Caltech

UNIVERSITY OF COPENHAGEN

FINITE CORRELATION LENGTH IMPLIES EFFICIENT PREPARATION OF GIBBS STATES

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CARLSBERGFONDET

VILLUM FONDEN

CONTENTS

Motivation

Local recovery for many body systems

Exact and approximate recovery

State preparation

- Evaluating local expectation values
- Efficient state preparation

Further Applications

MOTIVATION

Finite temperature quantum simulations

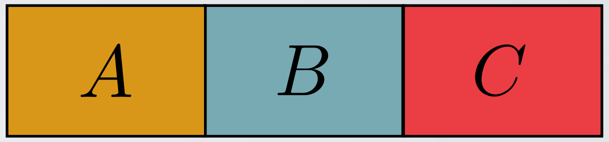
- Strongly correlated/frustrated materials
- Quantum SDP solvers, Quantum machine learning

New tools for the analysis of many body systems

- Local recovery in many body systems
- Exotic phases/topological order

STRONG SUB-ADDITIVITY

Strong sub-additivity (SSA):



$$I_\rho(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0$$

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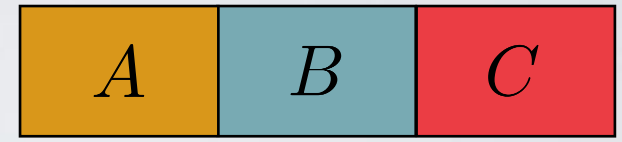

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Area Law for mixed states:

$$I(A:A^c) \equiv S(A) + S(A^c) - S(AA^c) \leq c|\partial A|$$

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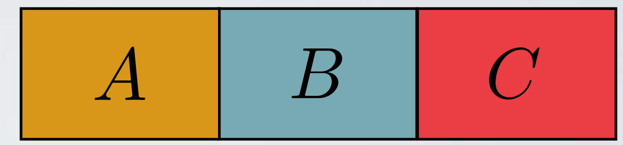

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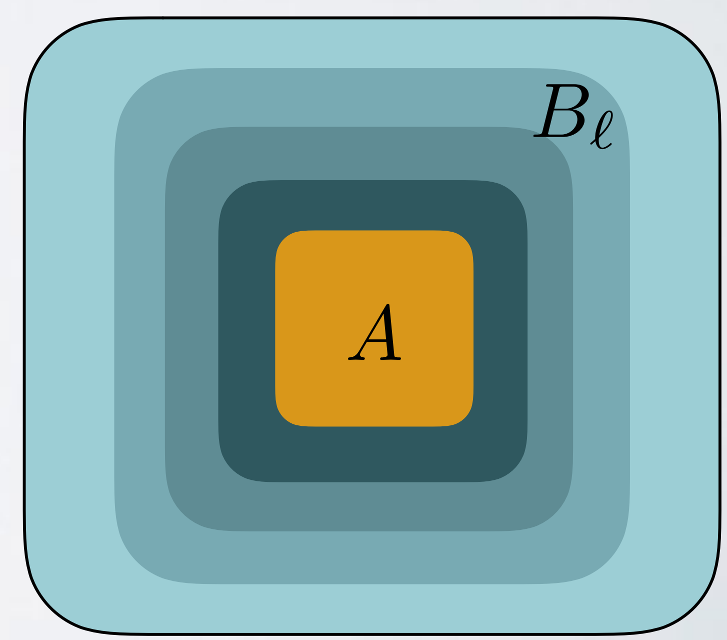
Area Law for mixed states:

$$I(A:A^c) \equiv S(A) + S(A^c) - S(AA^c) \leq c|\partial A|$$

➡ Quantitative extension to the Area Law

$$I(A:B_1 \cdots B_{n+1}) - I(A:B_1 \cdots B_n) = I(A:B_{n+1}|B_1 \cdots B_n)$$



Tells us how rapidly the area law is saturated

LOCAL RECOVERY MAPS

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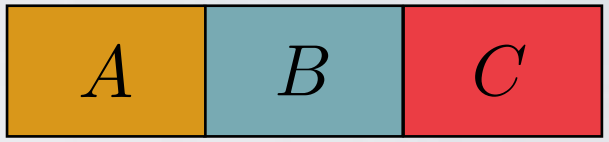


$$I_\rho(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0$$

Equality

$$I_\rho(A:C|B) = 0 \qquad \Leftrightarrow \qquad R_{AB}(\rho_{BC}) = \rho$$

Petz map

$$R_{AB}(\sigma) = \rho_{AB}^{1/2} \rho_B^{-1/2} \sigma \rho_B^{-1/2} \rho_{AB}^{1/2}$$

M. Ohya and D. Petz, (2004)

Markov State

$$\rho = \oplus_j \rho_{AB_j^L} \otimes \rho_{B_j^R C}$$

P. Hayden, et. al., CMP 246 (2004)

there exists a disentangling unitary on B.

Approximately LOCAL RECOVERY MAPS

Strengthening SSA:

$$I_\rho(A:C|B) \geq -2\log F(\rho, R_{AB}(\rho_{BC}))$$

O. Fawzi and R. Renner, CMP 340 (2015)

Rotated Petz map

$$R_{AB}(\sigma) = \int dt\beta(t)\rho_{AB}^{\frac{1}{2}+it}\rho_B^{-\frac{1}{2}-it}\sigma\rho_B^{-\frac{1}{2}+it}\rho_{AB}^{\frac{1}{2}-it}$$

M. Junge, et. al. arXiv:1509.07127
D. Sutter, et. al. arXiv:1604.03023

ABC are arbitrary

➡ Related to theory of approximate error correction (subspaces)

S. Flammia et. al. , arXiv:1610.06169

CLASSIFICATION

Exact recovery

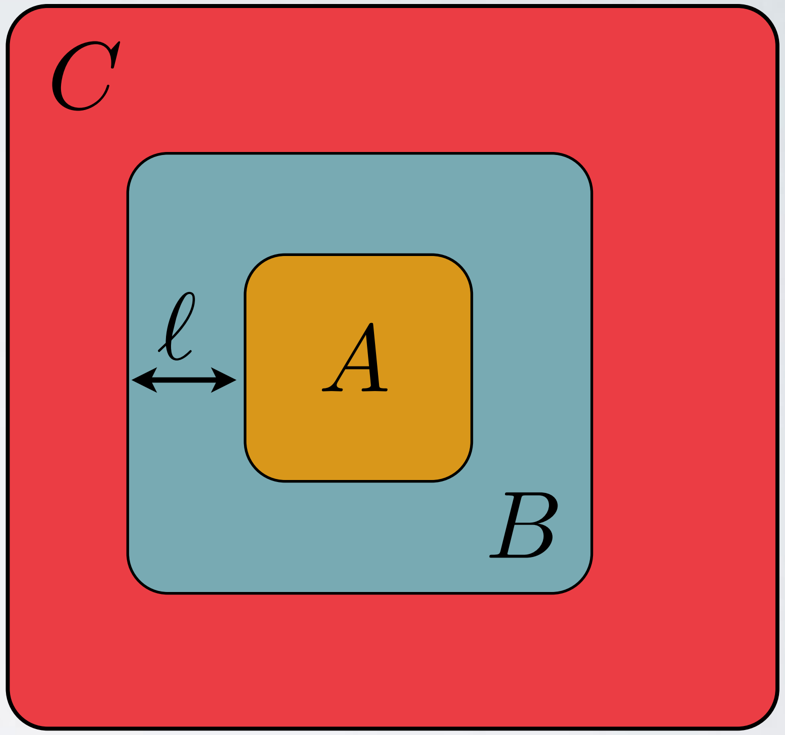
For any A, and B shielding A:
$$I_\rho(A : C|B) = 0$$

⇔ $\rho > 0$ is the Gibbs state of a local commuting H

W. Brown, D. Poulin, arXiv:1206.0755

⇔ $\rho = |\psi\rangle\langle\psi|$ is the ground state of a local commuting H

$$\mathcal{H} = \mathcal{H}_2^{\otimes N}$$



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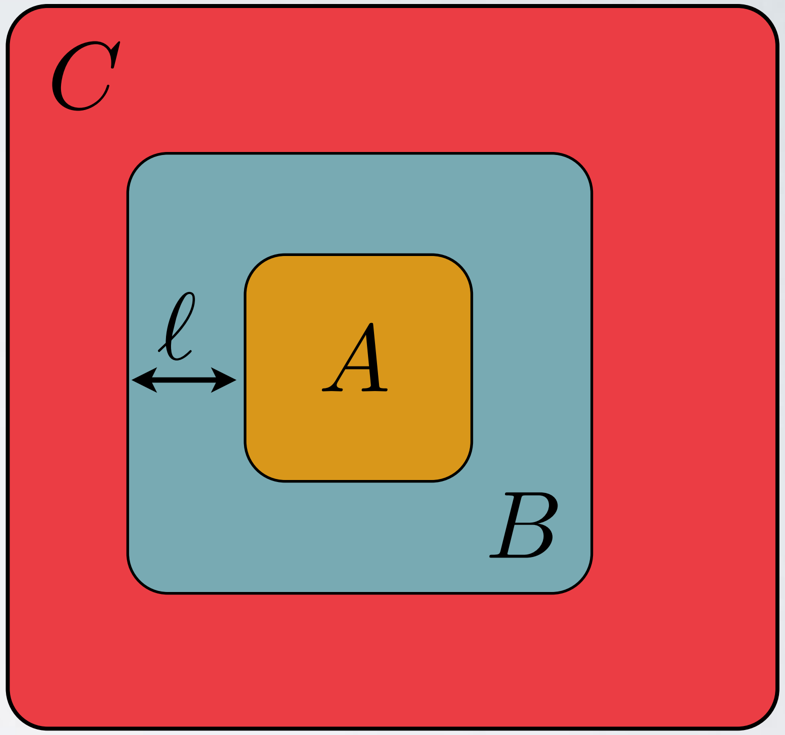

Approximate recovery

For any A, and B shielding A: $I_\rho(A:C|B) \leq Ke^{-c\ell}$

⇎ $\rho > 0$ is the Gibbs state of a quasi-local Hamiltonian

K. Kato, F Brandao, arXiv:1609.06636

⇎ $\rho = |\psi\rangle\langle\psi|$ is the ground state of a gaped quasi-local Hamiltonian

Dynamics?

MONTE-CARLO SIMULATIONS

Want to evaluate: $\langle Q \rangle = \sum_x \pi(x) Q(x)$ $\pi \propto e^{-\beta H}$

classical Gibbs state

Idea: - obtain a sample configuration from the distribution π

- Set up a Markov chain with π as an approximate fixed point

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Metropolis algorithm: (- start with random configuration)

- Flip a spin at random, calculate energy
- If energy decreased, accept the flip
- If energy increased, accept the flip with probability $p_{\text{flip}} = e^{-\beta \Delta E}$
- Repeat until equilibrium is reached

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ANALYTIC RESULTS

Note: - Glauber dynamics (Metropolis) is modelled by a semigroup

$$P_t = e^{tL}$$

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Fundamental result for Glauber dynamics:

π has exponentially decaying correlations ⇔ P_t mixes in time $O(\log(N))$ / L is gapped

F. Martinelli, Lect. Prof. Theor. Stats , Springer
A. Guionnet, B. Zegarlinski, Sem. Prob., Springer

- independent of boundary conditions in 2D
- independent of specifics of the model
- no intermediate mixing

QUANTUM GIBBS SAMPLERS

Commuting Hamiltonian

Davies maps are another generalization of Glauber dynamics

MJK and K. Temme, arXiv:1505.07811

$$T_t = e^{t\mathcal{L}}$$

$$\mathcal{L} = \sum_{j\in\Lambda}(R_{j\partial} - id)$$

$R_{j\partial}$ is the Petz recovery map!

The exists a partial extension of the **statics = dynamics** theorem

MJK and F. Brandao, CMP 344 (2016)

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The exists a partial extension of the **statics = dynamics** theorem ✸

MJK and F. Brandao, CMP 344 (2016)

Non-commuting Hamiltonian

$$\mathcal{L} = \sum_{j\in\Lambda}(R_{j\partial} - id)$$

$R_{j\partial}$ is the rotated Petz map!

- ➡ no longer frustration-free
- ➡ Theorem ✸ does not hold
- ➡ Davies maps are non-local

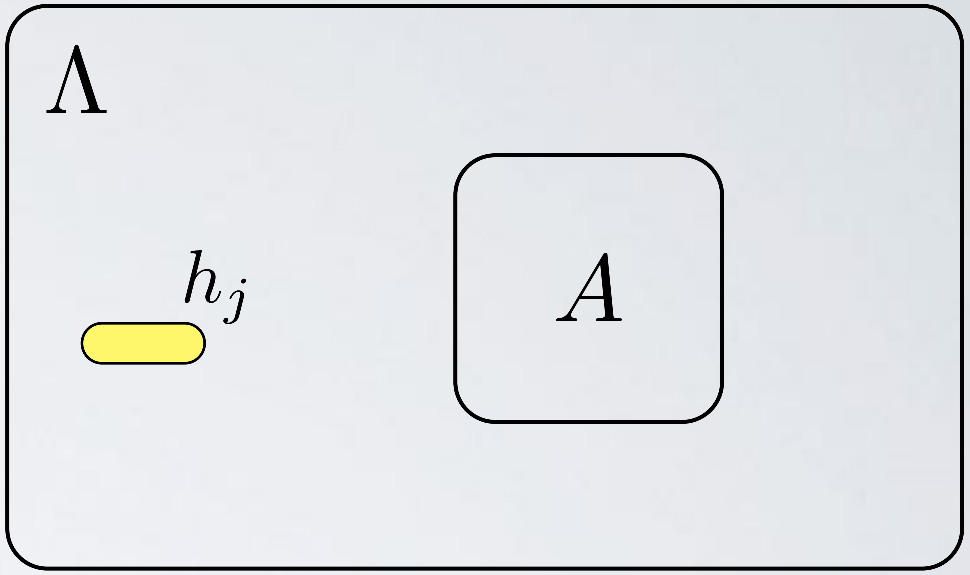
New approach

SETTING

Lattice: $A \subset \Lambda$

Hamiltonian: $H_A = \sum_{Z \subset A} h_Z$

$h_Z = 0 \text{ for } |Z| \geq K$



Gibbs states: $\rho^A = e^{-\beta H_A} / \mathrm{Tr}[e^{-\beta H_A}]$ is the Gibbs state restricted to A

Note: Superscript for domain of definition of Gibbs state, while subscript for partial trace.

THE MARKOV CONDITION

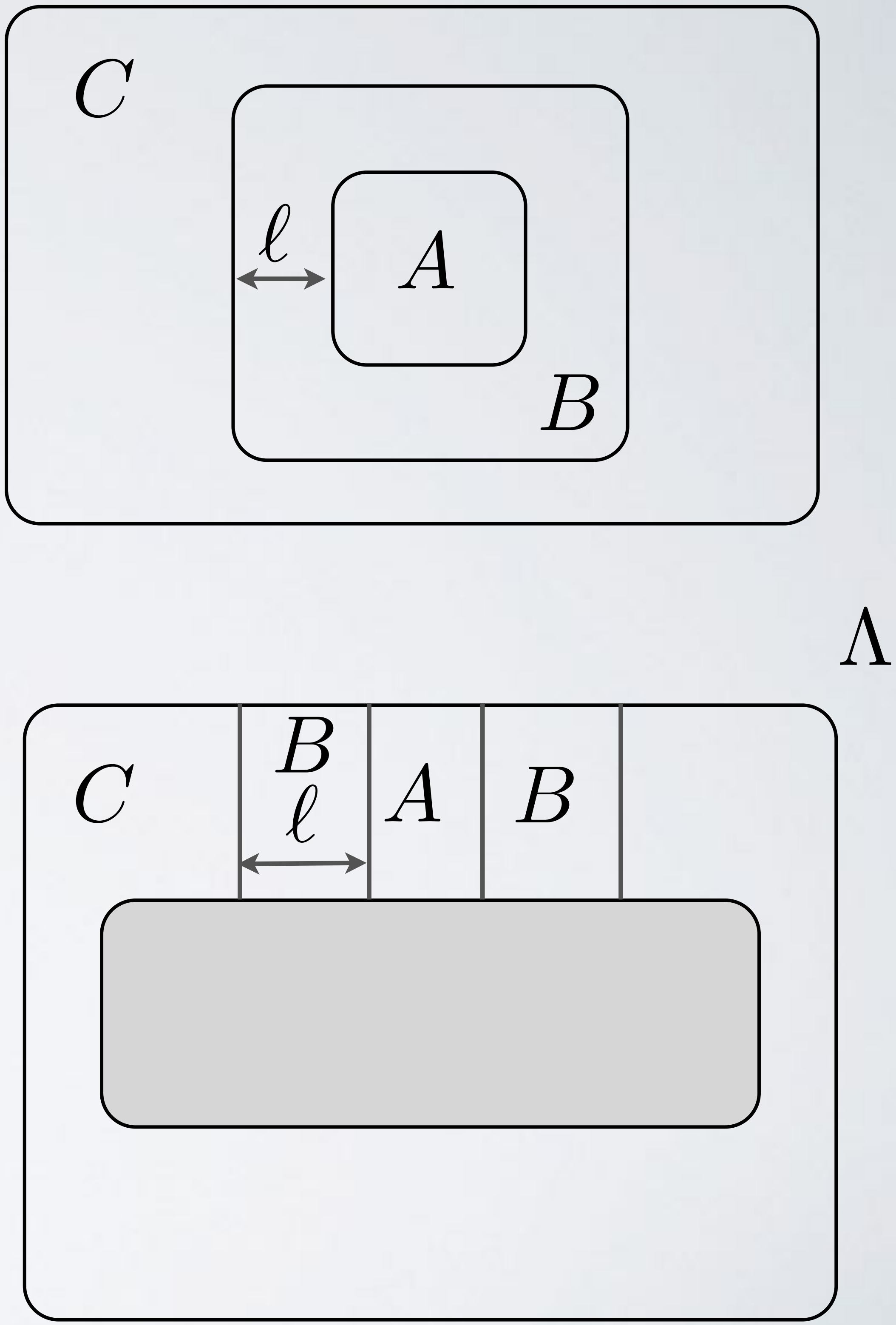
Uniform Markov:

Any subset $X = ABC \subset \Lambda$ with B shielding A from C in X, we have

$$I_{\rho^X}(A : C|B) \leq \delta(\ell)$$

Recall: $\rho^X = e^{-\beta H_X} / \mathrm{Tr}[e^{-\beta H_X}]$

Also must hold for non-contractible regions



CORRELATIONS

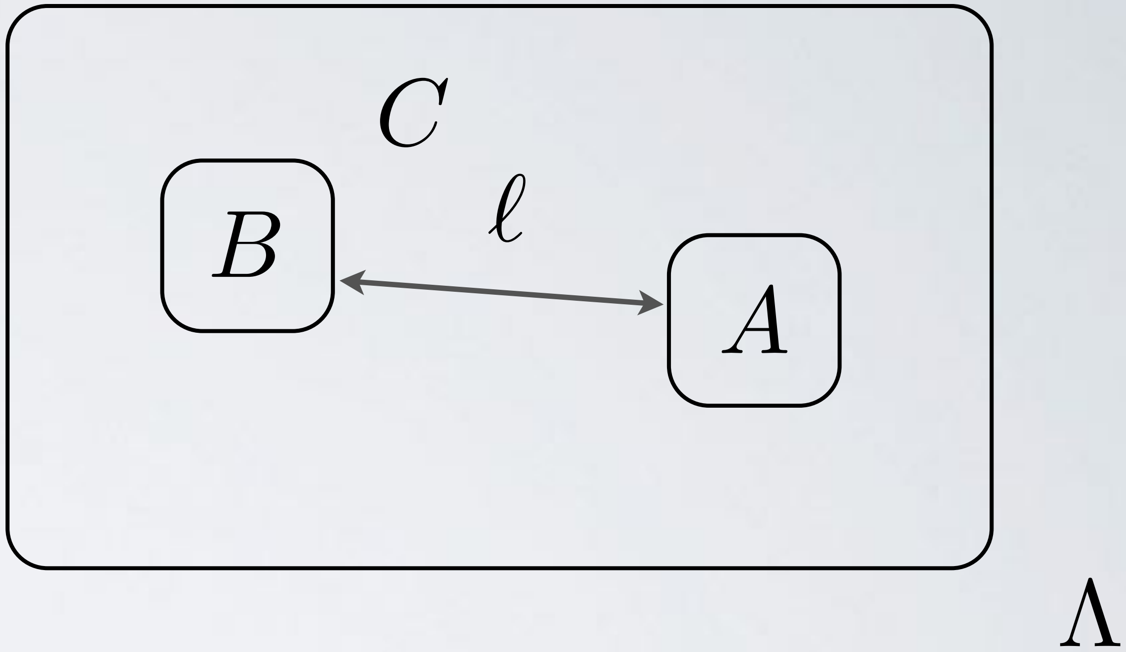
Uniform Clustering:

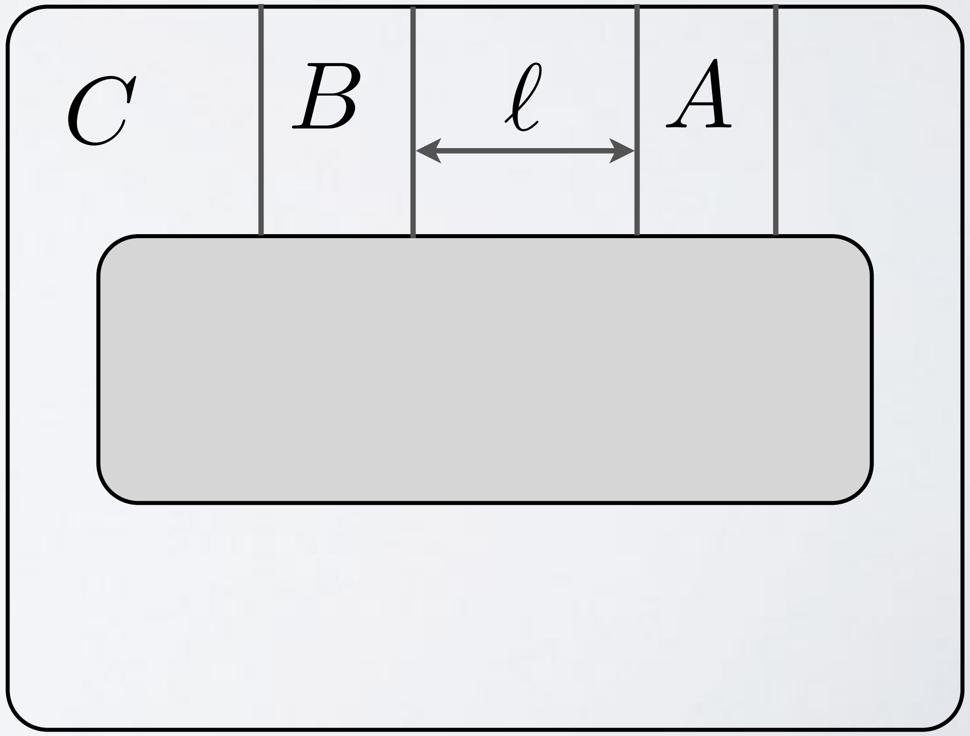
Any subset $X = ABC \subset \Lambda$ with $\mathrm{supp}(f) \subset A$ and $\mathrm{supp}(g) \subset B$

$$\mathrm{Cov}_{\rho^X}(f,g) \leq \epsilon(\ell)$$

$$\mathrm{Cov}_{\rho}(f,g) = |\mathrm{tr}[\rho f g] - \mathrm{tr}[\rho f]\mathrm{tr}[\rho g]|$$

Note: Uniform Clustering follows from uniform Gap



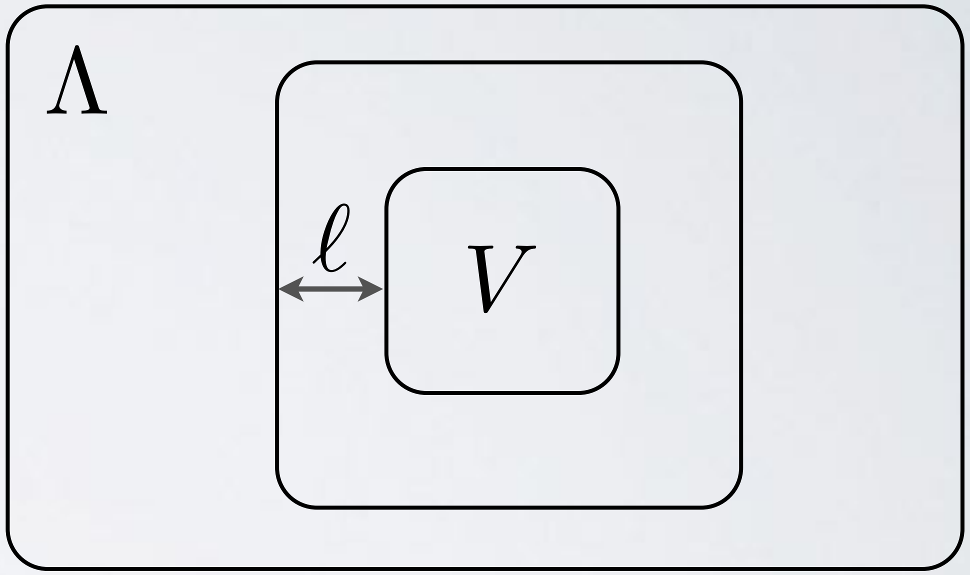


LOCAL PERTURBATIONS

Commuting Hamiltonian

$$e^{-\beta(H^A+H^B)} = e^{-\beta H^A} e^{-\beta H^B}$$

if $\quad [H^A, H^B] = 0$



Non-commuting Hamiltonian

General $\quad e^{-\beta(H+V)} = O_V e^{-\beta H} O_V^\dagger$

$$||O_V - O_V^\ell|| \leq c_1 e^{-c_2 \ell} \equiv \gamma(\ell) \qquad ||O_V|| \leq e^{\beta ||V||}$$

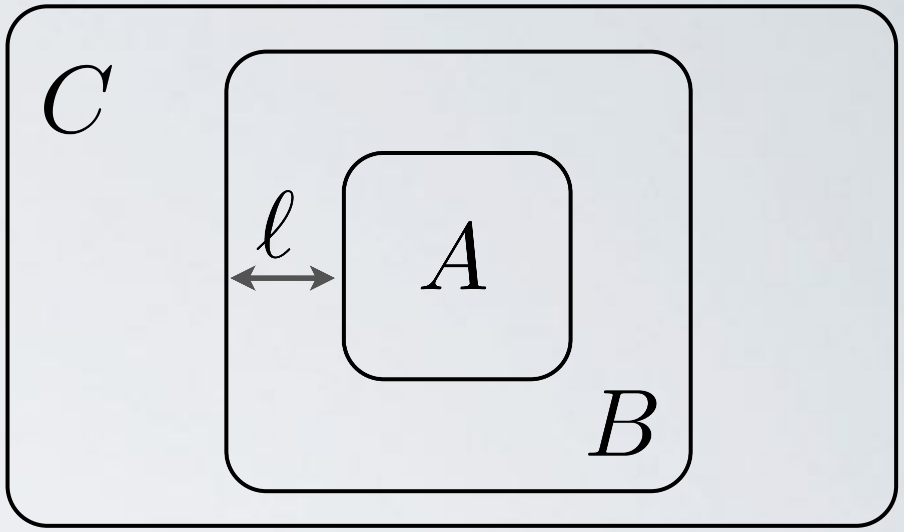
MB. Hastings, PRB 201102 (2007)

Only works if V is local!

APPROXIMATIONS

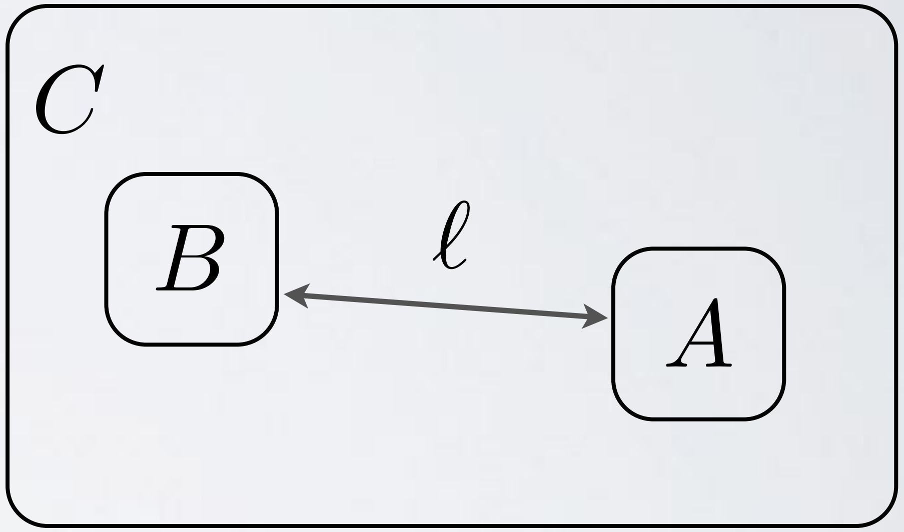
Uniform Markov

$$I_{\rho^X}(A:C|B) \leq \delta(\ell)$$



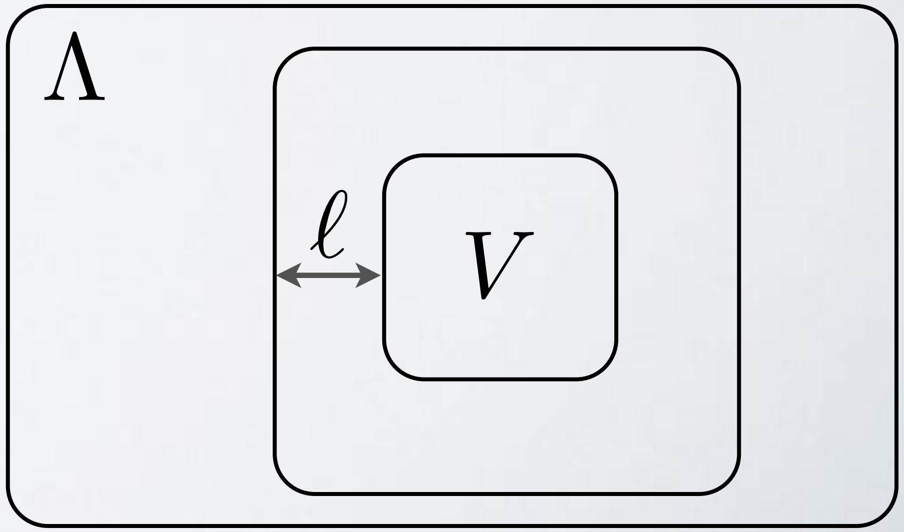
Uniform clustering

$$\mathrm{Cov}_{\rho^X}(f,g) \leq \epsilon(\ell)$$



Local perturbations

$$||e^{-\beta(H+V)} - O_V^\ell e^{-\beta H} O_V^\ell|| \leq c_1 e^{-c_2 \ell} \equiv \gamma(\ell)$$

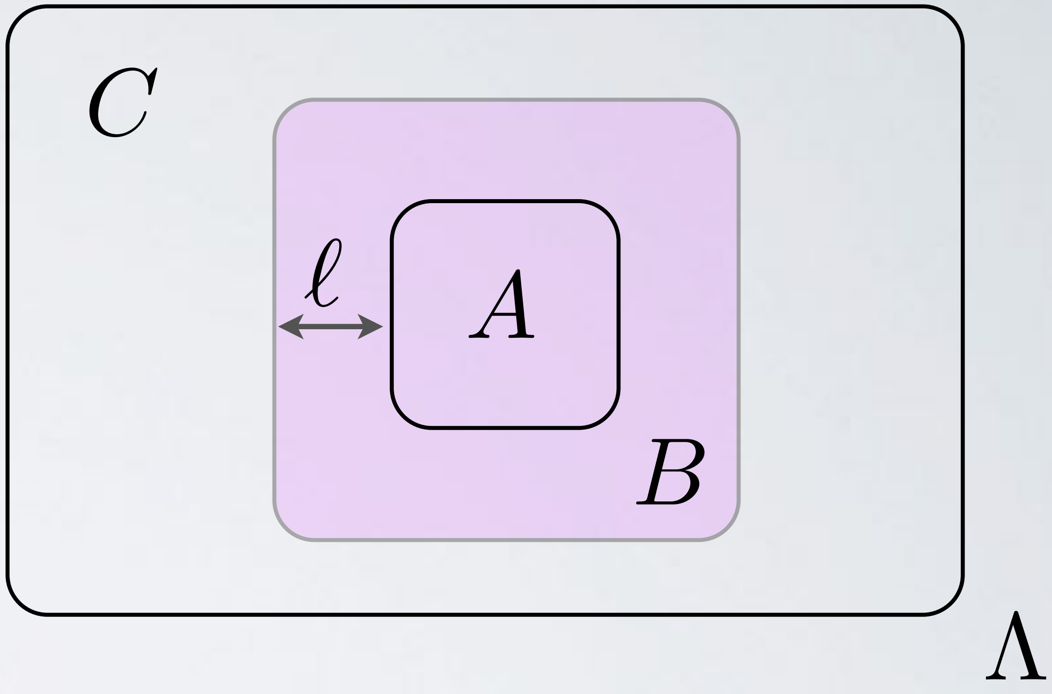


LOCAL INDISTINGUISHABILITY

Result 1:

Any subset $X = ABC \subset \Lambda$ with B shielding A from C in X, if ρ is uniformly clustering,

$$||\mathrm{tr}_{BC}[\rho^{ABC}] - \mathrm{tr}_B[\rho^{AB}]||_1 \leq c|AB|(\epsilon(\ell) + \gamma(\ell))$$



Consequence: Efficient evaluation of local expectation values

$$\langle O_A \rangle = \mathrm{tr}[\rho^\Lambda O_A] \approx \mathrm{tr}[\rho^{AB} O_A]$$

LOCAL INDISTINGUISHABILITY

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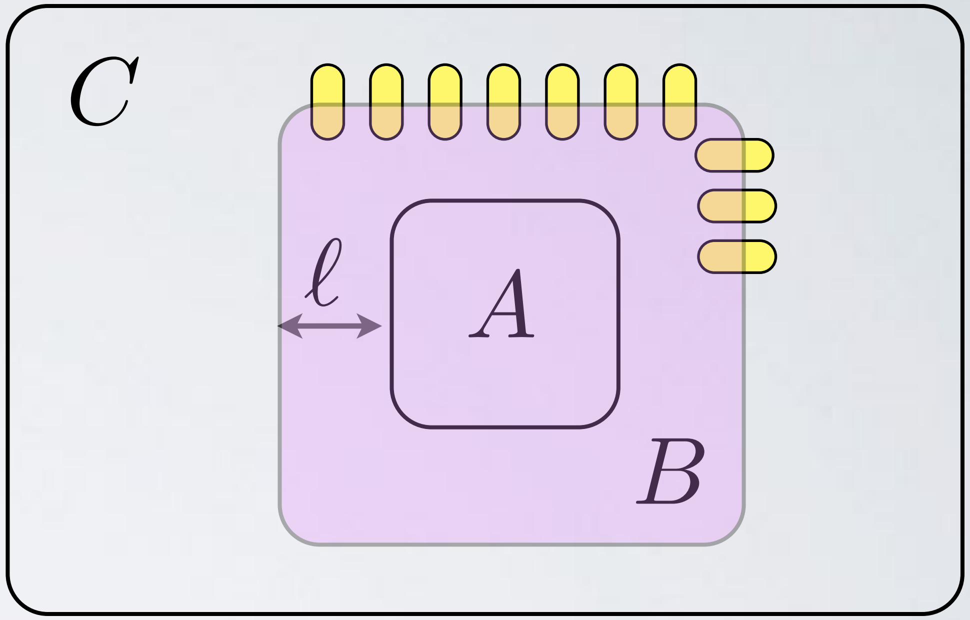
Proof idea:



Remove pieces of the boundary of B one by one

telescopic sum

$$||\mathrm{tr}_{BC}[\rho^{X} - \rho^{AB} \otimes \rho^{C}]||_1 \leq \sum_j ||\mathrm{tr}_{BC}[\rho^{X_{j+1}} - \rho^{X_j}]||_1$$

Bound each term

$$||\mathrm{tr}_{BC}[\rho^{X_{j+1}} - \rho^{X_j}]||_1 \approx \sup_{g_A} |\mathrm{tr}[g_A(O_j^{\ell}\rho^{X_j}O_j^{\ell,\dagger} - \rho^{X_j}]|$$
$$= \mathrm{Cov}_{\rho^{X_j}}(g_A, O_j^{\ell,\dagger}O_j^{\ell})$$

STATE PREPARATION

Main Result:

If ρ is uniformly clustering and uniformly Markov, then there exists a depth $D+1$ circuit of quantum channels $\mathbb{F} = \mathbb{F}_{D+1} \cdots \mathbb{F}_1$ of local range $O(\log(L))$, such that

$$||\mathbb{F}(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$

MJK, F. Brandao, arXiv:1609.07877

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MJK, F. Brandao, arXiv:1609.07877

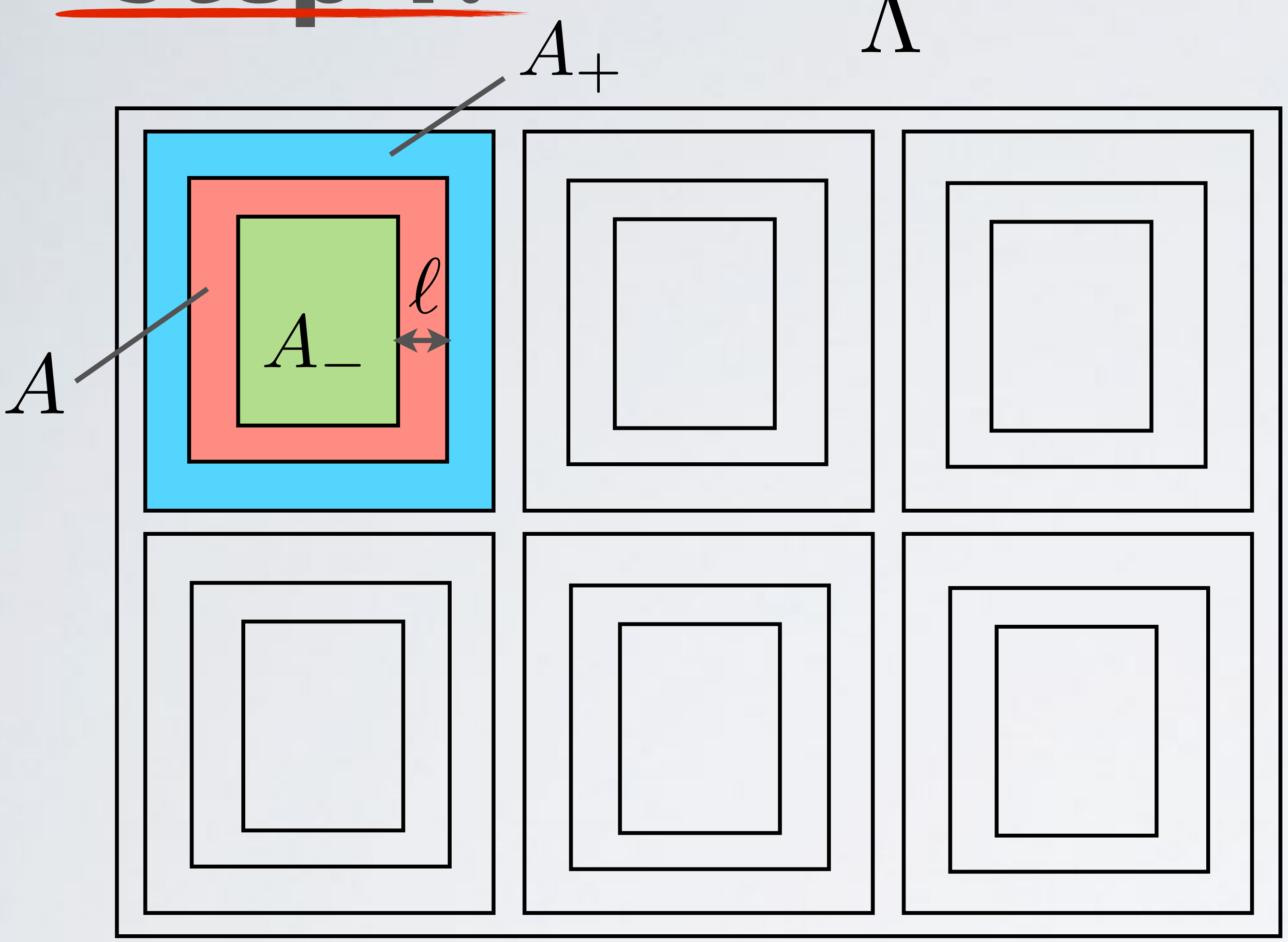
Corollary:

If ρ is uniformly clustering and uniformly Markov, then there exists a depth $M = O(\log(L))$ circuit of strictly local quantum channels $\mathbb{F} = \mathbb{F}_M \cdots \mathbb{F}_1$, such that

$$||\mathbb{F}(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$

PROOF OUTLINE (2D)

Step 1:



- Cover the lattice in concentric squares $A_- \subset A \subset A_+$
- By the Markov condition

$$||R^{\rho}_{A_+}(\rho_{A^c}) - \rho||_1 \leq N_A(\gamma(\ell) + \delta(\ell))$$

- By Local indistinguishability

$$||\mathrm{tr}_A[\rho^{A^c_-}_{A^c}] - \rho_{A^c}]||_1 \leq N_A \epsilon(\ell)$$

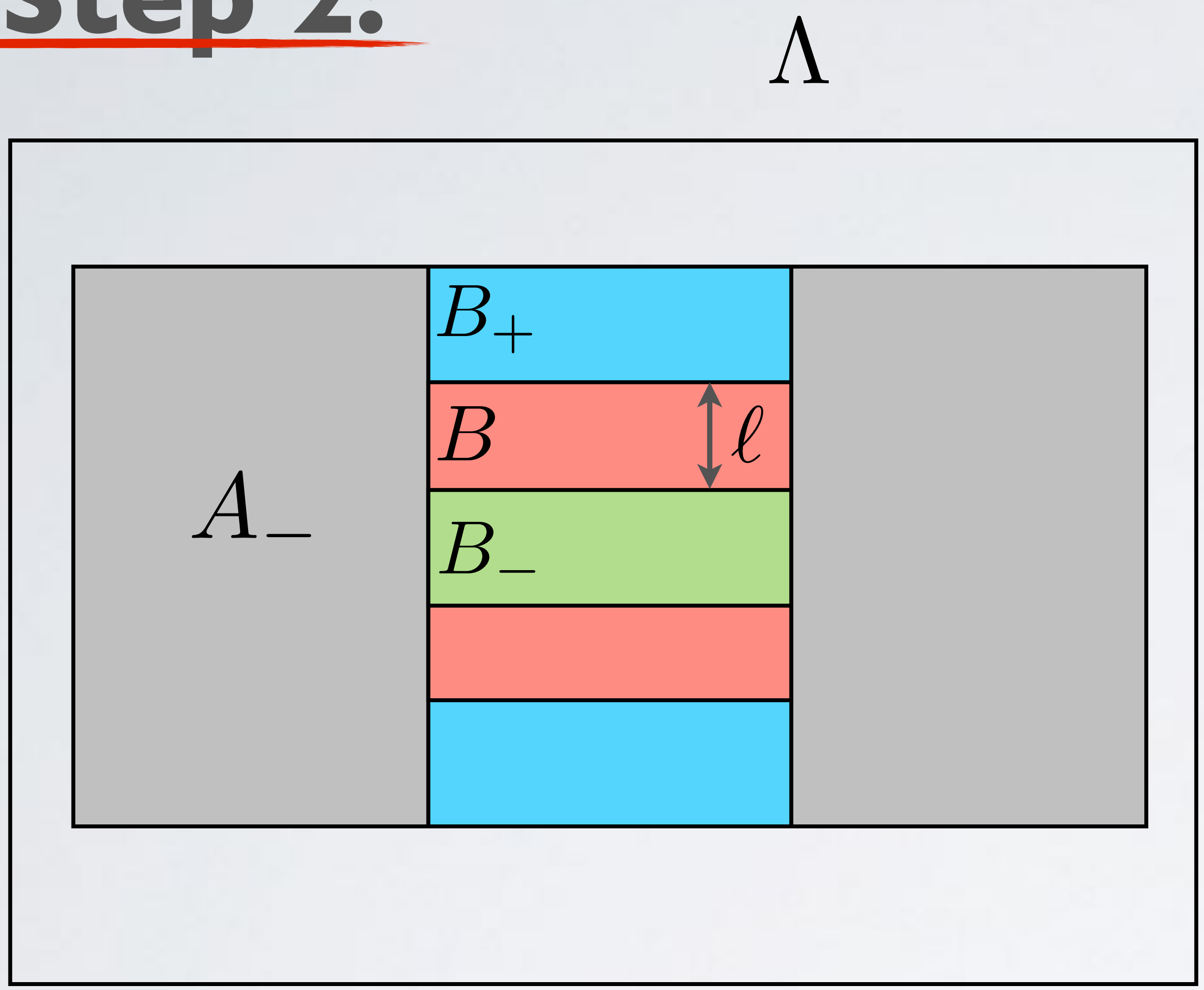
- Local cpt map $\mathbb{F}_A \equiv R^{\rho}_{A_+}\mathrm{tr}_A$

$$||\mathbb{F}_A(\rho^{A^c_-}) - \rho||_1 \leq N_A(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$$

➡ If we can build the lattice A^c_- with holes, then we can reconstruct the original lattice.

PROOF OUTLINE (2D)

Step 2:



- Break up the connecting regions

$$B_- \subset B \subset B_+$$

- By the Markov condition

$$||R_{B_+}^{\rho^{A^c_-}}(\rho_{B^c}^{A^c_-}) - \rho^{A^c_-}||_1 \leq N_B(\gamma(\ell) + \delta(\ell))$$

- By Local indistinguishability

$$||\mathrm{tr}_B[\rho^{(A_-B_-)^c}] - \rho_{B^c_-}^{A^c_-}]||_1 \leq N_B\epsilon(\ell)$$

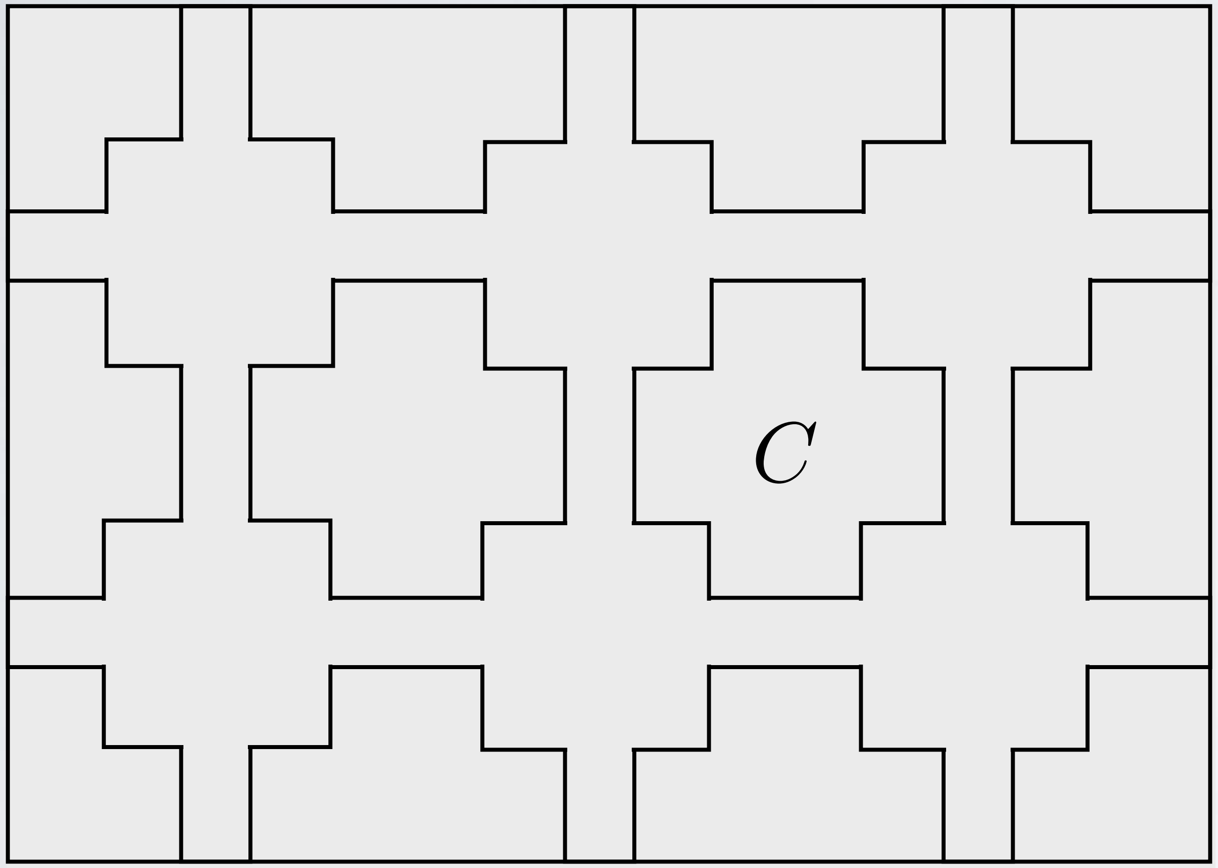
- Local cpt map $\mathbb{F}_B \equiv R_{B_+}^{\rho^{A^c_-}} \mathrm{tr}_B$

$$||\mathbb{F}_B\mathbb{F}_A(\rho^{(A_-B_-)^c}) - \rho||_1 \leq (N_A + N_B)(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$$

If we can build the lattice $(A_-B_-)^c$, then we can reconstruct the original lattice.

PROOF OUTLINE (2D)

Step 3:



- Project onto ρ^C
- By locality

$$\mathbb{F}_C(\psi) = \rho^c \mathrm{tr}_C[\psi]$$

- Finally $||\mathbb{F}_C \mathbb{F}_B \mathbb{F}_A(\psi) - \rho||_1 \leq (N_C + N_A + N_B)(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$

➡ The entire lattice can be built from a local circuit of cpt maps.

GROUND STATES?

Proof ingredients

- (uniform) Local indistinguishability
- (uniform) Markov condition
- Local definition of states

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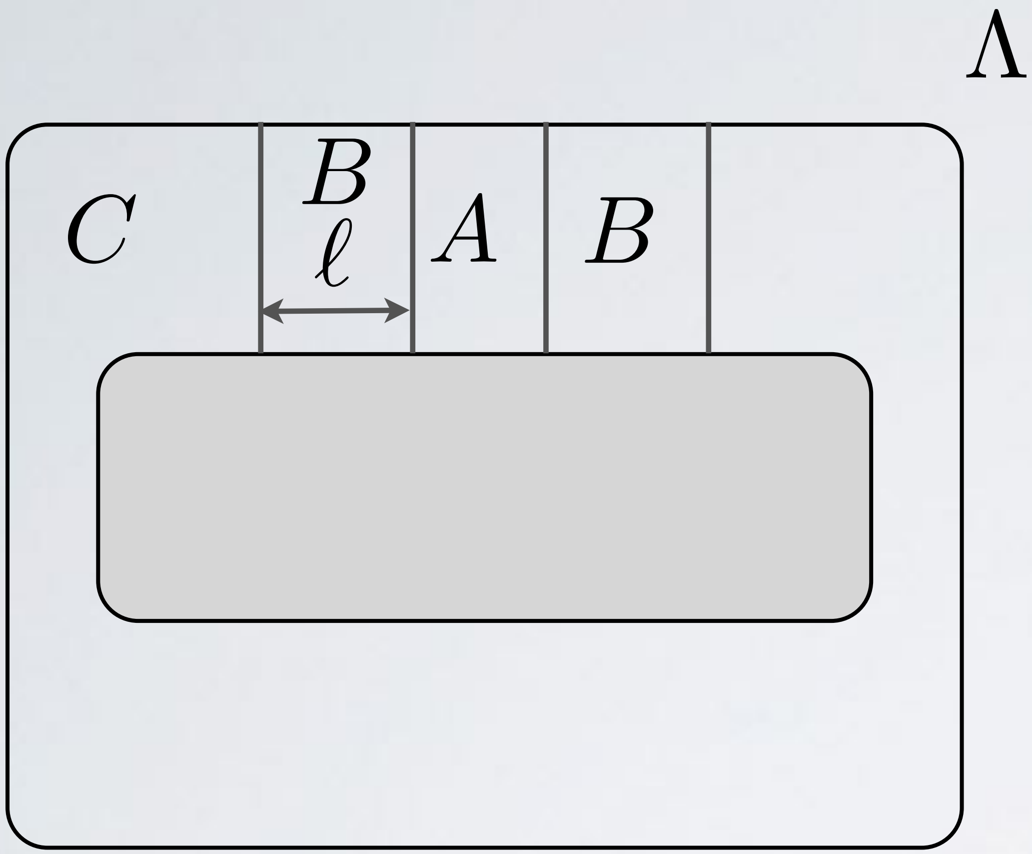
- (uniform) Markov condition
- Local definition of states

➡ For injective PEPS, proof can be reproduced exactly.

➡ Connection to the topological entanglement entropy

TOPOLOGICAL ENTANGLEMENT



Area law:

$$I(A:C|B) \leq \epsilon(\ell) + \nu$$

ν is a topological contribution

➡ Local indistinguishability and zero topological entanglement implies efficient preparation

OUTLOOK

Spectral gap analysis, entanglement spectrum

The same strategy might work for proving gaps of parent Hamiltonians of injective PEPS

Relaxing the assumption on uniform decay

More natural assumptions

Other applications of local indistinguishability to many body systems

Complete the classification

THANK YOU!

SPECTRAL GAP

We showed: $||\mathbb{F}_C\mathbb{F}_B\mathbb{F}_A(\psi) - \rho||_1 \leq L^D e^{-\ell/\xi}$

Define $\mathbb{F}_A = e^{t\mathcal{L}_A}$ $\mathcal{L}_A = \sum_j (\mathbb{F}_{A_i} - \text{id})$

If $\mathbb{F}_A, \mathbb{F}_B, \mathbb{F}_C$ had the same fixed point, then $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C$ is gaped, by the reverse detectability lemma.

A. Anshu, et. al., Phys. Rev. B 93, 205142 (2016)

- The same strategy might work for proving gaps of parent Hamiltonians of injective PEPS
- New strategy for proving the gap of the 2D AKLT model!!!

All about boundary conditions