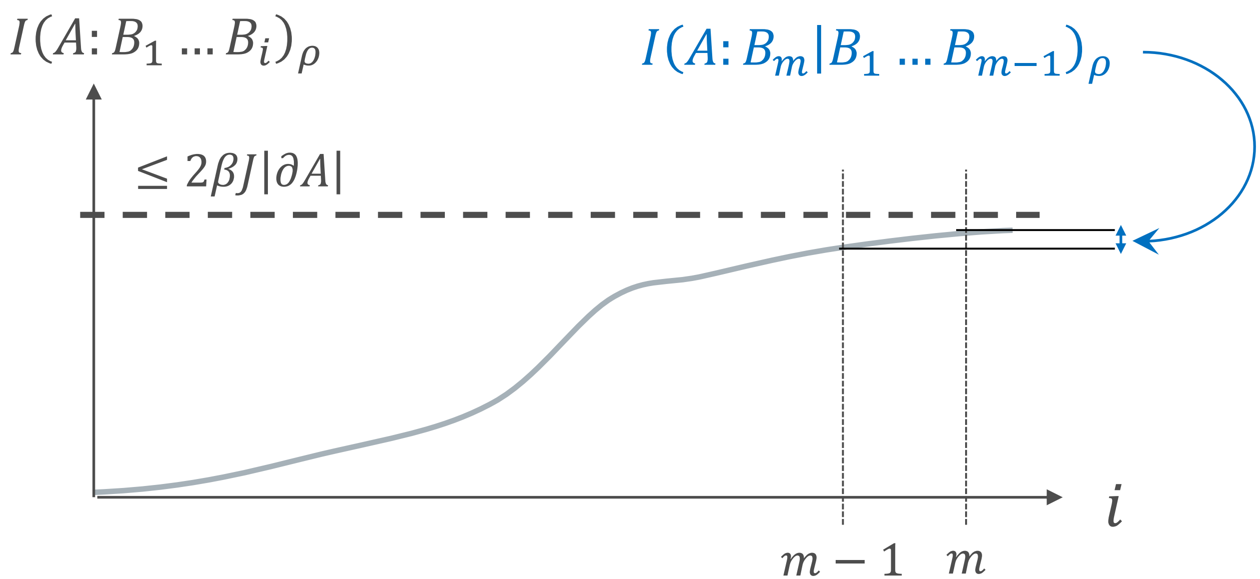
$$I(A\!:\!C|B)_\rho \coloneqq I(A\!:\!BC)_\rho - I(A\!:\!B)_\rho \geq 0$$

- Monotonicity of MI: $I(A\!:\!BC)_\rho \geq I(A\!:\!B)_\rho$

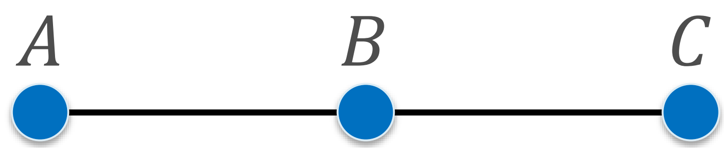
$$\rightarrow I(A\!:\!B_1)_\rho \leq I(A\!:\!B_1B_2)_\rho \leq \cdots \leq I(A\!:\!B_1 \ldots B_m)_\rho \leq 2\beta J|\partial A|$$



small for large m !



Quantum Markov Chain (for three systems)

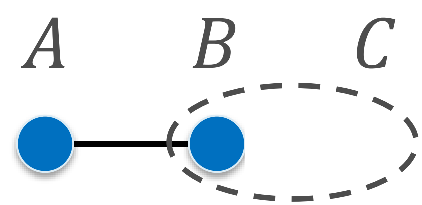


If $I(A\!:\!C|B)_\rho = 0$, quantum state ρ_{ABC} is called a *Quantum Markov Chain* $A - B - C$.

$\updownarrow$ [Hayden, et al., 03], [Brown & Poulin, '12]

1. There exists a CPTP-map $\Lambda_{B\to BC}\!: B \to BC$ s.t.

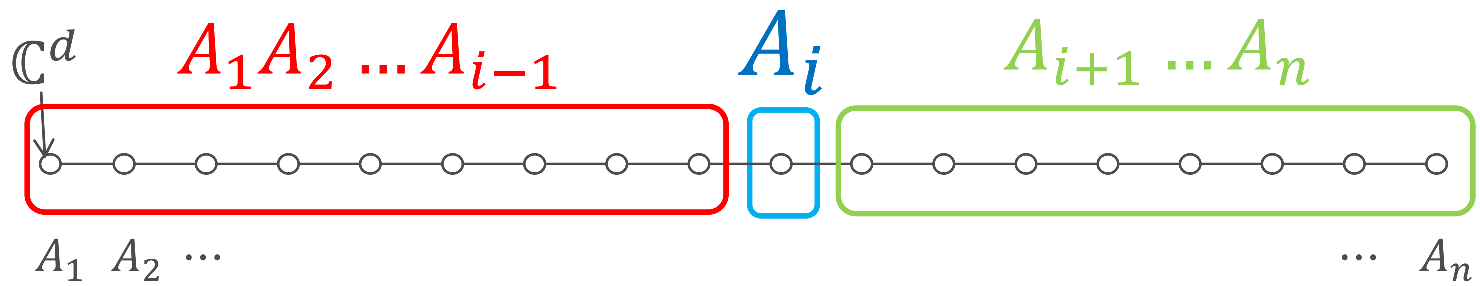
$$\rho_{ABC} = \mathrm{id}_{\mathrm{A}} \otimes \Lambda_{\mathrm{B}\to BC}(\rho_{AB})$$



2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.

$$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0 \ \ (\rho_{ABC} > 0)$$

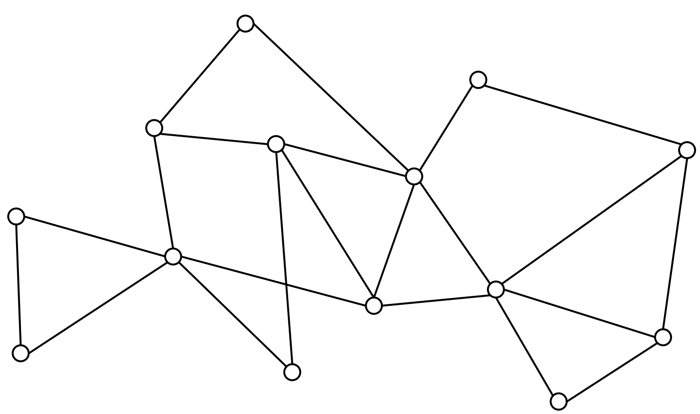
Longer Chains



ρ_A on the chain $A_1 A_2 \dots A_n$ is a (quantum) Markov chain if

$$I(A_1 \dots A_{i-1} : A_{i+1} \dots A_n | A_i)_\rho = 0$$

for arbitrary $i \in [n]$.



*We can generalize the concept of Markov chains to general graphs as *Markov networks*

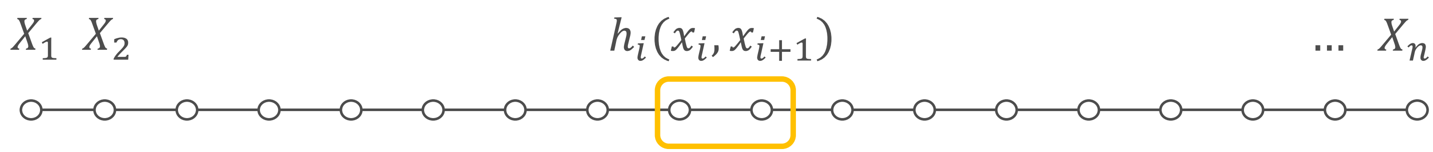
Hammersley-Clifford Theorem (1D)

[Hammersley&Clifford, '71]:

Random variables $X_1, X_2, \ldots, X_n$ forms a (positive) Markov chain

if, and only if, the distribution can be written as

$$p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \frac{1}{Z} \exp\left(-\sum_i h_i(x_i, x_{i+1})\right)$$



* also holds for Markov networks

Positive Markov chains

↕

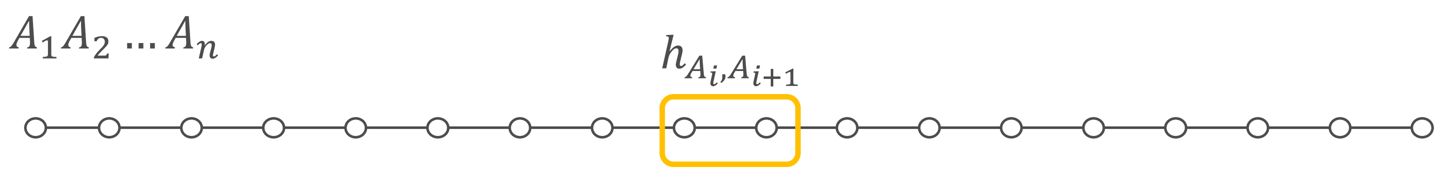
Gibbs distributions of 1D *short-range* Hamiltonians

Quantum Hammersley-Clifford Theorem (1D)

[Leifer & Poulin, '08], [Brown & Poulin, '12]:

A quantum state $\rho_{A_1 \ldots A_n} > 0$ on a chain forms a Markov chain if, and only if, the state can be written as

$$\rho_{A_1 \ldots A_n} = \frac{1}{Z} \exp\left(-\sum_i h_{A_i,A_{i+1}}\right), \qquad \left[h_{A_i,A_{i+1}}, h_{A_j,A_{j+1}}\right] = 0$$



* also holds for Markov networks

Positive **quantum** Markov chains

2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.

$$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0$$

Gibbs states of 1D **commuting** short-range Hamiltonians

Quantum Hammersley-Clifford Theorem (1D)

[Leifer & Poulin, '08], [Brown & Poulin, '12]:

A quantum state $\rho_{A_1 \dots A_n} > 0$ on a chain forms a Markov chain if, and only if, the state can be written as

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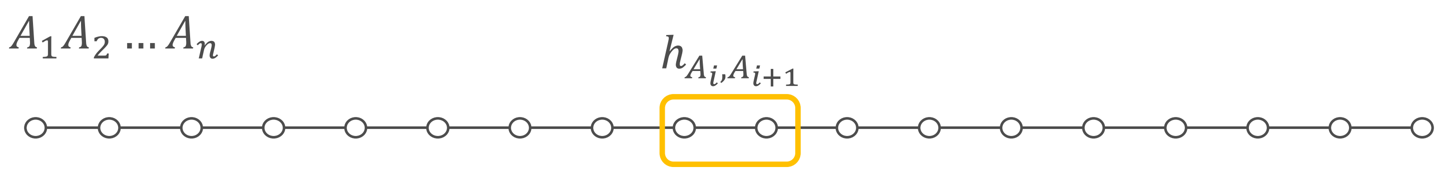

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$$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0$$

Properties of Approximate Markov Chains

How about states having small but non-zero CMI?

Naïve guess: all properties of Markov chains ***approximately*** hold for *approximate* Markov chains

Classical:

relative entropy

$$I(X\!:\!Z|Y)_p = \min_{q:\mathrm{Markov}} S(p_{XYZ}||q_{XYZ})$$

$$\Rightarrow \quad I(X\!:\!Z|Y)_p \leq \varepsilon \leftrightarrow p_{XYZ} \approx_{\varepsilon} q_{XYZ}$$

However…

Quantum:

$$I(A\!:\!C|B)_\rho \neq \min_{\sigma:\mathrm{Markov}} S(\rho_{ABC}||\sigma_{ABC})$$ [Ibinson, et al., '06]

∃ property of Markov chains which is invalid for approximate Markov chains

Local Recoverability of States with Small CMI

Some properties still approximately hold for approximate Markov chains

[Fawzi & Renner, '15]:
There exists a CPTP-map $\Lambda_{B\to BC}$ s.t.

$$I(A:C|B)_\rho \geq -2\log_2 F\left(\rho_{ABC}, \Lambda_{B\to BC}(\rho_{AB})\right)$$

$$I(A:C|B)_\rho \approx 0$$

$$\leftrightarrow$$

1. There exists a CPTP-map $\Lambda_{B\to BC}: B \to BC$ s.t.

$$\rho_{ABC} \approx \mathrm{id_A} \otimes \Lambda_{\mathrm{B}\to BC}(\rho_{AB})$$

*The converse part can be shown by using the Alicki-Fannes inequality.

Question

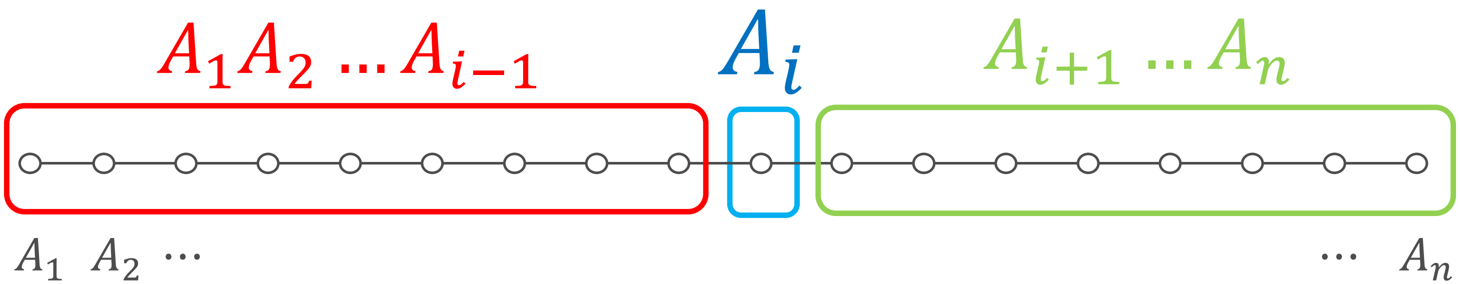
Q. How about the quantum Hammersley-Clifford theorem for approximate Markov chains ?

Quantum **approximate** Markov chains

↕

Gibbs states of 1D **short-range** Hamiltonians

Approximate Quantum HC Theorem (1D)



ρ_A is a ε −approximate Markov chain if

$$I(A_1 \dots A_{i-1} : A_{i+1} \dots A_n | A_i)_\rho \leq \varepsilon$$

for arbitrary $i \in [n]$.

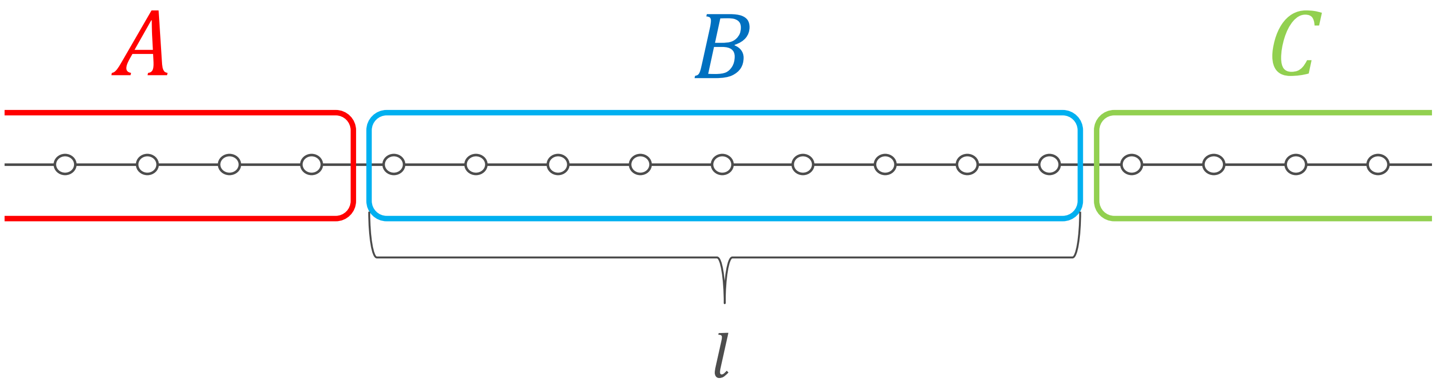
Result 1.
For any ε −approximate Markov chain $\rho_{A_1 A_2 \dots A_n}$, there exists a Hamiltonian $H_A = \sum h_{A_i A_{i+1}}$ s.t.,

$$S(\rho_A || e^{-H_A}) \leq n\varepsilon.$$

Application to gapped systems (next part)

Any approximate Markov chain can be approximated by local Gibbs states

Approximate Quantum HC Theorem (1D)



Result 2.
For any Gibbs state ρ of a short-range Hamiltonian H at temperature T

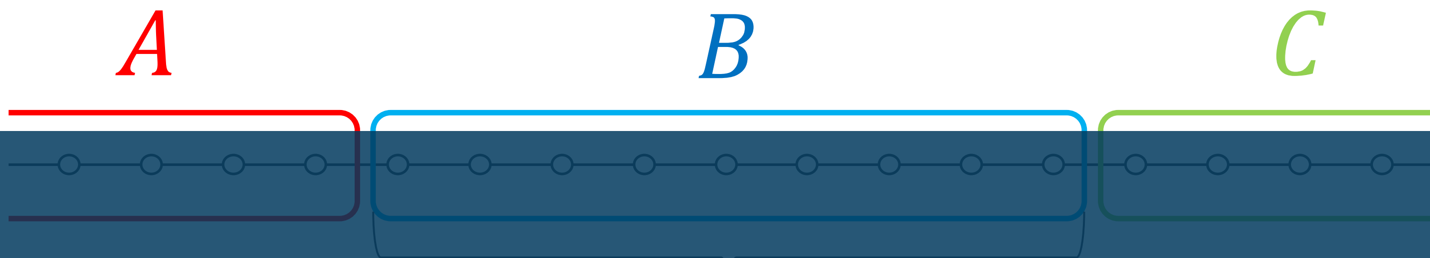
$$I(A:C|B)_\rho \leq ce^{-q(T)\sqrt{l}}$$

for $q(T) = e^{-c'T^{-1}}, c \geq 0, c' > 0$ and any partition ABC as in t

Application to Gibbs state preparation (see previous talk)

All 1D Gibbs states of short-range Hamiltonians are approximate Markov chains
(Strengthen the area law of 1D Gibbs states)

Approximate Quantum HC Theorem (1D)



Quantum **approximate** Markov chains

≈

Gibbs states of 1D **short-range Hamiltonian**

Result 2.
For any Gibbs state ρ of a short range Hamiltonian H at temperature T

$$I(A:C|B)_\rho \leq c e^{-q(T)\sqrt{l}}$$

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Application to Gibbs state preparation
(see previous talk)

All 1D Gibbs states of short-range Hamiltonians are approximate Markov chains
(Strengthen the area law of 1D Gibbs states)

PartII:
An application to entanglement spectrum in 2D systems

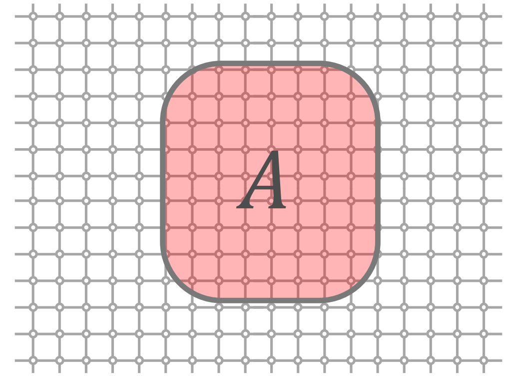
Area Law in 2D Gapped Systems

- Ground states of 2D gapped local Hamiltonians typically obey area law:

\# of boundary

$$S(A)_\rho = \alpha|\partial A| - n_{\partial A}\gamma + o(1)$$

$o(1) \to 0 \ (|\partial A| \to \infty)$



➢ γ: topological entanglement entropy
[Kitaev & Preskill, '06], [Levin & Wen '06]
($\gamma > 0 \leftrightarrow$ the g.s. is in a *topologically ordered phase* (?))

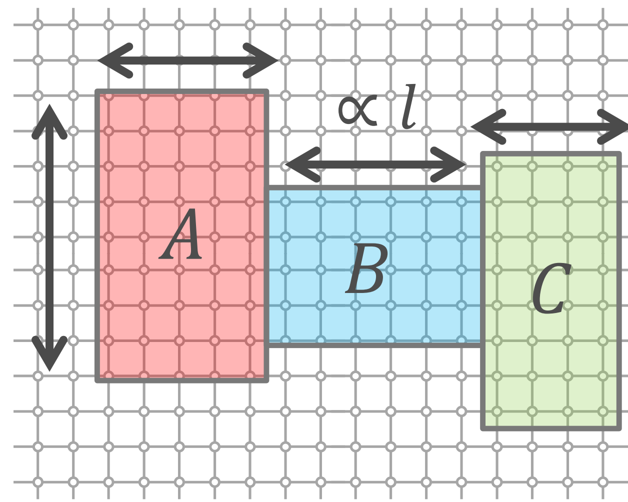
A strong type of area law (rest of this talk)

$$S(A)_\rho = \alpha|\partial A| - n_{\partial A}\gamma + e^{-|\partial A|/\xi}$$

⮡ For any ABC with no holes,

$$I(A{:}C|B)_\rho \le e^{-cl}$$

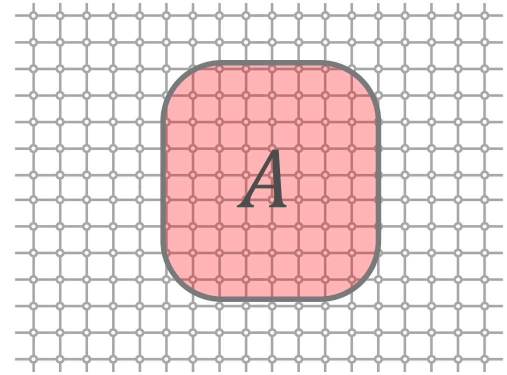
ρ_{ABC} is an approximate Markov chain



Entanglement Hamiltonian and Spectrum

- Other tools to study gapped g.s.

$$\rho_A =: e^{-H_A}$$ ← entanglement Hamiltonian



$\lambda(H_A)$: entanglement spectrum

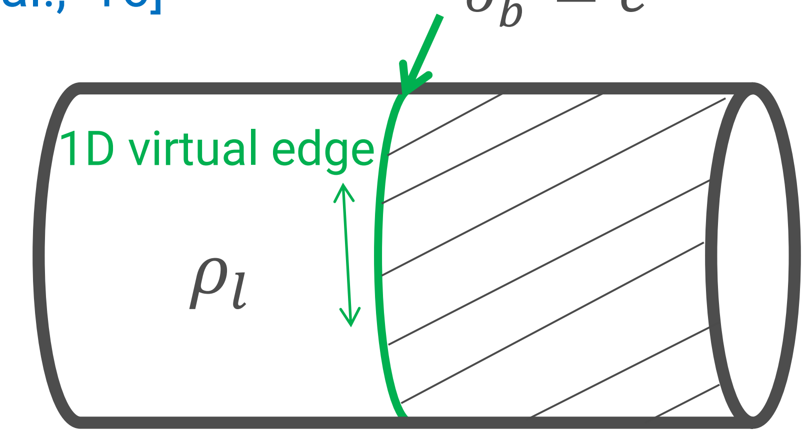
(logarithm of the Schmidt coefficients)

◆ Correspondence to edge theory in FQHE [Li & Haldane, '08]
also has been studied in other systems [Ali, et al., '09, Lauchli & Bergholtz, '10,…]

◆ Previous observations in the PEPS formalism
[Cirac et al., '11], [Schuch, et al., '13], [Cirac, et al., '16]

$$\sigma_b^2 = e^{-H_b}$$

$$\rho_l = V\sigma_b^2 V^\dagger \qquad V\text{:isometry}$$

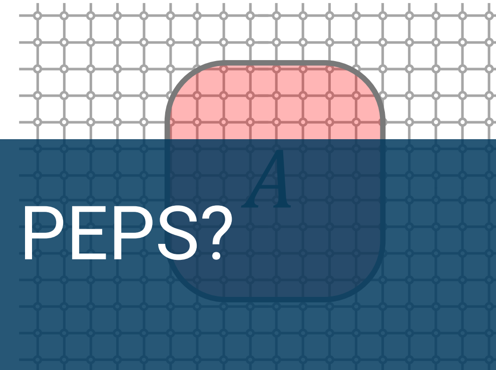


$H_b =$
- short-range (in trivial phase)
- short-range + **global interactions** (in topologically ordered phases)

Entanglement Hamiltonian and Spectrum

- Other tools to study gapped g.s.

$$\rho_A =: e^{-H_A}$$ ← entanglement Hamiltonian



Q. How general this observation in PEPS?

$\lambda(H_A)$: entanglement spectrum

(logarithm of the Schmidt coefficients)

This talk: connection to the *topological entanglement entropy*

◆ also has been studied in other systems [Ali, et al., '09, Lauchli & Bergholtz, '10,...]

◆ Previous observations in the PEPS formalism
[Cirac et al., '11], [Schuch, et al., '13], [Cirac, et al., '16]

$$\rho_l = V\sigma_b^2 V^\dagger \quad V\text{:isometry}$$

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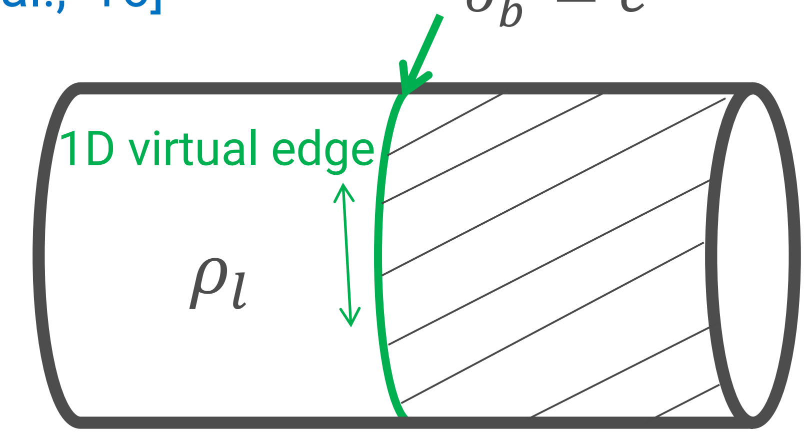


$H_b =$
- short-range (in trivial phase)
- short-range + **global interactions** (in topologically ordered phases)

Locality of Entanglement Spectrum ($\gamma = 0$)

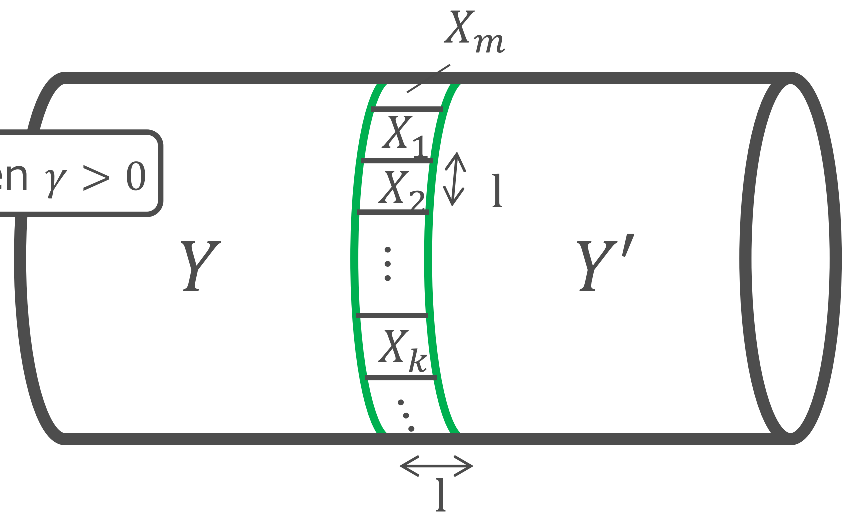
Suppose $|\psi_{YXY'}\rangle$ satisfies the area law and $\gamma = 0$ (trivial phase).

$\rightarrow \rho_{X_1 \dots X_m}$ is an approx. Markov chain

Not true when $\gamma > 0$

Result 1.

$\rightarrow \rho_{X_1 \dots X_m} \approx \frac{1}{Z} \exp(-\sum h_{X_i X_{i+1}})$



- $|\psi_{YXY'}\rangle$ is pure $\rightarrow \lambda(\rho_{YY'}) = \lambda(\rho_{X_1 \dots X_m})$
- $I(Y:Y')_\rho = I(Y:Y'|X)_\psi \approx 0 \rightarrow \rho_{YY'} \approx \rho_Y \otimes \rho_{Y'} = \rho_Y^{\otimes 2}$

assume reflection sym.

$$H_Y^{(2)} := \log\rho_Y \otimes I + I \otimes \log\rho_Y$$

$$\Rightarrow \left\| \lambda\left(H_Y^{(2)}\right) - \lambda\left(\sum h_{X_i X_{i+1}}\right) \right\|_1 \leq e^{-cl}$$ for some $c > 0$.

TEE and Non-Local Entanglement Hamiltonian

How about the case of $\gamma > 0$?

Result 3.
Under our assumption, for some $c > 0$ and sufficiently large l,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}) + e^{-cl} \geq 0 \ (l \gg 1)$$

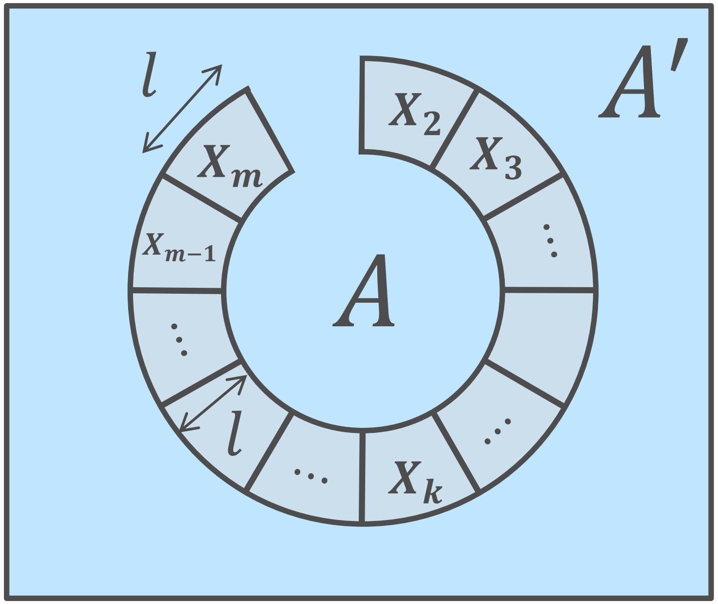
- $\mathcal{H}_2 := \{H = \sum h_{X_i X_{i+1}}, \|h_{X_i X_{i+1}}\| \leq \mathcal{O}(|X|)\}$

$\gamma > 0 \ \rightarrow -\log\rho_X$ is non-local

Note: EH is local after tracing out X_i.

$$\mathrm{tr}_{X_1} e^{-H_X} = \exp(-h_{X_2 X_3} \cdots - h_{X_{m-1} X_m})$$

Conjecture (no rigorous proof):
The non-local part is dominated by m-**body** interactions



Non-Locality of Entanglement Spectrum ($\gamma > 0$)

Result 3.
Under our assumption, for some $c > 0$ and sufficiently large l,

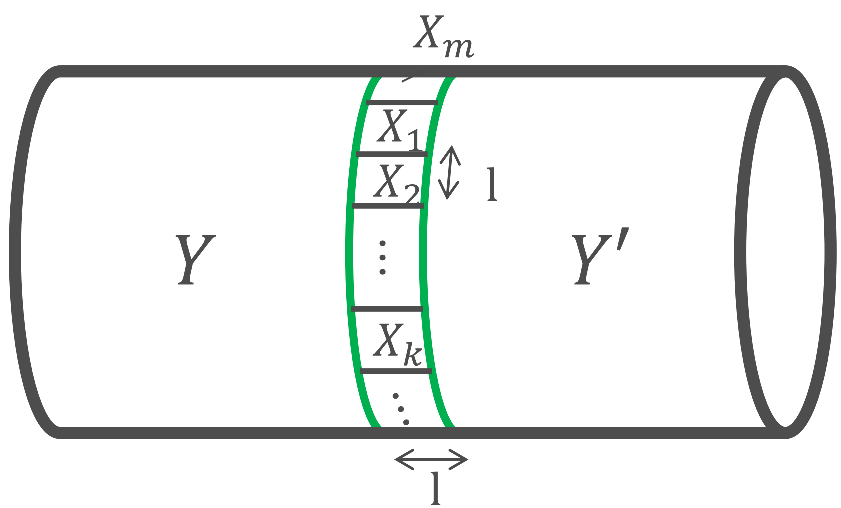
$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}) + e^{-cl}$$

➢ $\mathcal{H}_2 := \{H = \sum h_{X_i X_{i+1}}, \|h_{X_i X_{i+1}}\| \leq \mathcal{O}(|X|)\}$

⇓

$$\left\| \lambda\left(H_Y^{(2)}\right) - \lambda(H_X) \right\|_1 \leq e^{-cl}$$

for a **non-local** H_X.

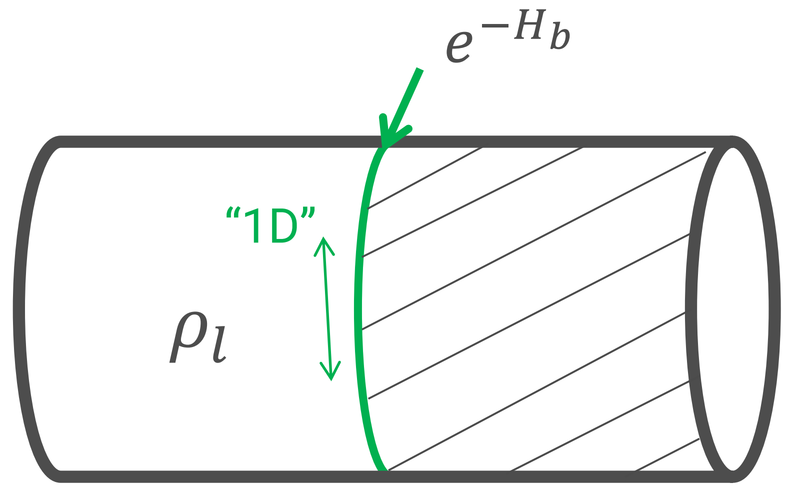


Difference to The Previous Results

Assumption: PEPS formalism (fixed-point) [Cirac et al., '11], [Schuch, et al., '13], [Cirac, et al., '16]

$$\lambda(-\log\rho_l) = \lambda(H_b)$$

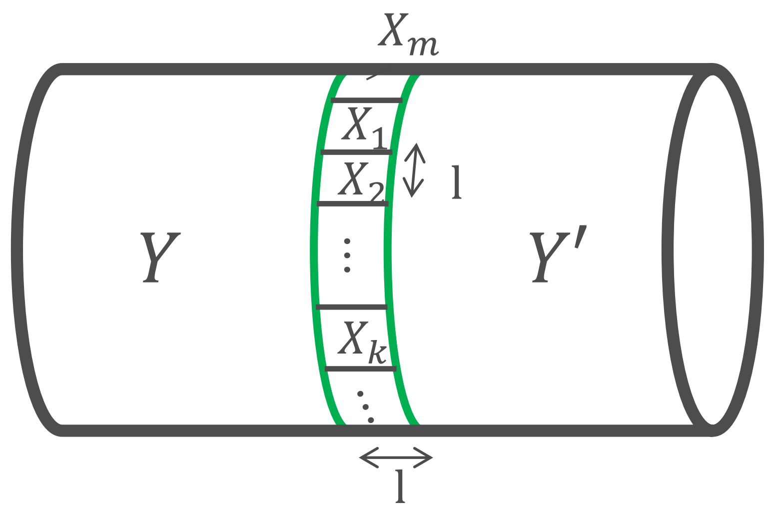
$H_b =$
- short-range (in trivial phase)
- short-range + **global interactions** (in topologically ordered phases)



Assumption: Strong type of area law (+ reflection symmetry) this talk

$$\left\|\lambda\left(H_Y^{(2)}\right) - \lambda(H_X)\right\|_1 \leq e^{-cl}$$

$H_X =$
- short-range ($\gamma = 0$)
- short-range + **global interactions** ($\gamma > 0$)



Summary

Take-home massages:

Part I: Quantum approximate Markov chains are Gibbs states of 1D short-range Hamiltonians.

Part II: The locality of the entanglement spectrum of gapped g.s. on a cylinder is related to the TEE.

Open problems:

Part I: Better bounds on CMI of 1D Gibbs states?
Generalization of the equivalence to Markov networks?
(→ application for Gibbs state preparation)

Part II: Weaker assumptions?
Do we really need double of the ES?
Consequences of the (non-)locality of ES?


Thank You!

Idea of the proof

Result 1.
For any ε −approximate Markov chain $\rho_{A_1 A_2 \dots A_n}$, there exists a Hamiltonian $H_A = \sum h_{A_i A_{i+1}}$ s.t.,

$$S(\rho_A || e^{-H_A}) \le n\varepsilon.$$

- The maximum entropy principle [Jaynes, '57]

The maximum entropy state σ_A satisfying

$$\sigma_{A_i A_{i+1}} = \rho_{A_i A_{i+1}}, \forall i$$

has the form

$$\sigma_{A_i A_{i+1}} = e^{-\sum h_{A_i A_{i+1}}}.$$

- A result from information geometry [Knauf & Weis, '10]

$$\inf_{H_A = \sum h_{A_i A_{i+1}}} S(\rho_A || e^{-H_A}) = S(A)_\rho - S(A)_\sigma$$

Small by the assumption + SSA

Idea of the proof

Result 2.
For any Gibbs state ρ of a short-range Hamiltonian H at temperature T,

$$I(A:C|B)_\rho \leq ce^{-q(T)\sqrt{l}}$$

for $q(T) = e^{-c'T^{-1}}$, $c \geq 0, c' > 0$ and any partition ABC as in the below.

Fannes inequality

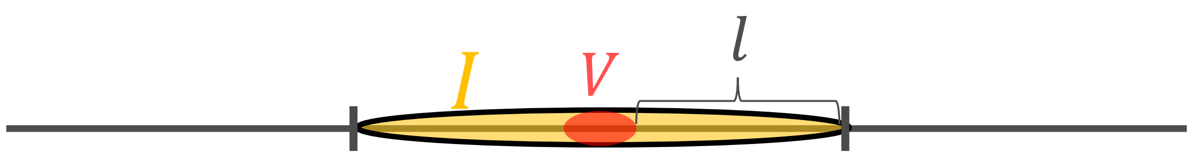
Explicitly construct a recovery map $\Lambda_{B\to BC}$ s.t.

$$\|\rho_{ABC} - \Lambda_{B\to BC}(\rho_{AB})\|_1 \leq c'e^{-q'\sqrt{l}}$$

- *Quantum belief propagation equation* [Hastings, '07][Kim, '11]

For 1D Hamiltonian with short-range H, $\exists O_I$ s.t.

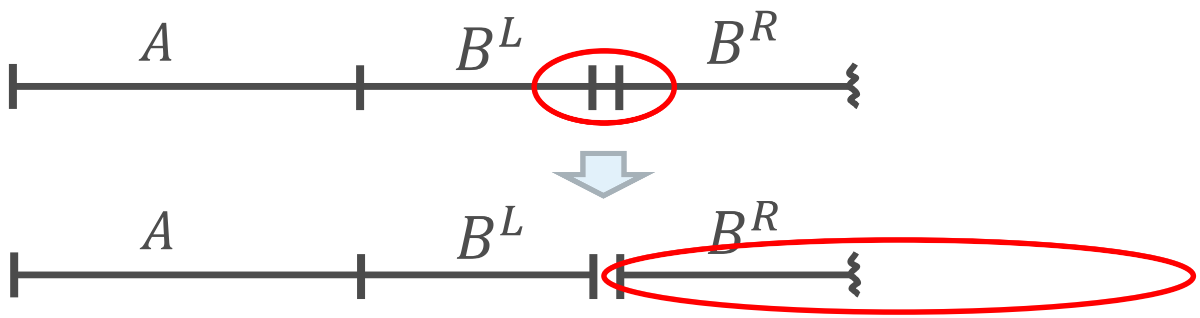
$$\left\|e^{-\beta(H+V)} - O_I e^{-\beta H} O_I^\dagger\right\| \leq e^{-q''l}$$



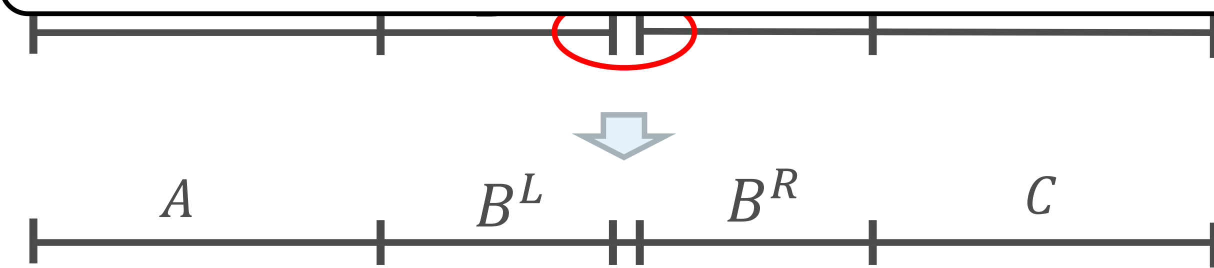
Idea of the proof

From the quantum belief propagation equation, there exists X_B s.t.

$$\rho_{ABC} \approx \kappa_{B\to \mathrm{BC}}(\rho_{AB}) = X_B\left(\mathrm{tr}_{B^R}\left[X_B^{-1}\rho_{AB}(X_B^{-1})^\dagger\right] \otimes \rho_{B^R C}\right)X_B^\dagger$$

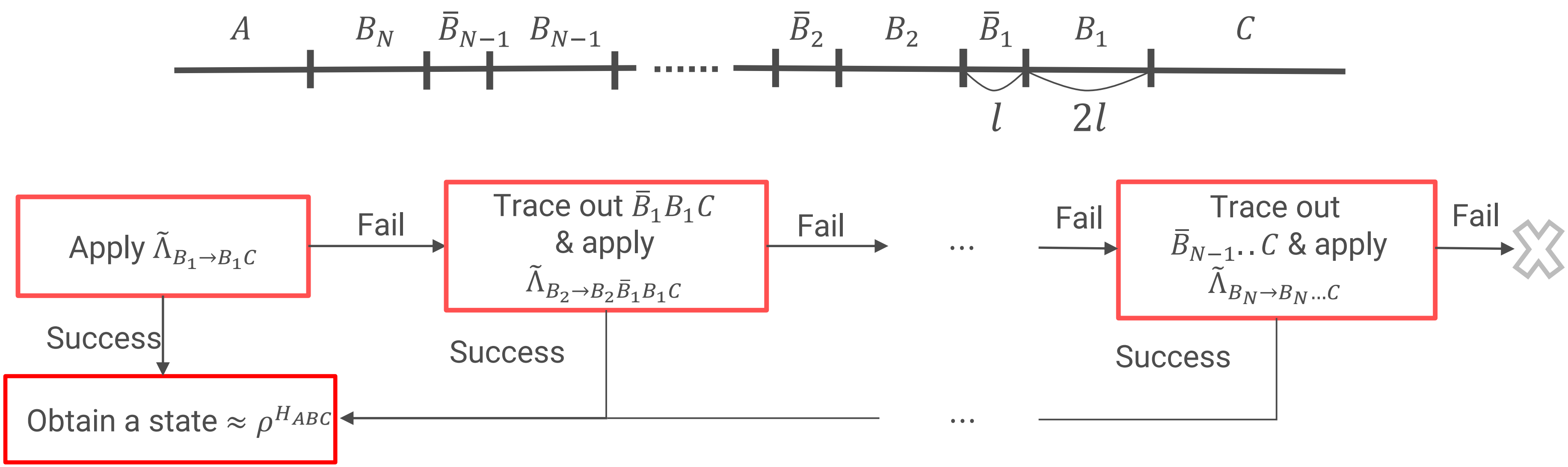


Note: Probably $\kappa_{B\to BC}$ is not a quantum operation



Repeat-until-success method

We normalize $\kappa_{B\to BC}$ and define a CPTD-map $\tilde{\Lambda}_{B\to BC}$.
→ Succeed to recover with a constant probability p (in 1D systems).



❑ Choose $N \sim l \left(|B| = \mathcal{O}(l^2)\right)$.

We can construct a CPTP-map $\Lambda_{B\to BC}$ satisfying

$$\|\rho_{ABC} - \mathrm{id}_A \otimes \Lambda_{B\to BC}(\rho_{AB})\|_1 \leq e^{-\mathcal{O}(l)}.$$

Idea of the proof

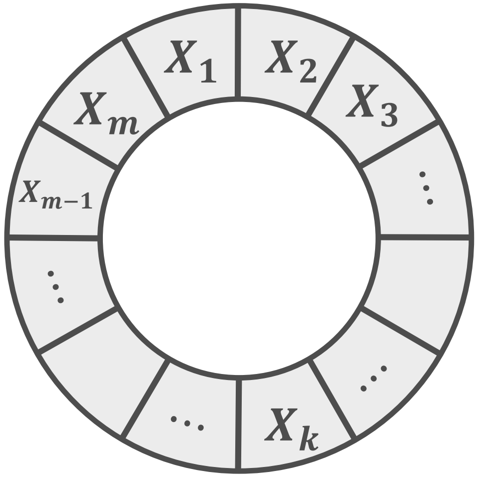
Result 3.
Under our assumption, for some $c > 0$ and sufficiently large l,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}) + e^{-cl}$$

- $\mathcal{H}_2 := \{H = \sum h_{X_i X_{i+1}}, \|h_{X_i X_{i+1}}\| \leq \mathcal{O}(|X|)\}$

By assumption, $I(X_1 : X_3 X_{m-1} | X_2 X_m)_\rho \approx 0$.
$\rightarrow \exists$ recovery map $\Lambda_{2m \rightarrow 12m} : X_2 X_m \rightarrow X_2 X_m X_1$

$$\sigma_X := \Lambda_{2m \rightarrow 12m}(\rho_{X_2 \ldots X_m})$$



Facts: $\sigma_{X_i X_{i+1}} \approx \rho_{X_i X_{i+1}}$

$$\rightarrow \sigma_X \approx \operatorname*{argmin}_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}), \qquad S(\rho_X || \sigma_X) \approx 2\gamma.$$